Chapter 1

The Lorenz Model

1.1 Introduction

Lorenz was interested in the predictability of the solutions to hydrodynamics equations. He was a meteorologist studying weather forecasting—and the question of the fundamental limitations to this endeavor. The model he introduced [1] can be thought of as a gross simplification of one feature of the atmosphere, namely the fluid motion driven by thermal buoyancy known as convection, although his original paper seems to use the model simply as a set of equations "whose solutions afford the simplest example of a deterministic nonperiodic flow of which the writer is aware".

The model describes the convection motion of a fluid in a small, idealized "Rayleigh-Bénard" cell. The idealization is that the boundary conditions of the fluid at the upper and lower plates are taken to "stress free" rather than the realistic "no-slip", the lateral boundary conditions are taken to be "periodic" rather than corresponding to a realistic side walls, and the motion is assumed to be two dimensional rather than three. These three modifications greatly simplify the mathematical analysis.

The full description of the motion of the fluid is replaced by dynamical equations for a few of the simplest modes of the system, so that the temperature T is approximated by

$$T(x, z, t) \simeq -rz + 9\pi^3\sqrt{3}Y(t)\cos(\pi z)\cos\left(\frac{\pi}{\sqrt{2}}x\right) + \frac{27\pi^3}{4}Z(t)\sin(2\pi z)$$
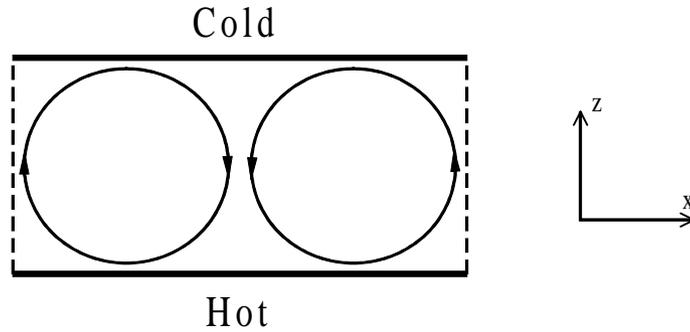
$$(1.1)$$

Figure 1.1: The physical system modelled by Lorenz

and the fluid velocity components u (x direction) and w (z direction) are given conveniently in terms of the stream function ψ

$$u = -\frac{\partial \psi}{\partial z}, \qquad w = \frac{\partial \psi}{\partial x}$$

with the stream function

$$\psi(x, z, t) = 2\sqrt{6}X(t)\cos(\pi z)\sin\left(\frac{\pi}{\sqrt{2}}x\right) \qquad . \qquad (1.2)$$

For the technically sophisticated it should be remarked that the variables have been "de-dimensionalized" with respect to a convenient length (the depth of the cell), time (the heat diffusion time across the depth of the cell) and temperature. The first term in the temperature equation is the linear profile that gives the conduction of heat in the absence of fluid motion, and the parameter r is the temperature difference across the cell (in the scaled units). The mode structure, which comes from solving the linearized fluid and heat equations, is sinusoidal in both horizontal and vertical directions. The horizontal wave length has been chosen as the one that optimizes the onset of convection, and the width of the system is fixed at one wavelength. The rather complicated looking numerical prefactors are simply a convenient choice of normalization of the modes that simplifies the dynamical equations.

The scheme now is to substitute these expressions into the coupled fluid and heat equations. Since these equations are non-linear, there will be terms coupling the

different modes, and also terms generating higher harmonics than are represented in equations (1.1),(1.2). The latter terms are ignored—the major approximation in the scheme. This reduces the complicated *partial differential equations* describing the fluid motion and heat flow to three *ordinary differential equations*

$$
\begin{aligned}
\dot{X} &= -\sigma(X - Y) \\
\dot{Y} &= rX - Y - XZ \\
\dot{Z} &= b(XY - Z)
\end{aligned}
\tag{1.3}
$$

where the "dot" denotes the time derivative d/dt. The parameter σ depends on the properties of the fluid (in fact the ratio of the viscous to thermal diffusivities): Lorenz took the value to be 10 in his paper; for water the value is typically between 1 and 4, for an ideal gas the value is about 0.7, and for oils the value is $10 - 1000$ or even higher. The number $b = 8/3$: this would be different for a different choice of horizontal wavelength or roll diameter. Again the temperature difference r appears and is the important control parameter: for $r < 1$ the solution at long times is asymptotic to $X = Y = Z = 0$, i.e. no convection. For $r > 1$ more interesting solutions occur!

The important features of these equations are:

- They are autonomous—time does not explicitly appear on the right hand side;

- They involve only first order time derivatives so that (with the autonomy) the evolution depends only on the instantaneous value of (X, Y, Z);

- They are non-linear, here through the quadratic terms XZ and XY in the second and third equations;

- They are dissipative—crudely the "diagonal" terms such as $\dot{X} = -\sigma X$ correspond to decaying motion, but more systematically we will see that "volumes in phase space" shrink in the dynamics;

- The solutions are bounded.

Lorenz simulated these equations and found chaos. (Actually he references his colleague Saltzman for telling him about "aperiodic solutions" to similar systems of equations, although a paper [2] by Saltzman published one year earlier has no mention of such dynamics. More importantly Lorenz realized the importance of the aperiodic motion, and developed diagnostic tools to make sense of it.)
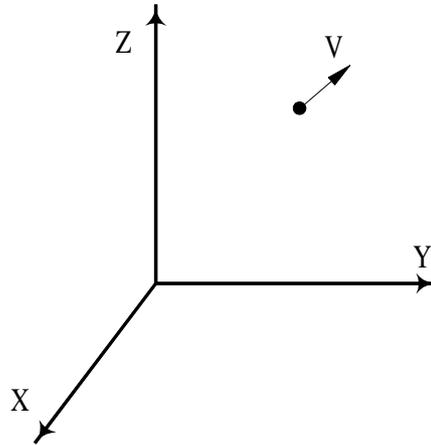
Figure 1.2: Phase Space for the Lorenz Model

We will now look at some examples of the dynamics. Since the equations for the dynamics of (X, Y, Z) are first order and autonomous we will call this set of variables the "phase space". The dynamics at each point in phase space, specified by the "velocity in phase space" vector $\vec{V} = (\dot{X}, \dot{Y}, \dot{Z})$, is unique. The evolution in time then traces out a path in the three dimensional phase space. An immediate result is that *phase space trajectories cannot cross*.

1.2 Demonstrations

Please see the accompanying programs

1.3 Final Words

The "sensitive dependence on initial conditions" found by Lorenz is now known affectionately as Lorenz's "butterfly effect". In fact in a later paper [3] Lorenz remarked:

> One meteorologist remarked that if the theory were correct, one flap of the sea gull's wings would be enough to alter the course of the weather

forever.

By the time of Lorenz's talk at the December 1972 meeting of the American Association for the Advancement of Science in Washington, D.C. the sea gull had evolved into the more poetic butterfly - the title of his talk was [4]:

> Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?

Lorenz's work was largely ignored for ten years, but can now be seen as a prescient beginning to the study of chaos. Using this model we have identified some characteristics of chaos, that we will want to quantify further:

- apparent "randomness" in the time variation;

- broad band components to the power spectrum;

- sensitive dependence on initial conditions.

In addition we have briefly looked at:

- interesting structures in phase space known as strange attractors;

- the Poincaré section to show the structure most simply;

- the use of 1d maps.

It is now known that the Lorenz equations are not an accurate description of the original (idealized) convection system for temperature differences, expressed by the dimensionless measure r, large enough to yield chaos—the equations may be derived systematically as an expansion in $r - 1$, but 27 is not a small number! For example Curry [5] has shown that if the mode truncation is not done, but instead sufficient modes are retained to give numerical convergence, the chaos disappears. In this work the two dimensionality of the flow (in the $x - z$ plane) was maintained. On the other hand McLauglin and Martin [6] showed that chaos is obtained for a three dimensional version—but of course the mode equations are then not simply the Lorenz equations.

An alternative approach has been to construct experimental systems for which the Lorenz equations *are* a good description. Since the Lorenz equations break down due to the excitation of higher spatial harmonics, one scheme has been to

investigate convection in a circular glass tube held vertically and heated over the lower half and cooled over the upper half. This is the thermosyphon. Indeed behavior qualitatively similar to the predictions of the Lorenz model are obtained, with chaotic reversal of the circulation around the tube. However since the boundary condition over the lower half is constant heat input, rather than constant temperature, it is found that a fourth mode (the mean temperature) must be added to make the predictions quantitative [7][8][9][10][11].

Another experimental system described by the Lorenz equations is the Rititake dynamo—a homopolar generator with the output fed back through inductors and resistors to the coil generating the magnetic field [11]. The disc is driven by a constant torque. The coupled circuit and rotation equations can be reduced to the Lorenz form, and experiments [12] indeed show an apparently chaotic reversal of the coil current and the magnetic field. Analogies with the earth's magnetic field, which shows irregular reversals on a time scale of millions of years, are certainly intriguing, although recent numerical simulations [13] suggest that a three mode truncation will not be a good approximation for the turbulent dynamics of the earth's interior.

1.4 Appendix: Numerical considerations

We need to solve ordinary differential equations (ODEs) of the form

$$\frac{dy}{dt} = f(y, t). \tag{1.4}$$

Coupled ODEs take this form if we think of y as a vector. The solution is based on the simple idea of dividing time up into small steps of size h: $t_n = nh$, $y_n = y(t_n)$ and then using a Taylor expansion about the present time:

$$y_{n+1} = y_n + h\, f(y_n, t_n) + \cdots \tag{1.5}$$

where the $\cdots$ represent higher terms in the expansion. Subtleties arise when trying to improve the *stability* and improve or control the *accuracy*.

A popular choice is the "4th order Runge Kutta" scheme where successive estimates of the increment are made, and combined in a way that effectively includes

higher order terms in the Taylor expansion. We evaluate

$$
\begin{aligned}
k_1 &= h\,f(y_n, t_n) \\
k_2 &= h\,f(y_n + \tfrac{1}{2}k_1, t_n + \tfrac{1}{2}h) \\
k_3 &= h\,f(y_n + \tfrac{1}{2}k_2, t_n + \tfrac{1}{2}h) \\
k_4 &= h\,f(y_n + k_3, t_n + h)
\end{aligned}
\tag{1.6}
$$

and combine them to give

$$
y_{n+1} = y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + \tfrac{1}{3}k_3 + \tfrac{1}{6}k_4 + O(h^5).
\tag{1.7}
$$

The size of the $O(h^5)$ error depends on the derivatives of f, i.e. the smoothness of the function.

Now we can choose a step size h, and iterate to get the solution. There are numerous tricks to improve the procedure and our knowledge of how well we are doing. An obvious improvement is to do at each time step an auxiliary calculation of two half steps, and to compare the two results to get an estimate of the error. We can then also combine the two results to get an estimate equivalent to a fifth order Runge Kutta! This is the routine used in the demonstrations. For more details see the book "Numerical Recipes" [14] chapter 15. A further enhancement is to dynamically vary the step size to keep the error estimate below some chosen limit: this is a good idea generally, but would further distort the loose connection between computer time and iteration time that is used in displaying the dynamic plots, so has not been implemented in the demonstrations. Completely different numerical schemes [14] may also have advantages.

In evolving ODEs displaying chaos the question of "numerical accuracy" is more subtle—the sensitive dependence on initial conditions implies that a small numerical error will be amplified over time to give an $O(1)$ error after times of order $\log(1/\text{error})$, usually well within the times of interest. Thus we cannot, with our usual care, expect to get the "correct" answer from a given initial condition, only one that is locally a good approximation and statistically representative in some sense. The question of how well numerical solutions "track" an accurate solution is still an active area of work [15] and is discussed further in chapter 26. This takes us back to Lorenz's original question, and is a good place to close this introduction!

December 24, 1999

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Chapter 2

The Language of Dynamical Systems

The well known example of the driven, damped pendulum provides a convenient introduction to some of the language of dynamical systems.

2.1 The ideal pendulum

If we define θ as the angular displacement of the pendulum from the equilibrium (hanging down) position, the equation of motion for the oscillations of an ideal pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \tag{2.1}$$

where l is the length and g is the gravitational acceleration. We can write (2.1) as two first order equations

$$\dot{\theta} = \omega \tag{2.2}$$
$$\dot{\omega} = -\frac{g}{l}\sin\theta$$

introducing the angular velocity ω, and then can use (θ, ω) as our phase space coordinates. Later, we will introduce a different pair of coordinates, using the angular momentum $J = Ml^2\omega$ as the second coordinate (with M the mass of the pendulum). The dynamics in the phase space is given by a series of trajectories, as shown in the figure: Since there is no dissipation in the equations, the energy is conserved, and we can imagine labelling each trajectory by its energy.

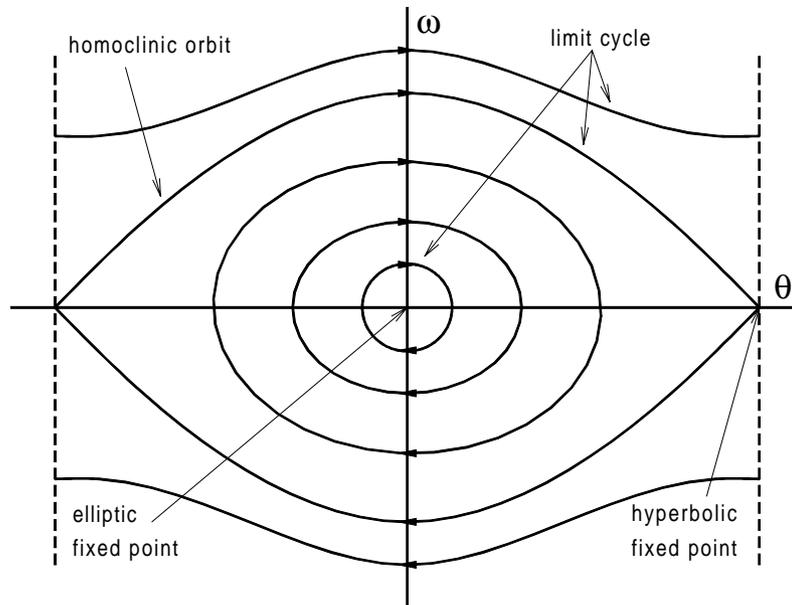Various features are marked on the figure

1

Figure 2.1: Phase space of the ideal pendulum

The rest position $\theta = \omega = 0$ is called a "fixed point". This is an "elliptic" fixed point, since nearby orbits take the form of ellipses (or circles in scaled coordinates). Naively we might call this a stable fixed point, but since there is no dissipation perturbations from the fixed point do not decay back to the fixed point.

The θ coordinate runs from $-\pi$ to π. There is a second fixed point at $(\pi, 0)$ corresponding to the pendulum pointing vertically up: this is a "hyperbolic" fixed point, because nearby trajectories take this form. These trajectories take an initial point near the fixed point far away, and we would naively call this an unstable fixed point.

The remaining orbits are periodic in time, and are called "limit cycles". For small energies, near (0, 0) the limit cycles are the familiar simple harmonic motion, represented by circles or ellipses (stretched circles) in the phase space. These would give a single peak in a power spectrum, and would sound like a pure musical tone. For larger energies, the orbit becomes distorted in the phase space and are no longer simple harmonic. The power spectrum would show harmonics, with additional frequencies at multiples of the fundamental, and the tone, although representing

one musical note, would sound more complex.

A special pair of orbits leave the hyperbolic fixed point, and then eventually return to it. (Remember the θ coordinate wraps around!) These are known as "homoclinic" orbits. The dynamics slows down approaching the fixed point, and the period of the limit cycle orbits diverge as their energy approaches the energy of the homoclinic orbit. (In other systems we might have a "heteroclinic" orbit connecting two different hyperbolic fixed points.)

We know that the ideal pendulum is a Hamiltonian system. This means we can use the energy to construct a Hamiltonian :

$$H = \frac{1}{2I} J^2 + Mgl(1 - \cos\theta) \qquad (2.3)$$

which is just the energy written as a function of the two "canonically conjugate" variables, the angular position θ and the angular momentum $J = I\omega$ with $I = Ml^2$ the moment of inertia. The Hamiltonian formulation of the dynamics is then

$$\dot{\theta} = \frac{\partial H}{\partial J} = \frac{J}{I} \qquad (2.4)$$
$$\dot{J} = -\frac{\partial H}{\partial \theta} = -Mgl\sin\theta \qquad .$$

It is easy to see that these are the same as (2.2).

A very important property of Hamiltonian systems is that the dynamics "preserves volumes in phase space". This means that if we start off many copies of the system, with initial conditions filling some small volume in phase space, then as the system evolves the volume of phase space containing the evolving points distorts in shape, but keeps a fixed volume.

We first define a velocity in phase space giving the time dependence of the phase space coordinates, here

$$\vec{V} = (\dot{\theta}, \dot{J}). \qquad (2.5)$$

Now it is easy to verify from the equations of motion that this "velocity" is divergence free:

$$div\vec{V} \equiv \frac{\partial\dot{\theta}}{\partial\theta} + \frac{\partial\dot{J}}{\partial J} = 0. \qquad (2.6)$$

This in fact is a general consequence of the form of the Hamilton equations of motion. Just as for an incompressible fluid, this is equivalent to volume conserving

flow, as can be seen by integrating over an arbitrary volume and using Gauss's theorem.

An immediate consequence of this result is that there are no attractors in Hamiltonian systems: there can be no attracting fixed point to which initial conditions distributed over some volume converge, since this would yield a volume of points in phase space contracting asymptotically to zero.

You can investigate the phase space of the ideal pendulum in demonstration 1.

2.2   The dissipative pendulum

If we add a dissipative force proportional to the velocity, the equation of motion becomes

$$\frac{d^2\theta}{dt^2} + \eta \frac{d\theta}{dt} + \frac{g}{l} \sin\theta = 0 \qquad . \tag{2.7}$$

It is easy to see that almost all phase space trajectories spiral into the fixed point at $(0, 0)$. This is now truly a "linearly stable" fixed point, since if a small perturbation is made from the fixed point, the perturbation decays in time (in fact exponentially for small enough perturbations). On the other hand the fixed point at $(\pi, 0)$ is "linearly unstable" because a small perturbation from this fixed point grows exponentially. Only very carefully tuned initial conditions will lead to a trajectory ending on the unstable fixed point, and almost all perturbations to the initial condition will lead to a trajectory that may approach close to the unstable fixed point, but eventually spirals into the stable fixed point. The $(0, 0)$ fixed point is "attracting", and in this case the "basin of attraction" i.e. the set of initial conditions leading to trajectories that approach the fixed point, is the whole phase space except for points on the "stable manifold" of the hyperbolic fixed point, which is a set of zero area in the phase space.

The dynamical behavior can be studied in demonstration 2.

2.3   The periodically driven, damped pendulum

The situation is more interesting if we also drive the pendulum, feeding in energy to resupply the energy dissipated. Simple harmonic driving leads to the following equation

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \sin\theta = g \cos(\omega_D t) \tag{2.8}$$

where we have rescaled time so that the period of small oscillations of the un-damped and undriven pendulum is unity, and we have written the scaled dissipation coefficient as γ.

For small amplitudes of driving g, and assuming a small initial condition, we can replace $\sin\theta$ by θ and solve the equation analytically:

$$\theta = \frac{g}{\sqrt{\left(1 - \omega_D^2\right)^2 + \gamma^2\omega_D^2}} \cos(\omega_D t + \phi) + Ae^{-\gamma t/2}\cos(\omega t + \phi_A) \qquad (2.9)$$

with

$$\tan\phi = -\frac{\gamma\omega_D}{\left(1 - \omega_D^2\right)}, \quad \omega = \sqrt{1 - \frac{\gamma^2}{4}}. \qquad (2.10)$$

This is the well known resonant response (the first term) oscillating at the applied frequency, together with decaying free oscillations (the second term) depending on the initial conditions. We would call this solution an attracting limit cycle.

What happens for large driving amplitudes? Here there are no analytic solutions, and we must proceed numerically. To gain some intuition we would like to view the dynamics in a phase space. To this we convert the equation to autonomous form by using *three* variables

$$\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\gamma\omega - \sin\theta + g\cos(\theta_D) \qquad (2.11) \\
\dot{\theta}_D &= \omega_D
\end{aligned}$$

where we have introduced the "phase of the driving" θ_D. This method of gaining an autonomous form at the expense of an extra equation is a common and useful trick. We again have a three dimensional phase space as in the Lorenz model: do we find chaos?

First it is useful to look again at volumes in phase space. Now we have for the divergence of the velocity $\vec{V} = (\dot{\theta}, \dot{\omega}, \dot{\theta}_D)$

$$div\vec{V} = \frac{\partial\dot{\theta}}{\partial\theta} + \frac{\partial\dot{\omega}}{\partial\omega} + \frac{\partial\dot{\theta}_D}{\partial\theta_D} = -\gamma \quad , \qquad (2.12)$$

a constant! This means that volumes contract at a constant proportional rate γ. (The Lorenz model shows this special feature too: there the proportional contraction rate is $\sigma + 1 + b$). Systems whose phase space volumes are not conserved, and

on some sort of average contract, are called dissipative systems. At first sight we might expect a volume of initial conditions must contract to a point, i.e. all orbits approach stable fixed points asymptotically—not very interesting. However this is not the only possibility. We already know from the small amplitude case that the orbits may approach an attracting limit cycle. Even more interesting, a phase space volume may be stretched in one or more directions, whilst it is contracting in the remaining ones so that overall the volume contracts. This is the crude description of how chaos may occur in purely contracting dissipative systems. How chaos occurs in perhaps this simplest and most familiar dynamical system is illustrated in demonstrations 3-7.

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