

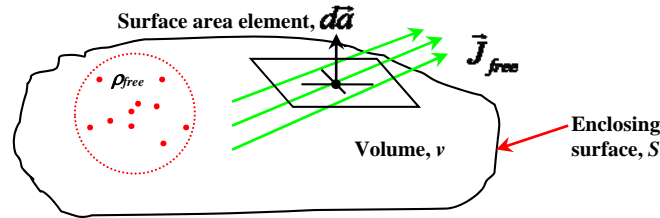
# LECTURE NOTES 1

## CONSERVATION LAWS

Conservation of energy  $E$ , linear momentum  $\vec{p}$ , angular momentum  $\vec{L}$  and electric charge  $q$  are of fundamental importance in electrodynamics (*n.b.* this is also true for all fundamental forces of nature – the weak, strong, *EM* and gravitational force, both microscopically (locally), and hence macroscopically (globally - *i.e.* the entire universe)!

### Electric Charge Conservation

Previously (*i.e.* last semester in Physics 435), we discussed electric charge conservation:



Electric current flowing outward from volume  $v$  through closed bounding surface  $S$  at time  $t$ :

$$I_{free}(t) = \oint_S \vec{J}_{free}(\vec{r}, t) \cdot d\vec{a} \quad (\text{Amperes})$$

Electric charge contained in volume  $v$  at time  $t$ :

$$Q_{free}(t) = \int_v \rho_{free}(\vec{r}, t) d\tau \quad (\text{Coulombs})$$

An outward flow of current through surface  $S$  corresponds to a decrease in charge in volume  $v$ :

$$I_{free}(t) = -\frac{dQ_{free}(t)}{dt} \quad (\text{Amperes} = \text{Coulombs/sec}) \quad \text{i.e.} \quad \frac{dQ_{free}(t)}{dt} < 0, \quad I_{free}(t) = -\frac{dQ_{free}(t)}{dt} > 0$$

Global conservation of electric charge:

$$I_{free}(t) = \oint_S \vec{J}_{free}(\vec{r}, t) \cdot d\vec{a} = -\frac{dQ_{free}(t)}{dt}$$

But: 
$$\frac{dQ_{free}(t)}{dt} = \frac{d}{dt} \int_v \rho_{free}(\vec{r}, t) d\tau = \int_v \frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} d\tau$$

Use the divergence theorem on the LHS of the global conservation of charge equation:

$$\int_v \vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) d\tau = -\int_v \frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} d\tau \quad \Leftarrow \text{Integral form of the continuity equation.}$$

This relation must hold for any arbitrary volume  $v$  associated with the enclosing surface  $S$ ; hence the integrands in the above equation must be equal – we thus obtain the continuity equation (in differential form), which expresses local conservation of electric charge at  $(\vec{r}, t)$ :

$$\vec{\nabla} \cdot \vec{J}_{free}(\vec{r}, t) = -\frac{\partial \rho_{free}(\vec{r}, t)}{\partial t} \quad \Leftarrow \text{Differential form of the continuity equation.}$$

*n.b.* The continuity equation doesn't explain why electric charge is conserved – it merely describes mathematically that electric charge is conserved!!

## Poynting's Theorem and Poynting's Vector $\vec{S}(\vec{r}, t)$

We know that the work required to assemble a *static* charge distribution is:

$$W_E(t) = \frac{\epsilon_o}{2} \int_v E^2(\vec{r}, t) d\tau = \frac{\epsilon_o}{2} \int_v (\vec{E}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)) d\tau = \frac{1}{2} \int_v (\vec{D}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)) d\tau \quad \text{SI units: Joules}$$

Linear Dielectric Media

Likewise, the work required to get electric currents flowing, *e.g.* against a back *EMF* is:

$$W_M(t) = \frac{1}{2\mu_o} \int_v B^2(\vec{r}, t) d\tau = \frac{1}{2\mu_o} \int_v (\vec{B}(\vec{r}, t) \cdot \vec{B}(\vec{r}, t)) d\tau = \frac{1}{2} \int_v (\vec{H}(\vec{r}, t) \cdot \vec{B}(\vec{r}, t)) d\tau \quad \text{SI units: Joules}$$

Linear Magnetic Media

Thus the total energy,  $U_{EM}$  stored in *EM* field(s) is (by energy conservation) = total work done:

$$U_{EM}(t) = W_{tot}(t) = W_{EM}(t) = W_E(t) + W_M(t) = \frac{1}{2} \int_v \left( \epsilon_o E^2(\vec{r}, t) + \frac{1}{\mu_o} B^2(\vec{r}, t) \right) d\tau = \int_v u_{EM}(\vec{r}, t) d\tau \quad \text{SI units: Joules}$$

$$U_{EM}(t) = \int_v u_{EM}(\vec{r}, t) d\tau = \frac{1}{2} \int_v \left( \epsilon_o E^2(\vec{r}, t) + \frac{1}{\mu_o} B^2(\vec{r}, t) \right) d\tau \quad \text{SI units: Joules}$$

where  $u_{EM}$  = total energy density:  $u_{EM}(\vec{r}, t) = \frac{1}{2} \left( \epsilon_o E^2(\vec{r}, t) + \frac{1}{\mu_o} B^2(\vec{r}, t) \right)$  (SI units: *Joules/m<sup>3</sup>*)

Suppose we have some charge density  $\rho(\vec{r}, t)$  and current density  $\vec{J}(\vec{r}, t)$  configuration(s) that at time  $t$  produce *EM* fields  $\vec{E}(\vec{r}, t)$  and  $\vec{B}(\vec{r}, t)$ . In the next instant  $dt$ , *i.e.* at time  $t + dt$ , the charge moves around. What is the amount of infinitesimal work  $dW$  done by *EM* forces acting on these charges / currents, in the time interval  $dt$ ?

The Lorentz Force Law is:  $\vec{F}(\vec{r}, t) = q(\vec{E}(\vec{r}, t) + \vec{v}(\vec{r}, t) \times \vec{B}(\vec{r}, t))$

The infinitesimal amount of work  $dW$  done on an electric charge  $q$  moving an infinitesimal distance  $d\vec{\ell} = \vec{v}dt$  in an infinitesimal time interval  $dt$  is:

$$dW = \vec{F} \cdot d\vec{\ell} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell} = q\vec{E} \cdot \vec{v}dt + q \underbrace{(\vec{v} \times \vec{B}) \cdot \vec{v}dt}_{=0} = q\vec{E} \cdot \vec{v}dt \quad \text{(n.b. magnetic forces do no work!!)}$$

But:  $\rho_{free}(\vec{r}, t) = \rho_{free}(\vec{r}, t) d\tau$  and:  $\rho_{free}(\vec{r}, t) \vec{v}(\vec{r}, t) = \vec{J}_{free}(\vec{r}, t)$

The (instantaneous) rate at which (total) work is done on all of the electric charges within the volume  $v$  is:

$$\begin{aligned} \frac{dW(t)}{dt} &= \int_v \vec{F}(\vec{r}, t) \cdot (d\vec{\ell}(\vec{r}, t)/dt) = \int_v \vec{F}(\vec{r}, t) \cdot \vec{v}(\vec{r}, t) = \int_v q_{free}(\vec{r}, t) \vec{E}(\vec{r}, t) \cdot \vec{v}(\vec{r}, t) \\ &= \int_v \rho_{free}(\vec{r}, t) d\tau \vec{E}(\vec{r}, t) \cdot \vec{v}(\vec{r}, t) \quad \text{using: } q_{free}(\vec{r}, t) = \rho_{free}(\vec{r}, t) d\tau \\ &= \int_v (\vec{E}(\vec{r}, t) \cdot \rho_{free}(\vec{r}, t) \vec{v}(\vec{r}, t)) d\tau \quad \text{but: } \vec{J}_{free}(\vec{r}, t) = \rho_{free}(\vec{r}, t) \vec{v}(\vec{r}, t) \end{aligned}$$

$$\therefore \boxed{\frac{dW(t)}{dt} = \int_v (\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t)) d\tau = P(t)} = \text{instantaneous power (SI units: Watts)}$$

The quantity  $\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t)$  is the (instantaneous) work done per unit time, per unit volume – *i.e.* the instantaneous power delivered per unit volume (aka the power density).

Thus: 
$$\boxed{P(t) = \frac{dW(t)}{dt} = \int_v (\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t)) d\tau} \quad (\text{SI units: Watts} = \frac{\text{Joules}}{\text{sec}})$$

We can express the quantity  $(\vec{E} \cdot \vec{J}_{free})$  in terms of the *EM* fields (alone) using the Ampere-Maxwell law (in differential form) to eliminate  $\vec{J}_{free}$ .

Ampere's Law with Maxwell's Displacement Current correction term (in differential form):

Thus: 
$$\boxed{\vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu_o \{ \vec{J}_{free}(\vec{r}, t) + \vec{J}_D(\vec{r}, t) \} = \mu_o \vec{J}_{free}(\vec{r}, t) + \mu_o \epsilon_o \frac{\partial \vec{E}(\vec{r}, t)}{\partial t}}$$

Then: 
$$\boxed{\vec{J}_{free}(\vec{r}, t) = \frac{1}{\mu_o} (\vec{\nabla} \times \vec{B}(\vec{r}, t)) - \epsilon_o \frac{\partial \vec{E}(\vec{r}, t)}{\partial t}}$$

Then: 
$$\boxed{\begin{aligned} \vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t) &= \vec{E}(\vec{r}, t) \cdot \left\{ \frac{1}{\mu_o} (\vec{\nabla} \times \vec{B}(\vec{r}, t)) - \epsilon_o \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} \right\} \\ &= \frac{1}{\mu_o} \vec{E}(\vec{r}, t) \cdot (\vec{\nabla} \times \vec{B}(\vec{r}, t)) - \epsilon_o \vec{E}(\vec{r}, t) \cdot \frac{\partial \vec{E}(\vec{r}, t)}{\partial t} \end{aligned}}$$

Now:  $\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$  Griffiths Product Rule #6 (see inside front cover)

Thus:  $\vec{E} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$

But Faraday's Law (in differential form) is: 
$$\boxed{\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}}$$

$$\therefore \vec{E} \cdot (\vec{\nabla} \times \vec{B}) = -\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$$

However:  $\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{B}) = \frac{1}{2} \frac{\partial}{\partial t} (B^2)$  and similarly:  $\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) = \frac{1}{2} \frac{\partial}{\partial t} (E^2)$

Therefore:

$$\begin{aligned}\vec{E}(\vec{r},t) \cdot \vec{J}_{free}(\vec{r},t) &= -\frac{1}{\mu_0} \left\{ -\frac{1}{2} \frac{\partial}{\partial t} (B^2(\vec{r},t)) - \vec{\nabla} \cdot (\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)) \right\} - \epsilon_0 \left\{ \frac{1}{2} \frac{\partial}{\partial t} (E^2(\vec{r},t)) \right\} \\ &= -\frac{1}{2} \frac{\partial}{\partial t} \left( \epsilon_0 E^2(\vec{r},t) + \frac{1}{\mu_0} B^2(\vec{r},t) \right) - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t))\end{aligned}$$

Then:

$$\begin{aligned}P(t) &= \frac{dW(t)}{dt} = \int_v (\vec{E}(\vec{r},t) \cdot \vec{J}_{free}(\vec{r},t)) d\tau \\ &= -\frac{1}{2} \frac{d}{dt} \int_v \left( \epsilon_0 E^2(\vec{r},t) + \frac{1}{\mu_0} B^2(\vec{r},t) \right) d\tau - \frac{1}{\mu_0} \underbrace{\int_v \vec{\nabla} \cdot (\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)) d\tau}_{\uparrow}\end{aligned}$$

Apply the divergence theorem to this term, get:

**Poynting's Theorem = "Work-Energy" Theorem of Electrodynamics:**

$$P(t) = \frac{dW(t)}{dt} = -\frac{d}{dt} \int_v \left\{ \frac{1}{2} \left( \epsilon_0 E^2(\vec{r},t) + \frac{1}{\mu_0} B^2(\vec{r},t) \right) \right\} d\tau - \frac{1}{\mu_0} \oint_S (\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)) \cdot d\vec{a}$$

Physically,  $\frac{1}{2} \int_v \left( \epsilon_0 E^2(\vec{r},t) + \frac{1}{\mu_0} B^2(\vec{r},t) \right) d\tau$  = instantaneous energy stored in the *EM* fields ( $\vec{E}(\vec{r},t)$  and  $\vec{B}(\vec{r},t)$ ) within the volume  $v$  (SI units: Joules)

Physically, the term  $-\frac{1}{\mu_0} \oint_S (\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)) \cdot d\vec{a}$  = instantaneous rate at which *EM* energy is carried / flows out of the volume  $v$  (carried microscopically by virtual (and/or real!) photons across the bounding/enclosing surface  $S$  by the *EM* fields  $\vec{E}$  and  $\vec{B}$  – *i.e.* this term represents/is the instantaneous *EM* power flowing across/through the bounding/enclosing surface  $S$  (SI units: *Watts = Joules/sec*).

**Poynting's Theorem says that:**

The instantaneous work done on the electric charges in the volume  $v$  by the *EM* force is equal to the decrease in the instantaneous energy stored in *EM* fields ( $\vec{E}$  and  $\vec{B}$ ), minus the energy that is instantaneously flowing out of/through the bounding surface  $S$ .

We define **Poynting's vector:**  $\vec{S}(\vec{r},t) \equiv \frac{1}{\mu_0} (\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t))$  = energy / unit time / unit area,

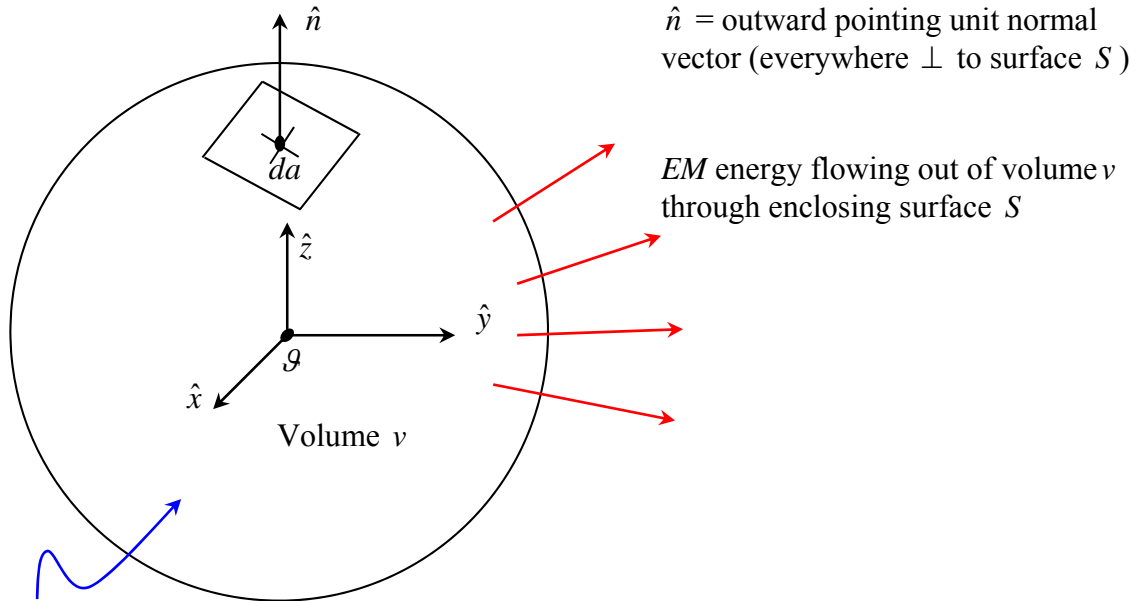
transported by the *EM* fields ( $\vec{E}$  and  $\vec{B}$ ) across/through the bounding surface  $S$

*n.b.* Poynting's vector  $\vec{S}$  has SI units of Watts/m<sup>2</sup> – *i.e.* an energy flux density.

Thus, we see that:

$$P(t) = \frac{dW(t)}{dt} = -\frac{dU_{EM}(t)}{dt} - \oint_S \vec{S}(\vec{r}, t) \cdot d\vec{a}$$

where  $\vec{S}(\vec{r}, t) \cdot d\vec{a}$  = instantaneous power (energy per unit time) crossing/passing through an infinitesimal surface area element  $d\vec{a} = \hat{n} da$ , as shown in the figure below:



Poynting's vector:  $\vec{S}(\vec{r}, t) \equiv \frac{1}{\mu_0} \vec{E}(\vec{r}, t) \times \vec{B}(\vec{r}, t)$  = Energy Flux Density (SI units: *Watts/m<sup>2</sup>*)

The work  $W$  done on the electrical charges contained within the volume  $v$  will increase their mechanical energy – kinetic and/or potential energy. Define the (instantaneous) mechanical energy density  $u_{mech}(\vec{r}, t)$  such that:

$$\frac{du_{mech}(\vec{r}, t)}{dt} = \vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t) \quad \text{Hence:} \quad \frac{dU_{mech}}{dt} = \int_v (\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t)) d\tau$$

Then:

$$P(t) = \frac{dW(t)}{dt} = \frac{dU_{mech}}{dt} = \frac{d}{dt} \int_v u_{mech}(\vec{r}, t) d\tau = \int_v (\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t)) d\tau$$

However, the (instantaneous) EM field energy density is:

$$u_{EM}(\vec{r}, t) = \frac{1}{2} \left( \epsilon_0 E^2(\vec{r}, t) + \frac{1}{\mu_0} B^2(\vec{r}, t) \right) \quad (\text{Joules/m}^3)$$

Then the (instantaneous) EM field energy contained within the volume  $v$  is:

$$U_{EM}(t) = \int_v u_{EM}(\vec{r}, t) d\tau \quad (\text{Joules})$$

Thus, we see that: 
$$\frac{d}{dt} \int_V (u_{mech}(\vec{r}, t) + u_{EM}(\vec{r}, t)) d\tau = -\oint_S \vec{S}(\vec{r}, t) \cdot d\vec{a} = -\int_V (\vec{\nabla} \cdot \vec{S}(\vec{r}, t)) d\tau$$

Using the Divergence theorem

The integrands of LHS vs. {far} RHS of the above equation **must** be equal for each/every space-time point  $(\vec{r}, t)$  within the source volume  $V$  associated with bounding surface  $S$ . Thus, we obtain:

**The Differential Form of Poynting's Theorem:** 
$$\frac{\partial}{\partial t} [u_{mech}(\vec{r}, t) + u_{EM}(\vec{r}, t)] = -\vec{\nabla} \cdot \vec{S}(\vec{r}, t)$$

**Poynting's theorem = Energy Conservation** "book-keeping" equation, *c.f.* with the **Continuity equation = Charge Conservation** "book-keeping" equation:

**The Differential Form of the Continuity Equation:** 
$$\frac{\partial}{\partial t} \rho(\vec{r}, t) = -\vec{\nabla} \cdot \vec{J}(\vec{r}, t)$$

Since  $\frac{\partial u_{mech}(\vec{r}, t)}{\partial t} = \vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t)$ , we can write the differential form of Poynting's theorem as:

$$\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t) + \frac{\partial u_{EM}(\vec{r}, t)}{\partial t} = -\vec{\nabla} \cdot \vec{S}(\vec{r}, t)$$

Or:

$$\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t) + \frac{\partial u_{EM}(\vec{r}, t)}{\partial t} + \vec{\nabla} \cdot \vec{S}(\vec{r}, t) = 0$$

Poynting's Theorem / Poynting's vector  $\vec{S}(\vec{r}, t)$  represents the (instantaneous) flow of *EM* energy in exactly the same/analogous way that the free current density  $\vec{J}_{free}(\vec{r}, t)$  represents the (instantaneous) flow of electric charge.

In the presence of *linear* dielectric / *linear* magnetic media, if one is ONLY interested in FREE charges and FREE currents, then:

$$u_{EM}^{free}(\vec{r}, t) = \frac{1}{2} (\vec{E}(\vec{r}, t) \cdot \vec{D}(\vec{r}, t) + \vec{B}(\vec{r}, t) \cdot \vec{H}(\vec{r}, t))$$

$$\vec{D}(\vec{r}, t) = \epsilon \vec{E}(\vec{r}, t)$$

$$\epsilon = \epsilon_o (1 + \chi_e)$$

$$\vec{S}(\vec{r}, t) = \frac{1}{\mu} \vec{E}(\vec{r}, t) \times \vec{B}(\vec{r}, t) = \vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t)$$

$$\vec{B}(\vec{r}, t) = \mu \vec{H}(\vec{r}, t)$$

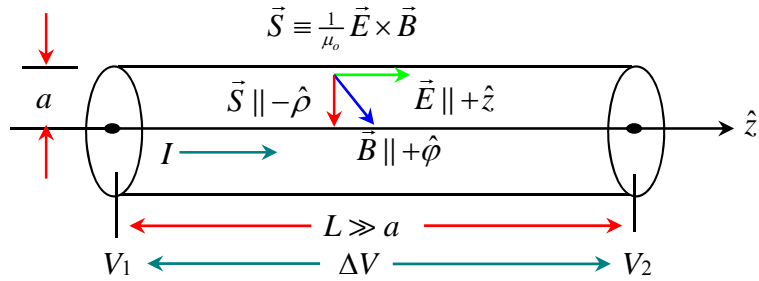
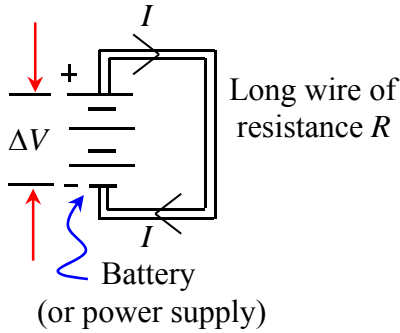
$$\mu = \mu_o (1 + \chi_m)$$

**Griffiths Example 8.1:**

 Poynting's vector  $\vec{S}$ , power dissipation and Joule heating of a long, current-carrying wire.

 When a steady, free electrical current  $I$  ( $\neq$  function of time,  $t$ ) flows down a long wire of length  $L \gg a$  ( $a$  = radius of wire) and resistance  $R$  ( $= L/\pi a^2 \sigma_c$ ), the electrical energy is dissipated as heat (*i.e.* thermal energy) in the wire.

Electrical power dissipation:  $P = \Delta V \cdot I = I^2 R$



Free Current Density:

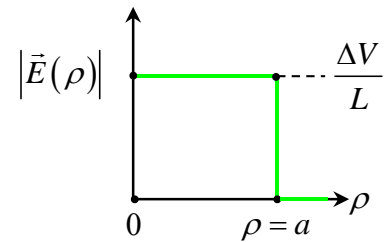
$$\vec{J}_{free} = \sigma_c \vec{E} = (I/\pi a^2) \hat{z} \quad (\text{Amps}/\text{m}^2)$$

Longitudinal Electric Field:

$$\vec{E} = \frac{\vec{J}_{free}}{\sigma_c} = \frac{\Delta V}{L} \hat{z} \quad (\text{Volts}/\text{m})$$

Potential Difference:

$$\Delta V = V_1 - V_2 (> 0) \quad (\text{Volts})$$



*n.b.* The {steady} free current density  $\vec{J}_{free} (= \sigma_c \vec{E} = I/\pi a^2)$  and the longitudinal electric field  $\vec{E} = (\Delta V/L) \hat{z}$  are uniform across (and along) the long wire, everywhere within the volume of the wire ( $\rho < a$ ).  $\Rightarrow$  Thus, this particular problem has no time-dependence...

From Ampere's Law:

$$\vec{B}^{inside} (\rho < a) = \frac{\mu_0 I \rho}{2\pi a^2} \hat{\phi} \quad \rho = \sqrt{x^2 + y^2} \text{ in cylindrical coordinates}$$

$$\left\{ \oint_C \vec{B}(\vec{r}) \cdot d\vec{\ell} = \mu_0 I_{encl} \right\}$$

$$\vec{B}^{outside} (\rho \geq a) = \frac{\mu_0 I}{2\pi \rho} \hat{\phi} \quad (\text{Tesla})$$

*n.b.* for simplicity's sake, we have approximated the finite length wire by an  $\infty$ -length wire. This will have unphysical, but understandable consequences later on....

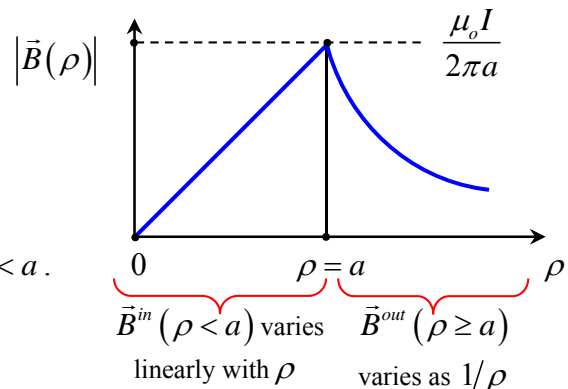
Poynting's Vector:

$$\vec{S}(\vec{r}) = \frac{1}{\mu_0} \vec{E}(\vec{r}) \times \vec{B}(\vec{r})$$

$$\vec{S}^{inside} (\rho < a) = \frac{\Delta V \cdot I \rho}{2\pi a^2 L} \overbrace{(\hat{z} \times \hat{\phi})}^{=-\hat{\rho}} = \frac{\Delta V \cdot I \rho}{2\pi a^2 L} (-\hat{\rho})$$

 Poynting's vector  $\vec{S}$  oriented radially inward for  $\rho < a$ .

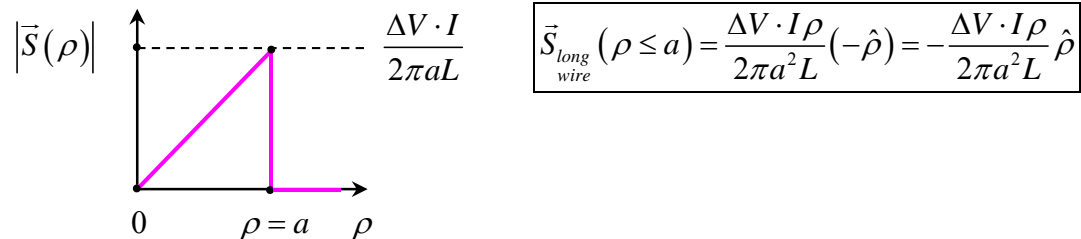
$$\vec{S}^{outside} (\rho > a) = 0 \quad \{\text{because } \vec{E}(\rho > a) = 0 \text{ !!!}\}$$



Note the following result for Poynting's vector evaluated at the surface of the long wire, *i.e.* @  $\rho = a$  :

$$\vec{S}^{inside}(\rho = a) = \frac{\Delta V \cdot I}{2\pi a L} (-\hat{\rho}) \quad (\text{SI units: Watts/m}^2)$$

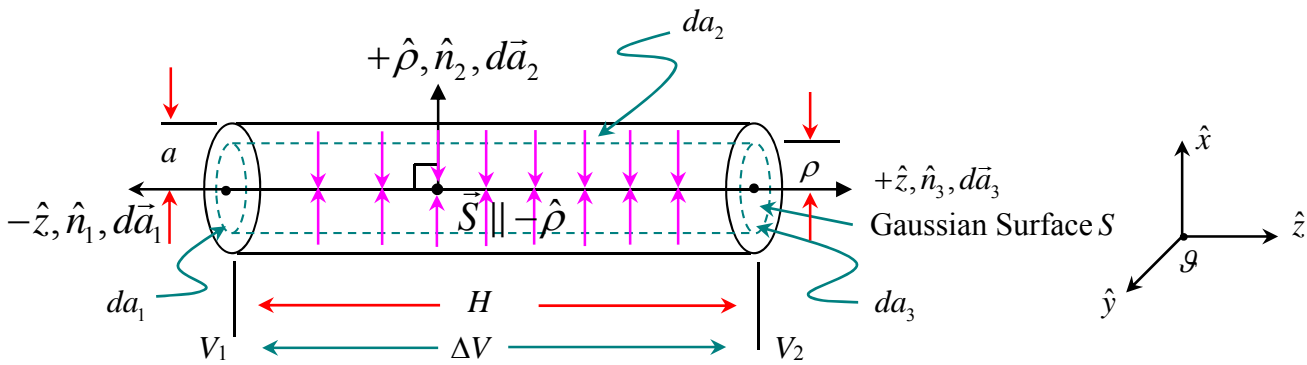
Since  $\vec{E}^{outside}(\rho \geq a) = 0$  :  $\vec{S}^{outside}(\rho = a) = 0 \Rightarrow \exists$  a **discontinuity** in  $\vec{S}$  at  $\rho = a$  !!!



Now let us use the *integral* version of Poynting's theorem to determine the *EM* energy flowing through an imaginary Gaussian cylindrical surface  $S$  of radius  $\rho < a$  and length  $H \ll L$  :

$$\begin{aligned} P(t) &= \frac{dW(t)}{dt} = \frac{dU_{mech}}{dt} = \frac{d}{dt} \int_v u_{mech}(\vec{r}, t) d\tau = \int_v (\vec{E}(\vec{r}, t) \cdot \vec{J}_{free}(\vec{r}, t)) d\tau \\ &= -\frac{dU_{EM}(t)}{dt} - \oint_S \vec{S}(\vec{r}, t) \cdot d\vec{a} = -\frac{d}{dt} \int_v u_{EM}(\vec{r}, t) d\tau - \int_v (\vec{\nabla} \cdot \vec{S}(\vec{r}, t)) d\tau \end{aligned}$$

Since this is a static/steady-state problem, we assume that  $dU_{EM}(t)/dt = 0$ .



Then for an imaginary Gaussian surface taken *inside* the long wire ( $\rho < a$ ) :

$$P_{wire} = -\oint_S \vec{S}_{wire} \cdot d\vec{a} = -\underbrace{\int_{LHS}^{=0} \vec{S} \cdot d\vec{a}_1}_{d\vec{a}_1 = da_1(-\hat{z})} - \underbrace{\int_{cyl} \vec{S} \cdot d\vec{a}_2}_{d\vec{a}_2 = da_2 \hat{\rho}} - \underbrace{\int_{RHS}^{=0} \vec{S} \cdot d\vec{a}_3}_{d\vec{a}_3 = da_3(+\hat{z})}$$

$\vec{S}(\parallel -\hat{\rho})$  is  $\perp$  to  $d\vec{a}_1(\parallel -\hat{z})$ ;  $\vec{S}(\parallel -\hat{\rho})$  is anti- $\parallel$  to  $d\vec{a}_2(\parallel +\hat{\rho})$ ;  $\vec{S}(\parallel -\hat{\rho})$  is  $\perp$  to  $d\vec{a}_3(\parallel +\hat{z})$

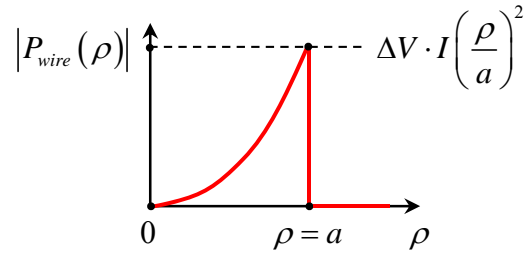
Only surviving term is:

$$P_{wire}(\rho) = -\int_{cyl} \vec{S}(\rho) \cdot d\vec{a}_2 = -\int_{z=-H/2}^{z=+H/2} \int_{\phi=0}^{\phi=2\pi} \left( -\frac{\Delta V \cdot I \rho}{2\pi a^2 H} \hat{\rho} \right) \rho d\phi dz \hat{\rho} = \left( \frac{\Delta V \cdot I}{2\pi a^2 H} \rho \right) (2\pi \rho H) = \Delta V \cdot I \left( \frac{\rho^2}{a^2} \right)$$



Thus: 
$$P_{\text{wire}}(\rho) = \Delta V \cdot I \left( \frac{\rho}{a} \right)^2 \quad (\text{Watts})$$

And: 
$$P_{\text{wire}}(\rho = a) = \Delta V \cdot I \quad (\text{Watts})$$



This *EM* energy is dissipated as heat (thermal energy) in the wire – also known as Joule heating of the wire. Since  $|P_{\text{wire}}(\rho)| \propto \rho^2$ , note also that the Joule heating of the wire occurs primarily at/on the outermost portions of the wire.

From Ohm's Law:  $\Delta V = I \cdot R_{\text{wire}}$  where  $R_{\text{wire}} = \text{resistance of wire} = \rho_c^{\text{wire}} L / A_{\perp}^{\text{wire}} = L / \sigma_c^{\text{wire}} A_{\perp}^{\text{wire}}$

Joule Heating  
of current-  
carrying wire

$$P_{\text{wire}}(\rho) = -I^2 R_{\text{wire}} \left( \frac{\rho}{a} \right)^2$$

$$P_{\text{wire}}(\rho = a) = -I^2 R_{\text{wire}}$$

Power losses in wire show up / result in Joule heating of wire. Electrical energy is converted into heat (thermal) energy – At the microscopic level, this is due to *kinetic* energy losses associated with the ensemble of individual drift/conduction/free electron scatterings inside the wire!

Again use the integral version of Poynting's theorem to determine the *EM* field energy flowing through an imaginary Gaussian cylindrical surface  $S$  of radius  $\rho \geq a$  and length  $H \ll L$ .

We expect that we should get the same answer as that obtained above, for the  $\rho < a$  Gaussian cylindrical surface. However, for  $\rho \geq a$ ,  $\vec{S}^{\text{outside}}(\rho > a) = 0$ , because  $\vec{E}^{\text{outside}}(\rho > a) = 0$  !!!

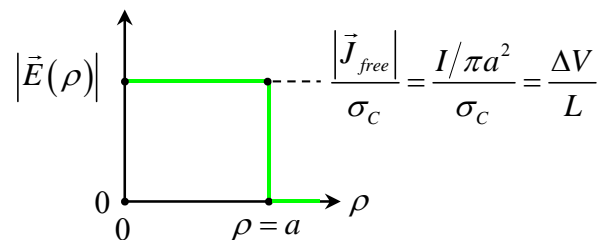
Thus, for a Gaussian cylindrical surface  $S$  taken with  $\rho \geq a$  we obtain:  $P_{\text{wire}} = -\oint_S \vec{S}_{\text{wire}} \cdot d\vec{a} = 0$  !!!

What??? How can we get two different  $P_{\text{wire}}$  answers for  $\rho < a$  vs.  $\rho \geq a$  ??? This can't be!!!

⇒ We need to re-assess our assumptions here...

It turns out that we have neglected an important, and somewhat subtle point...

The longitudinal electric field  $\vec{E} = (\Delta V/L)\hat{z}$  formally/mathematically has a discontinuity at  $\rho = a$ :

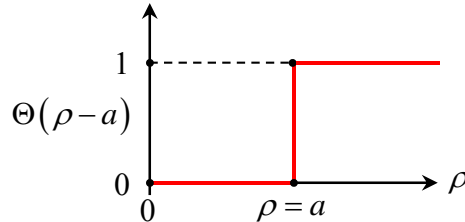


*i.e.* The tangential ( $\hat{z}$ ) component of  $\vec{E}$  is discontinuous at  $\rho = a$ .

Formally/mathematically, we need to write the longitudinal electric field for this situation as:

$$\vec{E}(\rho) = \frac{\vec{J}_{free}}{\sigma_c} [1 - \Theta(\rho - a)] = \frac{|\vec{J}_{free}|}{\sigma_c} [1 - \Theta(\rho - a)] \hat{z}$$

where the **Heaviside step function** is defined as:  $\Theta(\rho - a) \equiv \begin{cases} 0 & \text{for } \rho < a \\ 1 & \text{for } \rho \geq a \end{cases}$  as shown below:



Furthermore, note that:  $\Theta(x) = \int_{-\infty}^x \delta(t) dt$  and that:  $\frac{d}{dx} \Theta(x) = \delta(x)$ ,

where  $\delta(x)$  is the Dirac delta function.

Now, in the process of **deriving** Poynting's theorem (above), we used Griffith's Product Rule # 6 to obtain  $\vec{E} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{B})$ , and then used Faraday's law (in differential form)

$\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t$  and then used  $\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{B}) = \frac{1}{2} \frac{\partial}{\partial t} (B^2)$  and  $\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) = \frac{1}{2} \frac{\partial}{\partial t} (E^2)$

with  $u_{EM} = \frac{1}{2} (\epsilon_o E^2 + \frac{1}{\mu_o} B^2)$  to finally obtain:

$$\begin{aligned} P(t) &= \frac{dW(t)}{dt} = \frac{dU_{mech}}{dt} = \frac{d}{dt} \int_v u_{mech} d\tau = \int_v \vec{E} \cdot \vec{J}_{free} d\tau \\ &= -\frac{dU_{EM}(t)}{dt} - \oint_s \vec{S} \cdot d\vec{a} = -\frac{d}{dt} \int_v u_{EM} d\tau - \int_v \vec{\nabla} \cdot \vec{S}(\vec{r}, t) d\tau \end{aligned}$$

So here, in this specific problem, what is  $\vec{\nabla} \times \vec{E}$  ???

In cylindrical coordinates, the only non-vanishing term is:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial \rho} E_z \hat{\phi} = \frac{\partial}{\partial \rho} \left\{ -\frac{|\vec{J}_{free}|}{\sigma_c} [1 - \Theta(\rho - a)] \right\} \hat{\phi} = +\frac{|\vec{J}_{free}|}{\sigma_c} \frac{\partial \Theta(\rho - a)}{\partial \rho} \hat{\phi} = \frac{|\vec{J}_{free}|}{\sigma_c} \delta(\rho - a) \hat{\phi} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{In other words: } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \begin{cases} 0 & \text{for } \rho < a \\ \frac{|\vec{J}_{free}|}{\sigma_c} \hat{\phi} & \text{for } \rho = a \\ 0 & \text{for } \rho > a \end{cases}$$

Thus, {only} for  $\rho > a$  integration volumes, we {very definitely} need to {explicitly} include the  $\delta$ -function such that its contribution to the integral at  $\rho = a$  is properly taken into account!

$$\begin{aligned}
 P(t) &= \frac{dW(t)}{dt} = -\frac{d}{dt} \int_V u_{EM} d\tau - \oint_S \vec{S} \cdot d\vec{a} \\
 &= -\frac{d}{dt} \int_V \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) d\tau - \oint_S \vec{S} \cdot d\vec{a} \\
 &= -\frac{1}{2} \epsilon_0 \int_V \frac{d}{dt} E^2 d\tau - \frac{1}{2\mu_0} \int_V \frac{d}{dt} B^2 d\tau - \oint_S \vec{S} \cdot d\vec{a} \\
 &= -\epsilon_0 \int_V \vec{E} \cdot \frac{d\vec{E}}{dt} d\tau - \frac{1}{\mu_0} \int_V \vec{B} \cdot \frac{d\vec{B}}{dt} d\tau - \oint_S \vec{S} \cdot d\vec{a} \\
 &= -\epsilon_0 \int_V \vec{E} \cdot \frac{d\vec{E}}{dt} d\tau + \frac{1}{\mu_0} \int_V \vec{B} \cdot \vec{\nabla} \times \vec{E} d\tau - \oint_S \vec{S} \cdot d\vec{a} \\
 &= -\epsilon_0 \int_V \vec{E} \cdot \frac{d\vec{E}}{dt} d\tau + \frac{|\vec{J}_{free}|}{\mu_0 \sigma_C} \int_V \vec{B} \cdot \delta(\rho - a) \hat{\phi} d\tau - \oint_S \vec{S} \cdot d\vec{a}
 \end{aligned}$$

For this specific problem:  $d\vec{E}/dt = 0$  and for  $\rho > a$ ,  $\vec{S}(\rho > a) = \frac{1}{\mu_0} \underbrace{\vec{E}(\rho > a)}_{=0} \times \vec{B}(\rho > a) = 0$ .

Thus for  $\rho > a$ :

$$P(t) = \frac{|\vec{J}_{free}|}{\mu_0 \sigma_C} \int_V \vec{B} \cdot \delta(\rho - a) \hat{\phi} d\tau = 2\pi a L \frac{|\vec{J}_{free}|}{\mu_0 \sigma_C} |\vec{B}(\rho = a)| = \cancel{2\pi a L} \frac{|\vec{J}_{free}|}{\mu_0 \sigma_C} \frac{\cancel{\mu_0} I}{\cancel{2\pi a}} = \frac{|\vec{J}_{free}|}{\sigma_C} I \cdot L$$

But:  $\vec{E} = \frac{\vec{J}_{free}}{\sigma_C} = \frac{\Delta V}{L} \hat{z}$ , and thus, finally we obtain, for  $\rho > a$ :  $P(t) = \frac{\Delta V}{L} I \cdot L = \Delta V \cdot I$ ,

which agrees precisely with that obtained earlier for  $\rho < a$ :  $P(t) = \Delta V \cdot I$ !!!

For an *E&M* problem that nominally has a **steady-state** current  $I$  present, it is indeed curious that

$\vec{\nabla} \times \vec{E} = \frac{|\vec{J}_{free}|}{\sigma_C} \delta(\rho - a) \hat{\phi} = -\frac{\partial \vec{B}}{\partial t}$  is non-zero, and in fact singular {at  $\rho = a$  }! The singularity is a consequence of the discontinuity in  $\vec{E}$  on the  $\rho = a$  surface of the long, current-carrying wire.

The relativistic nature of the 4-dimensional space-time world that we live in is **encrypted** into Faraday's law; here is one example where we come face-to-face with it!

Let's pursue the physics of this problem a bit further – and calculate the magnetic vector potential  $\vec{A}(\vec{r})$  inside ( $\rho < a$ ) and outside ( $\rho > a$ ) the long wire...

In general, we know/anticipate that {here}:  $\vec{A}(\vec{r}) \parallel \vec{J}(\vec{r}) \parallel +\hat{z}$  since:  $\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}')}{r} d\tau'$

where  $r = |\vec{r}| \equiv |\vec{r} - \vec{r}'|$ .

We don't need to carry out the above integral to obtain  $\vec{A}(\vec{r})$  – a simpler method is to use  $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$  in cylindrical coordinates. Since  $\vec{A}(\vec{r}) = A_z(\vec{r}) \hat{z}$  (only, here), the only non-zero contribution to this curl is:  $\vec{B}(\vec{r}) = -\frac{\partial A_z(\vec{r})}{\partial \rho} \hat{\phi}$ .

$$\text{For } \rho < a: \vec{B}(\rho < a) = \frac{\mu_o I \rho}{2\pi a^2} \hat{\phi} = \frac{1}{2} \mu_o J \rho \hat{\phi} = -\frac{\partial A_z(\rho < a)}{\partial \rho} \hat{\phi} \Rightarrow \frac{\partial \vec{A}(\rho < a)}{\partial \rho} = -\frac{1}{2} \mu_o J \rho \hat{z}$$

$$\text{For } \rho \geq a: \vec{B}(\rho \geq a) = \frac{\mu_o I}{2\pi \rho} \hat{\phi} = \frac{1}{2} \mu_o J a^2 \left(\frac{1}{\rho}\right) \hat{\phi} = -\frac{\partial A_z(\rho \geq a)}{\partial \rho} \hat{\phi} \Rightarrow \frac{\partial \vec{A}(\rho \geq a)}{\partial \rho} = -\frac{1}{2} \mu_o J a^2 \left(\frac{1}{\rho}\right) \hat{z}$$

Using  $\rho = a$  as our reference point for carrying out the integration {and noting that as in the case for the scalar potential  $V(\vec{r})$ , we similarly have the freedom to *e.g.* add any constant vector to  $\vec{A}(\vec{r})$  }:

$$\vec{A}(\rho < a) = -\frac{1}{2} \mu_o J \int \rho d\rho \hat{z} = -\frac{1}{2} \mu_o J \frac{1}{2} (\rho^2 - c_1^2) \hat{z} = -\frac{1}{4} \mu_o J (\rho^2 - c_1^2) \hat{z}$$

$$\vec{A}(\rho \geq a) = -\frac{1}{2} \mu_o J a^2 \int \left(\frac{1}{\rho}\right) d\rho \hat{z} = -\frac{1}{2} \mu_o J a^2 \ln(\rho/c_2) \hat{z}$$

where  $c_1$  and  $c_2$  are constants of the integration(s).

Physically, we demand that  $\vec{A}(\rho)$  be continuous at  $\rho = a$ , thus we must have:

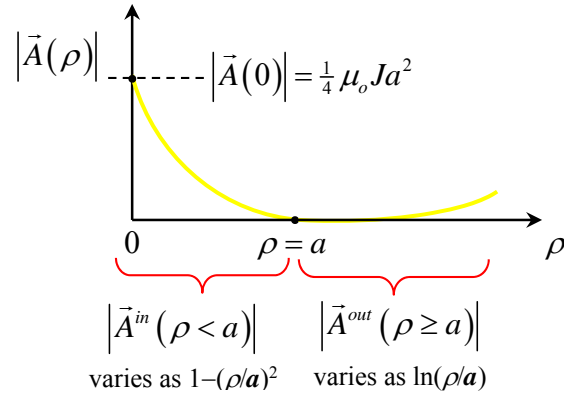
$$\vec{A}(\rho = a) = -\frac{1}{4} \mu_o J (a^2 - c_1^2) \hat{z} = -\frac{1}{2} \mu_o J a^2 \ln(a/c_2) \hat{z}$$

Obviously, the only way that this relation can be satisfied is if  $c_1 = c_2 = \pm a$ , because then  $\vec{A}(\rho = a) = 0$  {*n.b.*  $\ln(1) = \ln e^0 = 0$ }.

Additionally, we demand that  $\vec{A}(\vec{r}) \parallel \vec{J}(\vec{r}) \parallel +\hat{z}$ , hence the physically acceptable solution is  $c_1 = c_2 = -a$ , and thus the solutions for the magnetic vector potential  $\vec{A}(\vec{r})$  for this problem are:

$$\vec{A}(\rho < a) = -\frac{1}{4} \mu_o J (\rho^2 - a^2) \hat{z} = +\frac{1}{4} \mu_o J (a^2 - \rho^2) \hat{z}$$

$$\vec{A}(\rho \geq a) = -\frac{1}{2} \mu_o J a^2 \ln(\rho/-a) \hat{z} = +\frac{1}{2} \mu_o J a^2 \ln(\rho/a) \hat{z}$$



Note that:  $\vec{A}(\rho \geq a) = \frac{1}{2} \mu_0 J \ln(\rho/a) \hat{z}$  has a {logarithmic} divergence as  $\rho \rightarrow \infty$ , whereas:

$$\vec{B}(\rho \rightarrow \infty) = \nabla \times \vec{A}(\rho \rightarrow \infty) = \frac{1}{2} \mu_0 J a^2 \left( \frac{1}{\rho} \right) \hat{\phi} \rightarrow 0$$

This is merely a consequence associated with the {computationally-simplifying} choice that we made at the beginning of this problem, that of an *infinitely* long wire – which is unphysical. It takes *infinite EM* energy to power an *infinitely* long wire... For a *finite* length wire carrying a steady current  $I$ , the magnetic vector potential is mathematically well-behaved {but has a correspondingly more complicated mathematical expression}.

It is easy to show that both of the solutions for the magnetic vector potential  $\vec{A}(\rho \lesseqgtr a)$  satisfy the Coulomb gauge condition:  $\vec{\nabla} \cdot \vec{A}(\vec{r}) = 0$ , by noting that since  $\vec{A}(\rho \lesseqgtr a) = A_z(\rho \lesseqgtr a) \hat{z}$  are functions only of  $\rho$ , then in cylindrical coordinates:  $\vec{\nabla} \cdot \vec{A}(\rho \lesseqgtr a) = \partial A_z(\rho \lesseqgtr a) / \partial z = 0$ .

Let us now investigate the ramifications of the non-zero curl result associated with Faraday's law at  $\rho = a$  for the  $\vec{A}$ -field at that radial location:

$$\boxed{\vec{\nabla} \times \vec{E} = \frac{|\vec{J}_{free}|}{\sigma_c} \delta(\rho - a) \hat{\phi} = -\frac{\partial \vec{B}}{\partial t}}$$

Since  $\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial \rho} \hat{\phi}$  {here, in this problem}, then:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial (\vec{\nabla} \times \vec{A})}{\partial t} = -\frac{\partial}{\partial t} \left( \frac{\partial A_z}{\partial \rho} \right) \hat{\phi} = -\frac{|\vec{J}_{free}|}{\sigma_c} \delta(\rho - a) \hat{\phi} \quad \text{or:} \quad \frac{\partial}{\partial t} \left( \frac{\partial A_z}{\partial \rho} \right) = \frac{|\vec{J}_{free}|}{\sigma_c} \delta(\rho - a)$$

$$\text{Then: } \frac{\partial A_z}{\partial t} = \frac{|\vec{J}_{free}|}{\sigma_c} \underbrace{\int \delta(\rho - a) d\rho}_{\equiv \Theta(\rho - a)} = \frac{|\vec{J}_{free}|}{\sigma_c} \Theta(\rho - a) \quad \text{or:} \quad \boxed{\frac{\partial \vec{A}}{\partial t} = \frac{|\vec{J}_{free}|}{\sigma_c} \Theta(\rho - a) \hat{z}}$$

Now, recall that the {correct!} electric field for this problem is:

$$\vec{E}(\rho) = \frac{|\vec{J}_{free}|}{\sigma_c} [1 - \Theta(\rho - a)] \hat{z}$$

However, in general, the electric field is defined in terms of the scalar and vector potentials as:

$$\vec{E}(\vec{r}, t) = -\vec{\nabla}V(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}$$

Since {here, in this problem}:  $\frac{\partial \vec{A}}{\partial t} = \frac{|\vec{J}_{free}|}{\sigma_c} \Theta(\rho - a) \hat{z}$ , we see that:  $-\vec{\nabla}V = \frac{|\vec{J}_{free}|}{\sigma_c} \hat{z}$

and hence {in cylindrical coordinates} that:  $V(z) = -\frac{|\vec{J}_{free}|}{\sigma_c} z$ , then:

$$-\vec{\nabla}V = +\frac{\partial}{\partial z} \left( \frac{|\vec{J}_{free}|}{\sigma_c} z \right) \hat{z} = \frac{|\vec{J}_{free}|}{\sigma_c} \frac{\partial}{\partial z} (z) \hat{z} = \frac{|\vec{J}_{free}|}{\sigma_c} \hat{z}.$$

Note that the {static} scalar field  $V(z) = -\frac{|\vec{J}_{free}|}{\sigma_c} z$  pervades **all** space, as does  $\vec{A}(\rho \lesseqgtr a) \parallel \hat{z}$ .

Explicitly, due to the behavior of the Heaviside step function  $\Theta(\rho - a)$  we see that the electric

field contribution  $\frac{\partial \vec{A}}{\partial t} = \frac{|\vec{J}_{free}|}{\sigma_c} \Theta(\rho - a) \hat{z}$  is:  $\frac{\partial \vec{A}}{\partial t} = \begin{cases} 0 & \text{for } \rho < a \\ \frac{|\vec{J}_{free}|}{\sigma_c} \hat{z} & \text{for } \rho \geq a \end{cases}$ .

Explicitly writing out the electric field in this manner, we see that:

$$\vec{E}(\rho \lesseqgtr a) = -\vec{\nabla}V(\rho \lesseqgtr a) - \frac{\partial \vec{A}(\rho \lesseqgtr a)}{\partial t} = \begin{cases} \frac{|\vec{J}_{free}|}{\sigma_c} \hat{z} + 0 = \frac{|\vec{J}_{free}|}{\sigma_c} \hat{z} & \text{for } \rho < a \\ \frac{|\vec{J}_{free}|}{\sigma_c} \hat{z} - \frac{|\vec{J}_{free}|}{\sigma_c} \hat{z} = 0 & \text{for } \rho \geq a \end{cases}$$

Thus, for  $\rho \geq a$  we see that the  $-\partial \vec{A}(\rho \geq a)/\partial t$  contribution to the  $\vec{E}$ -field outside the wire {which arises from the non-zero  $\vec{\nabla} \times \vec{E}$  of Faraday's law at  $\rho = a$ } **exactly** cancels the  $-\vec{\nabla}V(\rho \geq a)$  contribution to the  $\vec{E}$ -field outside the wire, **everywhere** in space outside the wire, despite the fact that  $\vec{A}(\rho \geq a)$  varies logarithmically outside the wire!!!!

The long, current-carrying wire can thus also be equivalently viewed as an **electric flux tube**:

$$\Phi_E = \int_S \vec{E} \cdot d\vec{a} = \left( \left| \vec{J}_{free} \right| / \sigma_C \right) \int_S [1 - \Theta(\rho - a)] \hat{z} \cdot d\vec{a} = I / \sigma_C$$

The electric field  $\vec{E}$  is **confined** within the tube (= the long, current carrying wire) by the  $-\partial\vec{A}(\rho \geq a)/\partial t$  contribution arising from the Faraday's law effect on the  $\rho = a$  boundary of the flux tube, due to the {matter geometry-induced} discontinuity in the electric field at  $\rho = a$ !

The  $\vec{\nabla} \times \vec{E} = \left( \left| \vec{J}_{free} \right| / \sigma_C \right) \delta(\rho - a) \hat{\phi} = -\partial\vec{B}/\partial t$  effect at  $\rho = a$  also predicts a **non-zero** “induced” *EMF* in a loop/coil of wire:  $\mathcal{E} = -\partial\Phi_m/\partial t$ . The magnetic flux through a loop of wire is:

$\Phi_m = \oint_C \vec{A} \cdot d\vec{\ell} = \int_S \vec{B} \cdot d\vec{a} \approx B \cdot A_{\perp}^{loop}$  where  $A_{\perp}^{loop}$  is the cross-sectional area of a loop of wire {whose plane is perpendicular to the magnetic field at that point}. Note further that the width,  $w$  of the coil only needs to be large enough for the coil to accept the  $\partial\vec{B}/\partial t$  contribution from the  $\delta$ -function at  $\rho = a$ . Then, here in **this** problem, since the magnetic field at the surface of the wire is oriented in the  $\hat{\phi}$ -direction, and:

$$\frac{\partial\vec{B}}{\partial t} = -\frac{\left| \vec{J}_{free} \right|}{\sigma_C} \delta(\rho - a) \hat{\phi}, \text{ then we see that: } \mathcal{E} = -\frac{\partial\Phi_m}{\partial t} = -\frac{\partial\vec{B} \cdot A_{\perp}^{loop}}{\partial t} = \frac{\left| \vec{J}_{free} \right| \cdot A_{\perp}^{loop}}{\sigma_C} \delta(\rho - a)$$

For a **real** wire, *e.g.* made of copper, how large will this *EMF* be – is it something *e.g.* that we could actually measure/observe in the laboratory with garden-variety/every-day lab equipment???

A number 8 AWG (American Wire Gauge) copper wire has a diameter  $D = 0.1285'' = 0.00162 \text{ m}$  ( $\sim 1/8'' = 0.125''$ ) and can easily carry  $I = 10$  Amps of current through it.

The current density in an 8 AWG copper wire carrying a steady current of  $I = 10$  Amps is:

$$J_{8AWG} = \frac{I}{\pi a^2} = \frac{4 \cdot I}{\pi D^2} = \frac{4 \cdot 10}{\pi (0.001632)^2} \approx 4.8 \times 10^6 \text{ (Amps/m}^2\text{)}$$

The electrical conductivity of {pure} copper is:  $\sigma_C^{Cu} = 5.96 \times 10^7 \text{ (Siemens/m)}$ .

If our “long” 1/8” diameter copper wire is  $L = 1 \text{ m}$  long, and if we can *e.g.* make a loop of ultra-fine gauge wire that penetrates the surface of the wire and runs parallel to the surface, then if we approximate the radial delta function  $\delta(\rho - a)$  at  $\rho = a$  as  $\sim$  a narrow Gaussian of width

$w \sim 10 \text{ \AA} = 1 \text{ nm} = 10^{-9} \text{ m}$  (*i.e.*  $\sim$  the order of the inter-atomic distance/spacing of atoms in the copper lattice { $3.61 \text{ \AA}$ }), noting also that the delta function  $\delta(\rho - a)$  has physical SI units of inverse length (*i.e.*  $\text{m}^{-1}$ ) and, neglecting the sign of the *EMF*, an estimate of the magnitude of the “induced” *EMF* is:

$$\mathcal{E}_{Cu} = \frac{J_{8AWG} \cdot A_{\perp}^{loop}}{\sigma_C^{Cu}} \delta(\rho - a) \approx \frac{J_{8AWG} \cdot L \cdot \cancel{w}}{\sigma_C^{Cu}} \cdot \cancel{w} = \left( \frac{J_{8AWG}}{\sigma_C^{Cu}} \right) \cdot L \approx \left( \frac{4.8 \times 10^6 \text{ (Amps/m}^2\text{)}}{6 \times 10^7 \text{ (Siemens/m)}} \right) \cdot 1 \text{ m} \approx 80 \text{ mV !!!}$$

This size of an *EMF* is **easily** measureable with a modern *DVM*...

Using Ohm's Law:  $V = I \cdot R$ , note that the voltage drop  $V_{drop}$  across a  $L = 1\text{ m}$  length of 8 AWG copper wire with  $I = 10\text{ Amps}$  of current flowing thru it is:

$$V_{drop}^{1m} = I \cdot R_{1m} = I \cdot \frac{\rho_C^{Cu} L}{A_{\perp}^{wire}} = \left( J_{8AWG} \cdot \cancel{A_{\perp}^{wire}} \right) \cdot \frac{L}{\cancel{\sigma_C^{Cu} A_{\perp}^{wire}}} = \frac{J_{8AWG}}{\sigma_C^{Cu}} \cdot L = \varepsilon_{Cu} !!!$$

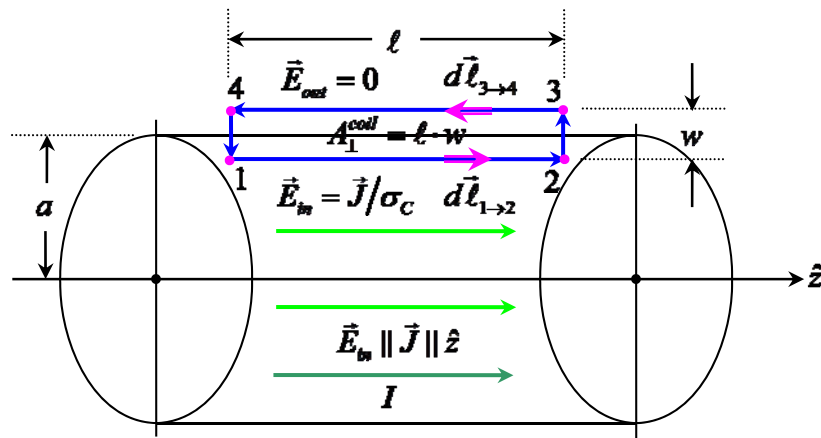
In other words, the “induced”  $EMF$ ,  $\varepsilon = \left( \left| \vec{J}_{free} \right| \cdot A_{\perp}^{loop} / \sigma_C \right) \delta(\rho - a)$  in the one-turn loop coil of length  $L$  {oriented as described above} is **precisely** equal to the voltage drop  $V_{drop} = \left( \left| \vec{J}_{free} \right| / \sigma_C \right) \cdot L$  along a length  $L$  of a portion of the long wire with steady current  $I$  flowing through it, even though the 1-turn loop coil is completely electrically isolated from the current-carrying wire!!!

This can be easily understood... Using Stoke's theorem, the surface integral of  $\vec{\nabla} \times \vec{E}$  can be converted to a line integral of  $\vec{E}$  along a closed contour  $C$  bounding the surface of integration  $S$ ; likewise, a surface integral of  $\partial \vec{B} / \partial t = \vec{\nabla} \times \partial \vec{A} / \partial t$  can be converted to a line integral of  $\partial \vec{A} / \partial t$  along a closed contour  $C$  bounding the surface of integration  $S$ :

$$\varepsilon = \int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi_m}{\partial t} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} = -\int_S \left( \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \right) \cdot d\vec{a} = -\oint_C \frac{\partial \vec{A}}{\partial t} \cdot d\vec{\ell}$$

$$n.b.: \oint_C -\vec{\nabla} V \cdot d\vec{\ell} \equiv 0$$

Then for any closed contour  $C$  associated with the surface  $S$  that encloses the Faraday law  $\vec{\nabla} \times \vec{E}$   $\delta$ -function singularity at  $\rho = a$ , e.g. as shown in the figure below:



the “induced”  $EMF$   $\varepsilon$  can thus also be calculated from the line integral  $\int_C \vec{E} \cdot d\vec{\ell}$  taken around the closed contour  $C$ . From the above discussion(s), the electric field inside (outside) the long current-carrying wire is  $\vec{E}_{in} = \vec{J} / \sigma_C$  ( $\vec{E}_{out} = 0$ ), respectively {n.b.  $\Rightarrow$  tangential  $\vec{E}$  is **discontinuous** across the boundary of a {volume} current-carrying conductor!}. Then:



$$\varepsilon = \int_C \vec{E} \cdot d\vec{\ell} = \int_1^2 \underbrace{\vec{E}_{in}^{1 \rightarrow 2} \cdot d\vec{\ell}_{1 \rightarrow 2}}_{=J/\sigma_C = E\ell = \Delta V_{1 \rightarrow 2}} + \int_2^3 \underbrace{\vec{E}^{2 \rightarrow 3} \cdot d\vec{\ell}_{2 \rightarrow 3}}_{=0} + \int_3^4 \underbrace{\vec{E}_{outside}^{3 \rightarrow 4} \cdot d\vec{\ell}_{3 \rightarrow 4}}_{=0} + \int_4^1 \underbrace{\vec{E}^{4 \rightarrow 1} \cdot d\vec{\ell}_{4 \rightarrow 1}}_{=0} = E \cdot \ell = \Delta V_{1 \rightarrow 2}$$

The presence of a non-zero Faraday's law  $\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t = \left( |\vec{J}_{free}| / \sigma_C \right) \delta(\rho - a) \hat{\phi}$  term at the surface of the long current-carrying wire implies that the “induced” *EMF*  $\varepsilon = \left( |\vec{J}_{free}| \cdot A_{\perp}^{loop} / \sigma_C \right) \delta(\rho - a)$  can also be viewed as arising from the **mutual** inductance  $M$  (*Henrys*) associated with the long wire and the coil {oriented as described above}, and a non-zero  $\partial I / \partial t$ :

$$\varepsilon = -\frac{\partial \Phi_m}{\partial t} = -\frac{\partial \vec{B} \cdot A_{\perp}^{loop}}{\partial t} = -M \frac{\partial I}{\partial t} = \frac{|\vec{J}_{free}| \cdot A_{\perp}^{loop}}{\sigma_C} \delta(\rho - a)$$

We can obtain a relation between  $\partial \vec{B} / \partial t$  and  $\partial I / \partial t$  using the integral form of Ampere's law:  $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_o I_{encl}$ . Taking the partial derivative of both sides of this equation with respect to time:

$$\frac{\partial}{\partial t} \left( \oint_C \vec{B} \cdot d\vec{\ell} \right) = \oint_C \frac{\partial \vec{B}}{\partial t} \cdot d\vec{\ell} = \mu_o \frac{\partial I_{encl}}{\partial t}$$

The contour of integration  $C$  needs to be taken just outside the surface of the long wire, along the  $\hat{\phi}$ -direction, since  $\vec{B} \parallel \hat{\phi}$  at  $\rho = a$ , i.e.  $d\vec{\ell} \parallel \hat{\phi}$  in order to include the non-zero Faraday's law effect at the surface of the long wire.

$$\text{Then: } \frac{\partial B}{\partial t} = \left( \frac{\mu_o}{2\pi a} \right) \frac{\partial I}{\partial t} = -\frac{|\vec{J}_{free}|}{\sigma_C} \delta(\rho - a) \quad \text{or:} \quad \frac{\partial I}{\partial t} = \left( \frac{2\pi a}{\mu_o} \right) \frac{\partial B}{\partial t} = -\left( \frac{2\pi a}{\mu_o} \right) \frac{|\vec{J}_{free}|}{\sigma_C} \delta(\rho - a)$$

$$\text{Then: } \varepsilon = -\frac{\partial \Phi_m}{\partial t} = -\frac{\partial \vec{B} \cdot A_{\perp}^{loop}}{\partial t} = -M \frac{\partial I}{\partial t} = \frac{|\vec{J}_{free}| \cdot A_{\perp}^{loop}}{\sigma_C} \delta(\rho - a)$$

Solving for the mutual inductance, we obtain a rather simple result:  $M = \mu_o \left( \frac{A_{\perp}^{loop}}{2\pi a} \right)$  (*Henrys*)

Note that the mutual inductance,  $M$  involves the magnetic permeability of free space  $\mu_o = 4\pi \times 10^{-7}$  (*Henrys/m*) {*n.b.* which has SI units of inductance/length} and geometrical aspects {only!} of the wire (its radius,  $a$ ) and the cross-sectional area of the loop,  $A_{\perp}^{loop}$ .

What is astonishing {and unique} *r.e.* the “induced” Faraday's law *EMF*  $\varepsilon = \left( |\vec{J}_{free}| \cdot A_{\perp}^{loop} / \sigma_C \right) \delta(\rho - a)$  associated with a long, steady current-carrying wire is that “normal” induced *EMF*'s **only** occur in electrical circuits that operate at **non-zero** frequencies, i.e.  $f > 0$  Hz. However, **here**, in **this** problem, we have an example of a **DC** induced *EMF* – i.e. an induced *EMF* that occurs at  $f \equiv 0$  Hz, arising from the non-zero Faraday's law effect  $\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t = \left( |\vec{J}_{free}| / \sigma_C \right) \delta(\rho - a) \hat{\phi}$  due to the longitudinal  $\vec{E}$ -field discontinuity at the surface ( $\rho = a$ ) of a long, **steady** current-carrying wire!!!

Instead of using a long wire to carry a steady current  $I$  to observe this effect, one might instead consider using *e.g.* a long, hollow steady current-carrying **pipe** of inner (outer) radius  $a$ , ( $b$ ) respectively. Following the above methodology, one can easily show that for such a long, hollow current-carrying pipe, **two opposing** non-zero Faraday law  $\vec{\nabla} \times \vec{E}$  radial  $\delta$ -function contributions occur – one located at the  $\rho = a$  inner surface, and the other located at the  $\rho = b$  outer surface of the long hollow current-carrying pipe:

$$\vec{\nabla} \times \vec{E} = -\partial \vec{B} / \partial t = -\left( |\vec{J}_{free}| / \sigma_C \right) [\delta(\rho - a) - \delta(\rho - b)] \hat{\phi}$$

The  $\vec{E}$  -field is:

$$\vec{E} = -\vec{\nabla} V - \partial \vec{A} / \partial t = \left( |\vec{J}_{free}| / \sigma_C \right) [1 + \bar{\Theta}(\rho - a) - \Theta(\rho - b)] \hat{z}$$


where:  $\bar{\Theta}(\rho - a) \equiv \begin{cases} 1 & \text{for } \rho < a \\ 0 & \text{for } \rho \geq a \end{cases}$  is the **complement** of the Heaviside step function, such that:

$d\Theta(x)/dx = -\delta(x)$  and:  $\bar{\Theta}(x) = -\int_{-\infty}^x \delta(t) dt$  where:  $\delta(x)$  is the Dirac delta-function.

Hence, a 1-turn coil {oriented as described above} enclosing the  $\rho = a$  inner surface **and** the  $\rho = b$  outer surface of a current-carrying hollow pipe will have a “null” induced *EMF*, *i.e.*  $\varepsilon = 0$  due to the wire loop **simultaneously** enclosing the **two opposing** non-zero Faraday law  $\vec{\nabla} \times \vec{E}$  radial  $\delta$ -function contributions, one located at  $\rho = a$ , the other at  $\rho = b$ :

$$\varepsilon = -\frac{\partial \Phi_m}{\partial t} = -\frac{\partial \vec{B} \cdot \vec{A}_{\perp}^{loop}}{\partial t} = \frac{|\vec{J}_{free}| \cdot \vec{A}_{\perp}^{loop}}{\sigma_C} (\delta(\rho - a) - \delta(\rho - b)) = 0$$

In general, **any** penetration/hole made into the metal conductor of a long, steady current-carrying wire will result in a non-zero Faraday law  $\vec{\nabla} \times \vec{E}$   $\delta$ -function on the boundary/surface of that penetration/hole! Since the current density  $\vec{J}_{free} = 0$  in the penetration/hole,  $\vec{E} = 0$  there and thus a discontinuity in  $\vec{E}$  exists on the boundary of the penetration/hole, hence a non-zero Faraday law  $\vec{\nabla} \times \vec{E}$   $\delta$ -function exists on the boundary of the penetration/hole!

This fact {unfortunately} has **important** ramifications for the experimental detection / observation of the predicted non-zero *DC* induced *EMF* in a coil {oriented as described above}, Embedding a portion of a physical wire loop inside the long, steady current-carrying wire requires making a penetration/hole {no matter how small} into the wire, which **will** result in a non-zero Faraday law  $\vec{\nabla} \times \vec{E}$   $\delta$ -function on the boundary/surface of that penetration/hole in the wire! Thus, the wire loop will in fact enclose **not only** the Faraday law  $\vec{\nabla} \times \vec{E}$  radial singularity at  $\rho = a$  on the surface of the wire, but will **also** enclose **another, opposing** singularity located on the boundary/surface of the penetration/hole made into the long wire {which was made to embed a portion of the wire loop in a long, steady current-carrying wire in the first place}, thus experimentally, a “null” induced *EMF*, *i.e.*  $\varepsilon = 0$  is expected/anticipated, because of this... 

Hence, in the **real** world of experimental physics, it appears that embedding a portion of a **real** wire loop in a long, steady current-carrying wire in an attempt to observe this effect is doomed to failure...