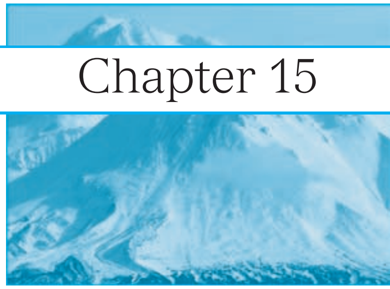


Chapter 15

Integral Transforms

15.1 Introduction and Definitions

Frequently in physics we encounter pairs of functions related by an integral of the form

$$F(\alpha) = \int_a^b f(t)K(\alpha, t)dt. \tag{15.1}$$

The function $F(\alpha)$ is called the (integral) transform of $f(t)$ by the kernel $K(\alpha, t)$. The operation may also be described as mapping a function $f(t)$ in t-space into another function $F(\alpha)$ in α-space. This interpretation takes on physical significance in the time–frequency relation of Fourier transforms, such as Example 15.4.4.

Linearity

These integral transforms are linear operators; that is,

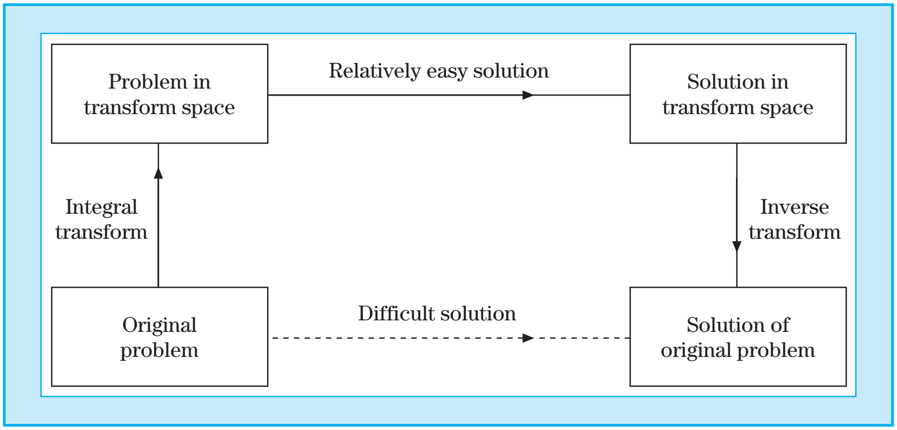
$$\begin{aligned}\int_a^b &[c_1 f_1(t) + c_2 f_2(t)]K(\alpha, t)dt \\ &= c_1 \int_a^b f_1(t)K(\alpha, t)dt + c_2 \int_a^b f_2(t)K(\alpha, t)dt, \\ &= c_1 F_1(\alpha) + c_2 F_2(\alpha),\end{aligned} \tag{15.2}$$

$$\int_a^b cf(t)K(\alpha, t)dt = c\int_a^b f(t)K(\alpha, t)dt, \tag{15.3}$$

where c_1, c_2, c are constants, and $f_1(t)$, $f_2(t)$ are functions for which the integral transform is well defined.

Figure 15.1

Schematic Integral Transforms



Representing our linear integral transform by the operator $\mathcal{L}$, we obtain

$$F = \mathcal{L} f. \tag{15.4}$$

We expect an inverse operator $\mathcal{L}^{-1}$ exists, such that[1]

$$f = \mathcal{L}^{-1} F. \tag{15.5}$$

For our three Fourier transforms $\mathcal{L}^{-1}$ is given in Section 15.4. In general, the evaluation of the inverse transform is the main problem in using integral transforms. The inverse Laplace transform is discussed in Section 15.12.

Integral transforms have many special physical applications and interpretations that are noted in the remainder of this chapter. The most common application is outlined in Fig. 15.1. Perhaps an original problem can be solved only with difficulty, if at all, in the original coordinates (space). It often happens that the transform of the problem can be solved relatively easily. Then, the inverse transform returns the solution from the transform coordinates to the original system. Examples 15.5.2 and 15.5.4 illustrate this technique.

15.2 Fourier Transform

One of the most useful of the infinite number of possible transforms is the Fourier transform, given by

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \tag{15.6}$$

[1]Expectation is not proof, and here proof of existence is complicated because we are actually in an **infinite**-dimensional space. We shall prove existence in the special cases of interest by actual construction.

Two modifications of this form, developed in Section 15.4, are the Fourier cosine and Fourier sine transforms:

$$F_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos\omega t\, dt, \tag{15.7}$$

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin\omega t\, dt. \tag{15.8}$$

All these integrals exist if $\int_{-\infty}^{\infty} |f(t)| dt < \infty$, a condition denoted $f \in L(-\infty, \infty)$ in the mathematical literature and meaning that the function f belongs to the space of absolutely integrable functions. Moreover, then Riemann's lemma holds

$$\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt \to 0, \qquad \int_{-\infty}^{\infty} f(t)\sin\omega t\, dt \to 0, \quad \text{as } \omega \to \infty.$$

The Fourier transform is based on the kernel $e^{i\omega t}$ and its real and imaginary parts taken separately, $\cos\omega t$ and $\sin\omega t$. Because these kernels are the functions used to describe waves, due to their periodicity, Fourier transforms appear frequently in studies of waves and the extraction of information from waves, particularly when phase information is involved. The output of a stellar interferometer, for instance, involves a Fourier transform of the brightness across a stellar disk. The electron charge distribution in an atom may be obtained from a Fourier transform of the amplitude of scattered X-rays. In quantum mechanics the physical origin of the Fourier relations of Section 15.7 is the wave nature of matter and our description of matter in terms of waves.

If we differentiate the Fourier transform

$$\frac{dF(\omega)}{d\omega} = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} tf(t)e^{i\omega t} dt,$$

we see that the original function $f(t)$ is multiplied by it. This is one way of generating new Fourier transforms.

If we differentiate a cosine transform with respect to ω, we are led to a sine transform and vice versa. Many examples are given by Titchmarsh (see Additional Reading).

EXAMPLE 15.2.1

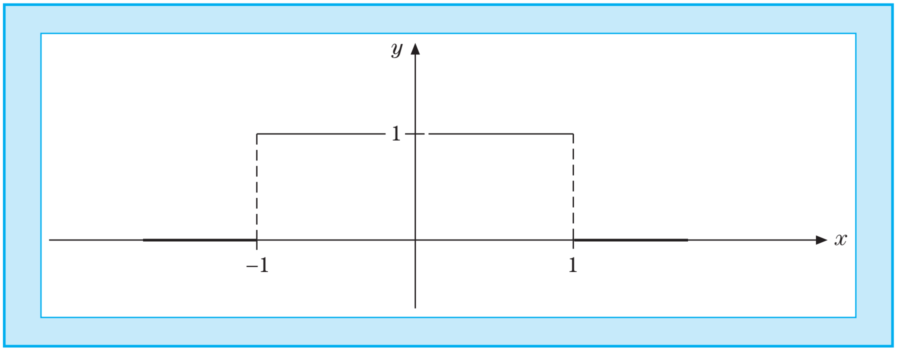
Square Pulse Let us find the Fourier transform for the shape in Fig. 15.2:

$$f(t) = \begin{cases} 1, & |t| < 1, \\ 0, & |t| > 1, \end{cases}$$

which is an even function of t. This is the single slit diffraction problem of physical optics. The slit is described by $f(t)$. The diffraction pattern **amplitude**

Figure 15.2

Square Pulse



is given by the Fourier transform $F(\omega)$. Starting from Eq.(15.6),

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega t}}{i\omega}\Big|_{-1}^{1}$$
$$= \frac{e^{i\omega} - e^{-i\omega}}{i\omega\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega},$$

which is an even function of ω. ■

EXAMPLE 15.2.2 **Fourier Transform of Gaussian** The Fourier transform of a Gaussian,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2t^2} e^{i\omega t} dt, \tag{15.9}$$

can be done analytically by completing the square in the exponent,

$$-a^2t^2 + i\omega t = -a^2\left(t - \frac{i\omega}{2a^2}\right)^2 - \frac{\omega^2}{4a^2},$$

which we check by evaluating the square. Substituting this identity we obtain

$$F(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a^2} \int_{-\infty}^{\infty} e^{-a^2t^2} dt$$

upon shifting the integration variable $t \to t + \frac{i\omega}{2a^2}$. This is justified by an application of Cauchy's theorem to the rectangle with vertices $-T$, T, $T + \frac{i\omega}{2a^2}$, $-T + \frac{i\omega}{2a^2}$ for $T \to \infty$, noting that the integrand has no singularities in this region and the integrals over the sides from $\pm T$ to $\pm T + \frac{i\omega}{2a^2}$ become negligible for $T \to \infty$. Finally, we rescale the integration variable as $\xi = at$ in the integral

$$\int_{-\infty}^{\infty} e^{-a^2t^2} dt = \frac{1}{a} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{a}.$$

Substituting these results we find

$$F(\omega) = \frac{1}{a\sqrt{2}} \exp\left(-\frac{\omega^2}{4a^2}\right), \tag{15.10}$$

again a Gaussian, but in ω-space. The smaller a is (i.e., the wider the original Gaussian $e^{-a^2t^2}$ is), the narrower is its Fourier transform $\sim e^{-\omega^2/4a^2}$. Differentiating $F(\omega)$, the Fourier transform of $i\omega e^{-\omega^2/4a^2}$ is $\sim te^{-a^2t^2}$, etc. ■

Laplace Transform

The equally important Laplace transform is related to a Fourier transform by replacing the frequency ω with an imaginary variable and changing the integration interval, that is, $\exp(i\omega x) \to \exp(-sx)$, which will be developed in Sections 15.8–15.12. The Laplace transform

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{15.11}$$

has the kernel e^{-st}. Clearly, the possible types of integral transforms are unlimited. The Laplace transform has been useful in mathematical analysis as well as in physics and engineering applications. The Laplace and Fourier transforms are by far the most used. For Mellin, Hankel, Bessel, and other transforms, see Additional Reading.

If the integrand of a Fourier integral is analytic, then the integral may be evaluated using the residue theorem according to Eq. (7.30). See Example 7.2.3 and Exercises 7.2.4 and 7.2.15. The same method applies to Laplace transforms.

EXAMPLE 15.2.3

Euler Integral as Laplace Transform If we generalize the Euler integral [Eq. (10.5)] to

$$\int_0^\infty e^{-st}t^z\,dt = \frac{1}{s^{z+1}}\int_0^\infty e^{-st}(st)^z\,d(st) = \frac{\Gamma(z+1)}{s^{z+1}},$$

where z is a complex parameter with $\Re(z) > -1$, the Laplace transform of the power t^z is the inverse power s^{-z-1} up to the normalization factor $\Gamma(z+1)$. ■

Laplace transforms will be treated in detail starting in Section 15.8.

EXERCISES

15.2.1 (a) Show that $F(-\omega) = F^*(\omega)$ is a necessary and sufficient condition for the Fourier transform $f(t)$ to be real.

(b) Show that $F(-\omega) = -F^*(\omega)$ is a necessary and sufficient condition for $f(t)$ to be pure imaginary.

15.2.2 Let $F(\omega)$ be the Fourier (exponential) transform of $f(t)$ and $G(\omega)$ the Fourier transform of $g(t) = f(t+a)$. Show that

$$G(\omega) = e^{-ia\omega}F(\omega).$$

15.2.3 Prove the identities involved in Exercise 6.5.15.

15.2.4 Find the Fourier sine and cosine transforms of $e^{-a|t|}$.

15.2.5 Find the Fourier exponential, sine, and cosine transforms of $e^{-a|t|}\cos bt$ and $e^{-a|t|}\sin bt$.

15.2.6 Find the Fourier exponential, sine, and cosine transforms of $1/(a^2+t^2)^n$ for $n = 2, 3$.

15.3 Development of the Inverse Fourier Transform

Fourier series, such as $\sum_n a_n \cos(n\pi t/L)$, that we studied in the previous chapter are sums of terms each involving a multiple $n\Delta\omega$ of a basic frequency $\omega = n\pi/L$. If we let the periodicity interval of length $L \to \infty$, then $n\Delta\omega$ becomes a continuous frequency variable ω, and the Fourier series goes over into a Fourier integral $A(t) = \int_{-\infty}^{\infty} a(\omega)\cos \omega t\, d\omega$ for a nonperiodic function $A(t)$. This transition from Fourier series to integral is now described in more detail.

In Chapter 14, it was shown that Fourier series are useful in representing certain functions over a limited range $[0, 2\pi]$, $[-L, L]$, and so on, if the function is periodic. We now turn our attention to the problem of representing a **nonperiodic function** over the infinite range, letting $L \to \infty$. Physically, this sometimes means resolving a single pulse or wave packet into sinusoidal waves or a temperature distribution that decays at $\pm\infty$ into wave components.

We have seen (Section 14.2) that for the interval $[-L, L]$ the coefficients a_n and b_n could be written as

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt, \tag{15.12}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt. \tag{15.13}$$

The resulting Fourier series

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$
$$+\frac{1}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt \tag{15.14}$$

or

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}(t-x)dt \tag{15.15}$$

is Eq. (14.28). However, we now let the parameter L approach infinity, transforming the finite interval $[-L, L]$ into the infinite interval $(-\infty, \infty)$. We set

$$\frac{n\pi}{L} = \omega, \qquad \frac{\pi}{L} = \Delta\omega, \quad \text{with } L \to \infty.$$

Then we have

$$f(x) \to \frac{1}{\pi} \sum_{n=1}^{\infty} \Delta\omega \int_{-\infty}^{\infty} f(t) \cos \omega(t-x) dt \tag{15.16}$$

or

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} d\omega \int_{-\infty}^{\infty} f(t) \cos \omega(t-x) dt, \tag{15.17}$$

replacing the infinite sum by the integral over ω. The first term (corresponding to a_0) has been absorbed at $\omega = 0$, assuming that $\int_{-\infty}^{\infty} f(t)dt$ exists. This **Fourier cosine formula** is valid if f is continuous at x. If f is only piecewise continuous, then $f(x)$ must be replaced by $\frac{1}{2}[f(x+0)+f(x-0)]$, which is the average of the limiting values of f to the left and right of the point x. Also, integrals $\int_{-\infty}^{\infty} f(t)dt$, etc., are always understood as the limit $\lim_{T\to\infty} \int_{-T}^{T} f(t)dt$.

It must be emphasized that this Fourier integral representation of $f(x)$ [Eq. (15.17)] is purely formal. It is not intended as a rigorous derivation, but it can be made rigorous (compare I. N. Sneddon, *Fourier Transforms*, Section 3.2; see Additional Reading). It is subject to the conditions that $f(x)$ is

- piecewise continuous;
- differentiable almost everywhere (of bounded variation); and
- absolutely integrable; that is, $\int_{-\infty}^{\infty} |f(x)|dx$ is finite.

Inverse Fourier Transform—Exponential Form

Our Fourier integral [Eq. (15.17)] may be put into exponential form by noting that because $\cos \omega(t-x)$ is an even function of ω and $\sin \omega(t-x)$ is an odd function of ω,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t) \cos \omega(t-x) dt, \tag{15.18}$$

whereas

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t) \sin \omega(t-x) dt = 0. \tag{15.19}$$

Adding Eqs. (15.18) and (15.19) (with a factor i), we obtain

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} d\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \tag{15.20}$$

or, in terms of the Fourier transform $F(\omega)$ of $f(t)$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega, \qquad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \tag{15.21}$$

The variable ω introduced here is an arbitrary mathematical variable. In many physical problems, however, t and x are time variables and then ω corresponds to a frequency. We may then interpret Eq. (15.18) or Eq. (15.20) as a representation of $f(x)$ in terms of a distribution of infinitely long sinusoidal wave trains of angular frequency ω in which this frequency is a **continuous** variable.

EXAMPLE 15.3.1

Inversion of Square Pulse Using the Fourier transform in Example 15.2.1, the square pulse can now be inverted as follows:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega} e^{-i\omega t} d\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} e^{-i\omega t} d\omega.$$

Splitting the integral into one over $(-\infty, 0)$ and another over $(0, \infty)$ gives

$$f(t) = \frac{1}{\pi} \int_0^{\infty} \frac{\sin\omega}{\omega} (e^{-i\omega t} + e^{i\omega t}) d\omega = \frac{2}{\pi} \int_0^{\infty} \frac{\sin\omega}{\omega} \cos\omega t \, d\omega,$$

an inverse cosine transform.

Alternatively, we can differentiate the Heaviside unit step function expression [using $\frac{du(x)}{dx} = \delta(x)$]

$$f(t) = u(t+1) - u(-1+t) \quad \text{giving} \quad \frac{df(t)}{dt} = \delta(t+1) - \delta(-1+t).$$

This yields

$$\begin{aligned} \frac{df(t)}{dt} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \int_{-\infty}^{\infty} [\delta(t'+1) - \delta(t'-1)] e^{-i\omega t'} dt' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} (e^{i\omega} - e^{-i\omega}) d\omega = \frac{i}{\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \sin\omega \, d\omega, \end{aligned}$$

and by integrating the result $f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \frac{\sin\omega}{\omega} d\omega$, as above. As a final check, Exercise 7.2.15 gives us

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} e^{-i\omega t} d\omega = \begin{cases} 0, & |t| > 1, \\ 1, & |t| < 1. \end{cases} \quad ■$$

Dirac Delta Function Derivation

If the order of integration of Eq. (15.20) is reversed, we may rewrite it as

$$f(x) = \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega \right\} dt. \tag{15.22}$$

Apparently, the quantity in curly brackets behaves as a delta function $\delta(t-x)$. We might take Eq. (15.22) as presenting us with a Fourier integral representation of the Dirac delta function. Alternatively, we take it as a clue to a new derivation of the Fourier integral theorem.

From Eq. (1.160) (shifting the singularity from $t = 0$ to $t = x$),

$$f(x) = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t) \delta_n(t-x) dt, \tag{15.23}$$

where $\delta_n(t-x)$ is a sequence defining the distribution $\delta(t-x)$. Note that Eq. (15.23) assumes that $f(t)$ is continuous at $t = x$. We take $\delta_n(t-x)$ to be

$$\delta_n(t-x) = \frac{\sin n(t-x)}{\pi(t-x)} = \frac{1}{2\pi} \int_{-n}^{n} e^{i\omega(t-x)} d\omega, \tag{15.24}$$

using Eq. (1.156). Substituting Eq. (15.24) into Eq. (15.23), we have

$$f(x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-n}^{n} e^{i\omega(t-x)} d\omega\, dt. \tag{15.25}$$

Interchanging the order of integration and then taking the limit, as $n \to \infty$, we have Eq. (15.20), the Fourier integral theorem.

With the understanding that it belongs under an integral sign as in Eq. (15.22), the identification

$$\delta(t-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega \tag{15.26}$$

provides a very useful **Fourier integral representation of the delta function**. It is used to great advantage in Sections 15.6 and 15.7.

EXERCISES

15.3.1 Prove that

$$\frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\omega t} d\omega}{E_0 - i\Gamma/2 - \hbar\omega} = \begin{cases} \exp(-\Gamma t/2\hbar)\exp(-iE_0 t/\hbar), & t > 0, \\ 0, & t < 0. \end{cases}$$

This Fourier integral appears in a variety of problems in quantum mechanics: Wentzel, Kramers, Brillouin (WKB) barrier penetration, scattering, time-dependent perturbation theory, and so on.
Hint. Try contour integration.

15.3.2 Find the Fourier transform of the triangular pulse (Fig. 15.3)

$$f(t) = \begin{cases} h(1 - a|t|), & |t| < 1/a, \\ 0, & |t| > 1/a. \end{cases}$$

Note. This function provides another delta sequence with $h = a$ and $a \to \infty$.

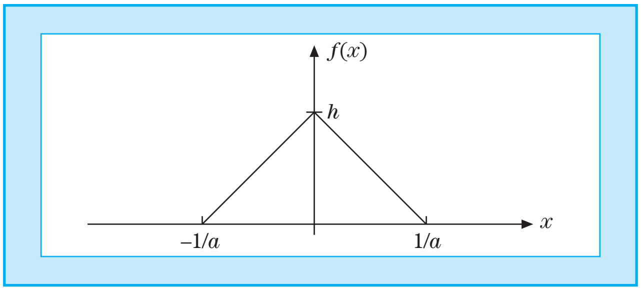
15.3.3 Prove

$$\int_0^{\infty} \frac{\cos \omega t\, d\omega}{a^2 + \omega^2} = \frac{\pi}{2a} e^{-a|t|}$$

by choosing a suitable contour and applying the residue theorem.

Figure 15.3

Triangular Pulse



15.3.4 Find the Fourier cosine, sine, and complex transforms of $e^{-a^2x^2}$.

15.3.5 Define a sequence

$$\delta_n(x) = \begin{cases} n, & |x| < 1/2n, \\ 0, & |x| > 1/2n. \end{cases}$$

[This is Eq.(1.153).] Express $\delta_n(x)$ as a Fourier integral and show that we may write

$$\delta(x) = \lim_{n\to\infty} \delta_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk.$$

15.3.6 Using the sequence

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\exp(-n^2x^2),$$

show that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}dk.$$

Note. Remember that $\delta(x)$ is defined in terms of its behavior as part of an integrand [Section 1.14, especially Eqs. (1.151) and (1.152)].

15.4 Fourier Transforms—Inversion Theorem

Let us **define** $F(\omega)$, the Fourier transform of the function $f(t)$, by

$$F(\omega) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt. \qquad (15.27)$$

Exponential Transform

Then from Eq. (15.20) we have the inverse relation

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega. \qquad (15.28)$$

Note that Eqs. (15.27) and (15.28) are almost but not quite symmetrical, differing only in the sign of i.

Here two points deserve comment. First, the $1/\sqrt{2\pi}$ symmetry is a matter of choice, not of necessity. Many authors attach the entire $1/(2\pi)$ factor of Eq. (15.20) to either Eq. (15.27) or Eq. (15.28). Second, although the Fourier integral [Eq. (15.20)] has received much attention in the mathematics literature, we shall be primarily interested in the Fourier transform and its inverse. They are the equations with physical significance.

When we move the Fourier transform pair to three-dimensional space, it becomes

$$F(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3 r, \tag{15.29}$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3 k. \tag{15.30}$$

The integrals are over all space. Verification, if desired, follows immediately by substituting the left-hand side of one equation into the integrand of the other equation and using the three-dimensional delta function.[2] Equation (15.30) may be interpreted as an expansion of a function $f(\mathbf{r})$ in a continuum of plane wave eigenfunctions; $F(\mathbf{k})$ then becomes the amplitude of the wave $\exp(-i\mathbf{k}\cdot\mathbf{r})$.

Cosine Transform

If $f(x)$ is odd or even, these transforms may be expressed in a different form. Consider, first, $f_c(x) = f_c(-x)$, even. Writing the exponential of Eq. (15.27) in trigonometric form, we have

$$\begin{aligned} F_c(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_c(t)(\cos\omega t + i\sin\omega t)\,dt \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(t)\cos\omega t\, dt, \end{aligned} \tag{15.31}$$

the $\sin\omega t$ dependence vanishing on integration over the symmetric interval $(-\infty, \infty)$. Similarly, since $\cos\omega t$ is even, Eq. (15.27) transforms to

$$f_c(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\omega)\cos\omega t\, d\omega. \tag{15.32}$$

Equations (15.31) and (15.32) are known as Fourier cosine transforms.

EXAMPLE 15.4.1

Evaluation of Fourier Cosine Transform Evaluate the Fourier cosine integral of e^{-ax} with a a positive constant. Integrating by parts twice, we obtain

$$\begin{aligned} \int_0^{\infty} e^{-ax}\cos\omega x\, dx &= -\frac{1}{a} e^{-ax}\cos\omega x \Big|_0^{\infty} - \frac{\omega}{a}\int_0^{\infty} e^{-ax}\sin\omega x\, dx \\ &= \frac{1}{a} - \frac{\omega}{a}\left[-\frac{1}{a} e^{-ax}\sin\omega x \Big|_0^{\infty} + \frac{\omega}{a}\int_0^{\infty} e^{-ax}\cos\omega x\, dx \right]. \end{aligned}$$

Now we combine the integral on the right-hand side with that on the left, giving

$$\left(1 + \frac{\omega^2}{a^2}\right)\int_0^{\infty} e^{-ax}\cos\omega x\, dx = \frac{1}{a}$$

[2] $\delta(\mathbf{r}_1 - \mathbf{r}_2) = \delta(x_1 - x_2)\delta(y_1 - y_2)\delta(z_1 - z_2)$ with Fourier integral $\delta(x_1 - x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[ik_1(x_1 - x_2)]dk_1$, etc.

or

$$\int_0^\infty e^{-ax}\cos\omega x\,dx = \frac{a}{a^2+\omega^2}. \quad ■$$

Sine Transform

The corresponding pair of Fourier sine transforms is obtained by assuming that $f_s(x) = -f_s(-x)$, odd, and applying the same symmetry arguments. The equations are

$$F_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_s(t)\sin\omega t\,dt,^3 \tag{15.33}$$

$$f_s(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(\omega)\sin\omega t\,d\omega. \tag{15.34}$$

From the last equation we may develop the physical interpretation that $f(t)$ is being described by a continuum of sine waves. The amplitude of $\sin\omega t$ is given by $\sqrt{2/\pi}\,F_s(\omega)$, in which $F_s(\omega)$ is the Fourier sine transform of $f(t)$. It will be seen that Eq. (15.34) is the integral analog of the summation [Eq. (14.23)]. Similar interpretations hold for the cosine and exponential cases.

EXAMPLE 15.4.2

Evaluation of Fourier Sine Transform Evaluate the Fourier sine integral of $\frac{\omega}{a^2+\omega^2}$ with a a positive constant. The denominator has the poles $\omega = \pm ia$, suggesting contour integration in the complex ω-plane. With this in mind, we replace $\omega \to -\omega$ and show that

$$-\int_0^\infty \frac{\omega e^{-i\omega t}d\omega}{a^2+\omega^2} = \int_{-\infty}^0 \frac{\omega e^{i\omega t}d\omega}{a^2+\omega^2}$$

so that our Fourier sine integral becomes

$$\int_0^\infty \frac{\omega\sin\omega t\,d\omega}{a^2+\omega^2} = \frac{1}{2i}\int_0^\infty \frac{\omega(e^{i\omega t}-e^{-i\omega x})}{a^2+\omega^2} = \frac{1}{2i}\int_{-\infty}^\infty \frac{\omega e^{i\omega t}}{a^2+\omega^2}.$$

For $t > 0$, we close the contour in the upper ω-plane by a large half-circle, which does not contribute to the integral as its radius goes to ∞. We pick up the residue $iae^{-at}/2ia$ at the pole $\omega = ia$ and find

$$\int_0^\infty \frac{\omega\sin\omega t\,d\omega}{a^2+\omega^2} = \frac{\pi}{2}e^{-at}. \quad ■$$

If we take Eqs. (15.27), (15.31), and (15.33) as the direct integral transforms, described by $\mathcal{L}$ in Eq. (15.4) (Section 15.1), the corresponding inverse transforms, $\mathcal{L}^{-1}$ of Eq. (15.5), are given by Eqs. (15.28), (15.32), and (15.34).

[3]Note that a factor $-i$ has been absorbed into this $F_s(\omega)$.

EXAMPLE 15.4.3

Proton Charge Form Factor The charge form factor

$$G_E(\mathbf{q}^2) = \int \rho(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} d^3r$$

of a particle is defined as the Fourier transform of its charge density ρ, except for the factor $(2\pi)^{-3/2}$; G_E can be measured by elastically scattering electrons from a target of particles (H atoms for the proton) because the (so-called Mott) cross section of a pointlike particle is modified by the charge form factor squared for a particle with finite size, if magnetic scattering is neglected. This is a good approximation at small scattering angle θ. The momentum transfer $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ is taken in units of $\hbar$, where $\mathbf{p}$ is the incident electron momentum and $\mathbf{p}'$ the scattered electron momentum in the laboratory frame (rest frame of the proton). For elastic scattering $p = |\mathbf{p}| = |\mathbf{p}'| = p'$, if recoil is neglected at low momentum p. Figure 15.4 shows that $q = |\mathbf{q}| = 2p \sin\theta/2$.

For a **pointlike particle** of charge Q, $\rho(\mathbf{r}) = Q\delta(\mathbf{r})$ so that the charge form factor

$$G_E(\mathbf{q}^2) = Q \int \delta(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} = Q$$

is constant.

At $q = 0$, $G_E(0) = \int \rho d^3r = Q$ is the total charge Q in units of the elementary charge $|e|$; for the proton $Q = 1$.

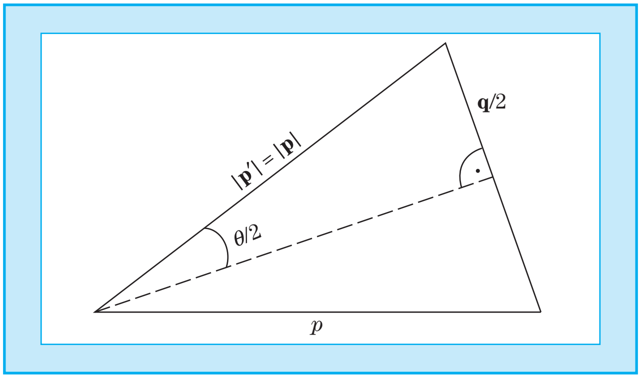
In case of spherical symmetry, we use polar coordinates r, θ, φ in which the charge form factor takes the form

$$G_E(q^2) = \int_0^\infty \rho(r) r^2 dr \int_{-1}^{1} e^{iqr\cos\theta} d\cos\theta \int_0^{2\pi} d\varphi \tag{15.35}$$

$$= \frac{2\pi}{iq} \int_0^\infty \rho(r) r dr e^{iqr\cos\theta} \Big|_{\cos\theta=-1}^{1} = \frac{4\pi}{q} \int_0^\infty \rho(r) \sin(qr) r\, dr.$$

Figure 15.4

Proton Charge Form Factor



Inverting this sine transform, one can extract the charge density from the measured charge form factor $G_E(q^2)$. This is how the proton, nuclear radii, and sizes of atoms and molecules are measured by electron scattering.

At small q compared to the inverse radius of the proton, we can use the power series for $\sin qr$ and obtain

$$G_E(q^2) = 4\pi \int_0^\infty \rho(r) r^2\, dr - \frac{q^2}{6} \int_0^\infty \rho(r) r^4\, dr + \cdots = 1 - \frac{q^2}{6}\langle r^2 \rangle + \cdots,$$

where the first term is the charge $Q = 1$ and the integral in the second term is the mean square radius $\langle r^2 \rangle$ of the proton, because the density $\rho = |\psi|^2$ is given by the quark wave function ψ, quarks being the constituents of the proton. Thus, the proton size can be extracted from the measured slope of the proton charge form factor,

$$\langle r^2 \rangle = -6 \frac{dG_E(q^2)}{dq^2}\bigg|_{q=0}.$$

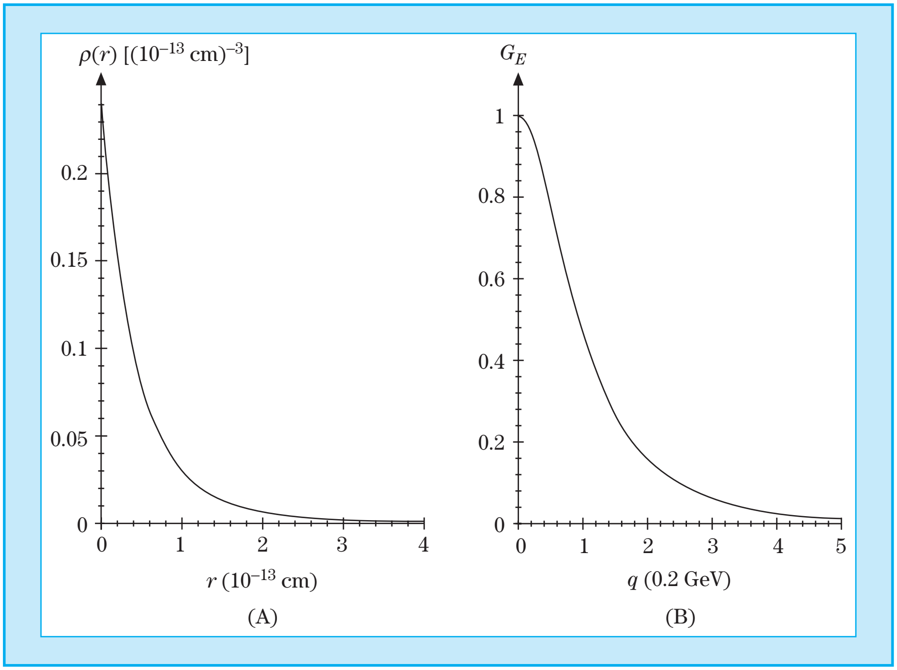
Since the proton has a finite radius of approximately 1 fm $= 10^{-13}$ cm, we consider a spherically symmetric model (Fig. 15.5A)

$$\rho(r) = \frac{N}{r}[e^{-r/R_1} - e^{-r/R}],$$

where $R \ll R_1$ are finite size parameters yet to be determined. The normalization N follows from the charge 1 of the proton, in units of the elementary charge e; that is, $G_E(0) = 1$. The same model applies to the charge density of the ^{3}He nucleus, except for the charge $G_E(0) = 2$ and its larger radius, because it is made up of two protons and one neutron instead of three quarks for the proton.

Figure 15.5

Charge Density (A) and Form Factor (B) of the Proton



Let us start by determining N from $G_E(0) = 1$ by integrating by parts as follows:

$$
\begin{aligned}
1 &= 4\pi \int_0^\infty \rho(r) r^2\, dr = 4\pi N \int_0^\infty [e^{-r/R_1} - e^{-r/R}] r\, dr \\
&= 4\pi N[-rR_1 e^{-r/R_1} + rRe^{-r/R}]\Big|_0^\infty + 4\pi N \int_0^\infty [R_1 e^{-r/R_1} - Re^{-r/R}] dr \\
&= 4\pi N\big(R_1^2 - R^2\big), \quad N = \frac{1}{4\pi\left(R_1^2 - R^2\right)}.
\end{aligned}
$$

A look at the sine transform for $G_E(q)$ [Eq. (15.35)] tells us that we also need to calculate the integral

$$
\begin{aligned}
\int_0^\infty e^{-r/R} \sin qr\, dr &= -Re^{-r/R} \sin qr\Big|_0^\infty + qR \int_0^\infty e^{-r/R} \cos qr\, dr \\
&= qR\left[-Re^{-r/R} \cos qr\Big|_0^\infty - qR \int_0^\infty e^{-r/R} \sin qr\, dr\right],
\end{aligned}
$$

which we do by integrating by parts twice. This yields the same integral on the right-hand side, which we combine with that on the left-hand side, so that

$$
\int_0^\infty e^{-r/R} \sin qr\, dr = \frac{qR^2}{1 + q^2R^2}.
$$

Substituting this result into the G_E sine transform formula [Eq. (15.35)] yields

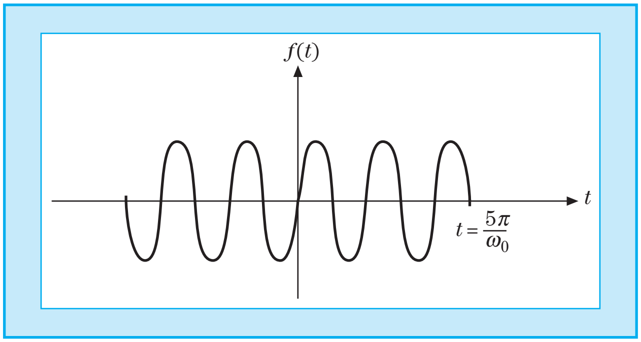
$$
\begin{aligned}
G_E(q^2) &= \frac{4\pi N}{q} \int_0^\infty [e^{-r/R_1} - e^{-r/R}] \sin qr\, dr \\
&= \frac{1}{R_1^2 - R^2}\left(\frac{R_1^2}{1 + q^2R_1^2} - \frac{R^2}{1 + q^2R^2}\right).
\end{aligned}
$$

Note that at $q = 0$ this charge form factor is properly normalized to unity, whereas at large q it falls like q^{-4}. This falloff is called quark counting and predicted by quantum chromodynamics, the quantum field theory of the strong interaction that binds quarks in the proton. Our nonrelativistic model simulates this behavior. Now we choose $R_1 = 1$ fm, approximately the size of the proton, and $R = 1/4$ fm; this is shown in Fig. 15.5. ■

Note that the Fourier cosine transforms and the Fourier sine transforms each involve only positive values (and zero) of the arguments. We use the parity of $f(t)$ to establish the transforms, but once the transforms are established, the behavior of the functions f and g for negative argument is irrelevant. In effect, the transform equations impose a **definite parity: even for the Fourier cosine** transform and **odd for the Fourier sine** transform.

Figure 15.6

Finite Wave Train



EXAMPLE 15.4.4

Finite Wave Train An important application of the Fourier transform is the resolution of a finite pulse into sinusoidal waves. Imagine that an infinite wave train $\sin\omega_0 t$ is clipped by Kerr cell or saturable dye cell shutters so that

$$f(t) = \begin{cases} \sin\omega_0 t, & |t| < \frac{N\pi}{\omega_0}, \\ 0, & |t| > \frac{N\pi}{\omega_0}. \end{cases} \tag{15.36}$$

This corresponds to N cycles of our original wave train (Fig. 15.6). Since $f(t)$ is odd, we may use the Fourier sine transform [Eq. (15.33)] to obtain

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{N\pi/\omega_0} \sin\omega_0 t \sin\omega t \, dt. \tag{15.37}$$

Integrating, we find our amplitude function

$$F_s(\omega) = \sqrt{\frac{2}{\pi}} \left[\frac{\sin[(\omega_0 - \omega)(N\pi/\omega_0)]}{2(\omega_0 - \omega)} - \frac{\sin[(\omega_0 + \omega)(N\pi/\omega_0)]}{2(\omega_0 + \omega)} \right]. \tag{15.38}$$

It is of considerable interest to see how $F_s(\omega)$ depends on frequency. For large ω_0 and $\omega \approx \omega_0$, only the first term will be of any importance because of the denominators. It is plotted in Fig. 15.7. This is the amplitude curve for the single slit diffraction pattern.

There are zeros at

$$\frac{\omega_0 - \omega}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \pm\frac{1}{N}, \pm\frac{2}{N}, \quad \text{and so on.} \tag{15.39}$$

For large N, $F_s(\omega)$ may also be interpreted as a Dirac delta distribution, as in Section 1.14. Since the contributions outside the central maximum are small in this case, we may take

$$\Delta\omega = \frac{\omega_0}{N} \tag{15.40}$$

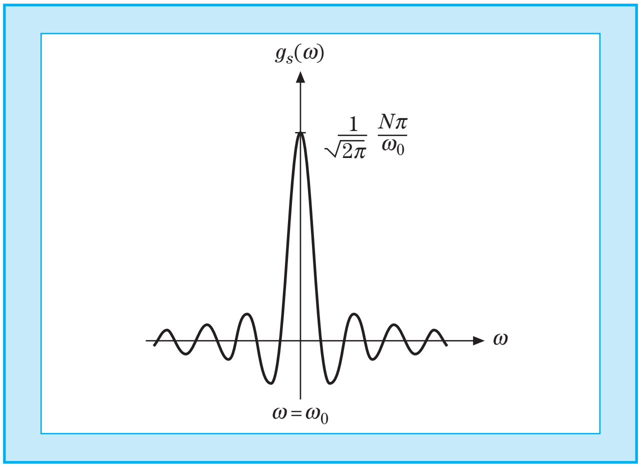


Figure 15.7

Fourier Transform of Finite Wave Train

as a good measure of the spread in frequency of our wave pulse. Clearly, if N is large (a long pulse), the frequency spread will be small. On the other hand, if our pulse is clipped short (N is small), the frequency distribution will be wider and the secondary maxima are more important. ■

EXERCISES

15.4.1 The function

$$f(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$

is a symmetrical finite step function.

(a) Find the $F_c(\omega)$, Fourier cosine transform of $f(t)$.

(b) Taking the inverse cosine transform, show that

$$f(t) = \frac{2}{\pi} \int_0^\infty \frac{\sin\omega \cos\omega t}{\omega} d\omega.$$

(c) From part (b) show that

$$\int_0^\infty \frac{\sin\omega \cos\omega t}{\omega} d\omega = \begin{cases} 0, & |t| > 1, \\ \frac{\pi}{4}, & |t| = 1, \\ \frac{\pi}{2}, & |t| < 1. \end{cases}$$

15.4.2 Derive sine and cosine representations of $\delta(t - x)$ that are comparable to the exponential representation [Eq. (15.26)].

ANS. $\frac{2}{\pi} \int_0^\infty \sin\omega t \sin\omega x \, d\omega, \quad \frac{2}{\pi} \int_0^\infty \cos\omega t \cos\omega x \, d\omega.$

15.4.3 In a resonant cavity, an electromagnetic oscillation of frequency ω_0 dies out as

$$A(t) = A_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t}, \quad t > 0.$$

(Take $A(t) = 0$ for $t < 0$.)
The parameter Q is a measure of the ratio of stored energy to energy loss per cycle. Calculate the frequency distribution of the oscillation, $a^*(\omega)a(\omega)$, where $a(\omega)$ is the Fourier transform of $A(t)$.
Note. The larger Q is, the sharper your resonance line will be.

$$\textit{ANS.} \quad a^*(\omega)a(\omega) = \frac{A_0^2}{2\pi} \frac{1}{(\omega - \omega_0)^2 + (\omega_0/2Q)^2}.$$

15.4.4 (a) Calculate the Fourier exponential transform of $f(t) = t^n e^{-a|t|}$ for $n = 1, 2, 3$.
(b) Calculate the inverse transform by employing the calculus of residues (Section 7.2).

15.5 Fourier Transform of Derivatives

Figure 15.1 outlines the overall technique of using Fourier transforms and inverse transforms to solve a problem. Here, we take an initial step in **solving a differential equation**—obtaining the Fourier transform of a derivative.

Using the exponential form, we determine that the Fourier transform of $f(t)$ is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \tag{15.41}$$

and for $df(t)/dt$

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(t)}{dt} e^{i\omega t} dt. \tag{15.42}$$

Integrating Eq. (15.42) by parts, we obtain

$$F_1(\omega) = \frac{e^{i\omega t}}{\sqrt{2\pi}} f(t)\Big|_{-\infty}^{\infty} - \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \tag{15.43}$$

If $f(t)$ vanishes[4] as $t \to \pm\infty$, we have

$$F_1(\omega) = -i\omega F(\omega); \tag{15.44}$$

that is, the transform of the derivative is $(-i\omega)$ times the transform of the original function. This may readily be generalized to the nth derivative to yield

$$F_n(\omega) = (-i\omega)^n F(\omega), \tag{15.45}$$

[4]Apart from cases such as Exercises 15.3.5 and 15.3.6, $f(t)$ must vanish as $t \to \pm\infty$ in order for the Fourier transform of $f(t)$ to exist.

provided all the integrated parts of Eq. (15.43) vanish as $t \to \pm\infty$. This is the power of the Fourier transform, the main reason it is so useful in solving (partial) differential equations. The operation of **differentiation in coordinate space has been replaced by a multiplication in ω space**. Such properties of the kernel are the key in applications of integral transforms to solving ordinary differential equations (ODEs) and partial differential equations (PDEs), developed next.

EXAMPLE 15.5.1

Driven Harmonic Oscillator If we substitute the Fourier integral $y(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}dt$ into the harmonic oscillator ODE $\frac{d^2y}{dt^2} + \Omega^2 y = A\cos(\omega_0 t)$, where t is the time now, we obtain an **algebraic** equation for $Y(\omega)$ called the Fourier transform of our solution $y(t)$,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(\omega)(\Omega^2 - \omega^2)e^{i\omega t}d\omega = \frac{A}{2}\int_{-\infty}^{\infty} e^{i\omega t}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]d\omega,$$

because differentiating twice corresponds to multiplying $Y(\omega)$ by $(i\omega)^2$, and we represent the driving term as a Fourier integral with the only frequencies $\pm\omega_0$. Upon comparing integrands, valid because the integrals are over the same interval in the same variable ω (or, more rigorously, using the inverse Fourier transform), we find

$$Y(\omega) = \sqrt{\frac{\pi}{2}}\frac{A}{\Omega^2-\omega^2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

The resulting integral,

$$y(t) = \frac{A}{2}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{\Omega^2-\omega^2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]d\omega = \frac{A}{\Omega^2-\omega_0^2}\cos(\omega_0 t),$$

is the steady-state and particular solution of our inhomogeneous ODE. Note that **the assumption that the end points in the partially integrated term** in Eq. (15.43) **do not contribute eliminates solutions of the homogeneous harmonic oscillator ODE (called transients in physics**; undamped $\sin\Omega t$, $\cos\Omega t$ solutions in our case).

Alternatively, we Fourier transform the ODE as follows:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{d^2y}{dt^2}+\Omega^2 y\right)e^{i\omega t}dt = \frac{A}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}(e^{i\omega_0 t}+e^{-i\omega_0 t})e^{i\omega t}dt$$
$$= A\sqrt{\pi}2[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)].$$

We integrate by parts twice,

$$\int_{-\infty}^{\infty}\frac{d^2y}{dt^2}e^{i\omega t}dt = \frac{dy}{dt}e^{i\omega t}\Big|_{-\infty}^{\infty} - i\omega\int_{-\infty}^{\infty}\frac{dy}{dt}e^{i\omega t}dt$$
$$= -i\omega\left[ye^{i\omega t}\Big|_{-\infty}^{\infty} - i\omega\int_{-\infty}^{\infty}ye^{i\omega t}dt\right] = -\omega^2\int_{-\infty}^{\infty}ye^{i\omega t}dt,$$

assuming that $y(t) \to 0$ and $\frac{dy(t)}{dt} \to 0$ sufficiently fast, as $t \to \pm\infty$. The result of comparing integrands (using the inverse Fourier transform) is the same as before:

$$(\Omega^2 - \omega^2)Y(\omega) = \sqrt{\frac{\pi}{2}}A[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad \blacksquare$$

Similarly, a PDE might become an ODE, such as the heat flow PDE considered next.

EXAMPLE 15.5.2

Heat Flow PDE To illustrate the transformation of a PDE into an ODE, let us Fourier transform the heat flow partial differential equation

$$\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2},$$

where the solution $\psi(x, t)$ is the temperature in space as a function of time. By substituting the Fourier integral solution

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(\omega, t) e^{-i\omega x}\, d\omega,$$

this yields an ODE for the Fourier transform Ψ of ψ,

$$\frac{\partial \Psi}{\partial t} = -a^2\omega^2\Psi(\omega, t),$$

in the time variable t. Alternatively and equivalently, apply the inverse Fourier transform to each side of the heat PDE. Integrating, we obtain

$$\ln \Psi = -a^2\omega^2 t + \ln C, \quad \text{or} \quad \Psi = Ce^{-a^2\omega^2 t},$$

where the integration constant C may still depend on ω and, in general, is determined by initial conditions. Putting this solution back into our inverse Fourier transform,

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} e^{-a^2\omega^2 t} d\omega,$$

yields a separation of the x and t variables. For simplicity, we here take C ω-independent (assuming appropriate initial conditions) and integrate by completing the square in ω, as in Example 15.2.2, making appropriate changes of variables and parameters ($a^2 \to a^2 t$, $\omega \to x$, $t \to -\omega$). This yields the particular solution of the heat flow PDE,

$$\psi(x, t) = \frac{C}{a\sqrt{2t}} \exp\left(-\frac{x^2}{4a^2 t}\right),$$

that appears as a clever guess in Section 16.2. In effect, we have shown that ψ is the inverse Fourier transform of $C \exp(-a^2\omega^2 t)$. $\blacksquare$

EXAMPLE 15.5.3

Inversion of PDE Derive a Fourier integral for the Green's function G_0 of Poisson's PDE, which is a solution of

$$\nabla^2 G_0(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

Once G_0 is known, the general solution of Poisson's PDE

$$\nabla^2 \Phi = -4\pi\rho(\mathbf{r})$$

of electrostatics is given as

$$\Phi(\mathbf{r}) = \int G_0(\mathbf{r}, \mathbf{r}')4\pi\rho(\mathbf{r}')d^3r'.$$

Applying ∇^2 to Φ and using the PDE the Green's function satisfies, we check that

$$\nabla^2\Phi(\mathbf{r}) = \int \nabla^2 G_0(\mathbf{r}, \mathbf{r}')4\pi\rho(\mathbf{r}')d^3r' = \int \delta(\mathbf{r} - \mathbf{r}')4\pi\rho(\mathbf{r}')d^3r' = 4\pi\rho(\mathbf{r}).$$

Now we use Fourier transforms of the δ function and G_0, writing

$$\nabla^2 \int g_0(\mathbf{p})e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\frac{d^3p}{(2\pi)^3} = \int e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\frac{d^3p}{(2\pi)^3}.$$

Because the integrands of equal Fourier integrals must be the same (almost) everywhere, which follows from the inverse Fourier transform, and with

$$\nabla e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} = i\mathbf{p}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')},$$

this yields $-\mathbf{p}^2 g_0(\mathbf{p}) = 1$. Substituting this solution into the inverse Fourier transform for G_0 gives

$$G_0(\mathbf{r}, \mathbf{r}') = -\int e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\frac{d^3p}{(2\pi)^3\mathbf{p}^2} = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

We can verify the last part of this result by applying ∇^2 to G_0 again and recalling from Chapter 1 that $\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}')$.

The inverse Fourier transform can be evaluated using polar coordinates exploiting the spherical symmetry of $\mathbf{p}^2$, similar to the charge form factor in Example 15.4.3 for a spherically symmetric charge density. For simplicity, we write $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and call θ the angle between $\mathbf{R}$ and $\mathbf{p}$ so that

$$\begin{aligned}
\int e^{i\mathbf{p}\cdot\mathbf{R}}\frac{d^3p}{p^2} &= \int_0^\infty dp \int_{-1}^1 e^{ipR\cos\theta}d\cos\theta \int_0^{2\pi} d\varphi \\
&= \frac{2\pi}{iR}\int_0^\infty \frac{dp}{p}e^{ipR\cos\theta}\Big|_{\cos\theta=-1}^1 = \frac{4\pi}{R}\int_0^\infty \frac{\sin pR}{p}dp \\
&= \frac{4\pi}{R}\int_0^\infty \frac{\sin pR}{pR}d(pR) = \frac{2\pi^2}{R},
\end{aligned}$$

where θ and φ are the angles of $\mathbf{p}$, and $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$ from Example 7.2.4. Dividing by $-(2\pi)^3$, we obtain $G_0(R) = -1/(4\pi R)$, as claimed. An evaluation of this Fourier transform by contour integration is given in Example 16.3.2. ■

EXAMPLE 15.5.4

Wave Equation The Fourier transform technique may be used to advantage in handling PDEs with constant coefficients. To illustrate the technique further, let us derive a familiar expression of elementary physics. An infinitely long string is vibrating freely. The amplitude y of the (small) vibrations satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \tag{15.46}$$

We shall assume an initial condition

$$y(x, 0) = f(x), \tag{15.47}$$

where f is localized, that is, approaches zero at large x.

Applying our Fourier transform to both sides of our PDE [Eq. (15.46)] means multiplying by $e^{i\alpha x}/\sqrt{2\pi}$ and integrating over x according to

$$Y(\alpha, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x, t)e^{i\alpha x}dx \tag{15.48}$$

and using Eq. (15.43) for the second derivative. Note that the integrated part of $\frac{\partial Y}{\partial x}$ and $\frac{\partial^2 Y}{\partial x^2}$ vanishes: The wave has not yet gone to $\pm\infty$, as it is propagating forward in time, and there is no source at infinity [$f(\pm\infty) = 0$]. We obtain

$$\int_{-\infty}^{\infty} \frac{\partial^2 y(x, t)}{\partial x^2}e^{i\alpha x}dx = \frac{1}{v^2}\int_{-\infty}^{\infty} \frac{\partial^2 y(x, t)}{\partial t^2}e^{i\alpha x}dx \tag{15.49}$$

or

$$(-i\alpha)^2 Y(\alpha, t) = \frac{1}{v^2}\frac{\partial^2 Y(\alpha, t)}{\partial t^2}. \tag{15.50}$$

Since no derivatives with respect to α appear, Eq. (15.50) is actually an ODE—in fact, it is the linear oscillator equation. This transformation, from a PDE to an ODE, is a significant simplification. We solve Eq. (15.50) subject to the appropriate initial conditions. At $t = 0$, applying Eq. (15.47), Eq. (15.48) reduces to

$$Y(\alpha, 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = F(\alpha), \tag{15.51}$$

where $F(\alpha)$ is the Fourier transform of the initial condition $f(x)$. The general solution of Eq. (15.50) in exponential form is

$$Y(\alpha, t) = F(\alpha)e^{\pm iv\alpha t}. \tag{15.52}$$

Using the inversion formula [Eq. (15.28)], we have

$$y(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(\alpha, t)e^{-i\alpha x}d\alpha, \tag{15.53}$$

and, by Eq. (15.52),

$$y(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha(x\mp vt)}d\alpha. \tag{15.54}$$

Since $f(x)$ is the Fourier inverse transform of $F(\alpha)$,

$$y(x, t) = f(x \mp vt), \tag{15.55}$$

corresponding to waves advancing in the $+x$- and $-x$-directions, respectively.
The boundary condition of Eq. (15.47) is built into these particular linear combinations of waves. ■

The accomplishment of the Fourier transform here deserves special emphasis.

- The Fourier transform converts a PDE into an ODE, where the "degree of transcendence" of the problem is reduced.

In Section 15.10, Laplace transforms are used to convert ODEs (with constant coefficients) into algebraic equations. Again, the degree of transcendence is reduced. The problem is simplified, as outlined in Fig. 15.1.

EXERCISES

15.5.1 Equation (15.45) yields

$$F_2(\omega) = -\omega^2 F(\omega)$$

for the Fourier transform of the second derivative of $f(x)$. The condition $f(x) \to 0$ for $x \to \pm\infty$ may be relaxed slightly. Find the least restrictive condition for the preceding equation for $F_2(\omega)$ to hold.

$$\textit{ANS.} \quad \left[\frac{df(x)}{dx} - i\omega f(x)\right] e^{i\omega x}\Bigg|_{-\infty}^{\infty} = 0.$$

15.5.2 (a) Given that $F(\mathbf{k})$ is the three-dimensional Fourier transform of $f(\mathbf{r})$ and $F_1(\mathbf{k})$ is the three-dimensional Fourier transform of $\nabla f(\mathbf{r})$, show that

$$F_1(\mathbf{k}) = (-i\mathbf{k})F(\mathbf{k}).$$

This is a three-dimensional generalization of Eq. (15.45) for $n = 1$.

(b) Show that the three-dimensional Fourier transform of $\nabla \cdot \nabla f(\mathbf{r})$ is

$$F_2(\mathbf{k}) = (-i\mathbf{k})^2 F(\mathbf{k}).$$

Note. Vector $\mathbf{k}$ is in the transform space. In Section 15.7, we shall have $\hbar\mathbf{k} = \mathbf{p}$, linear momentum.

15.5.3 Show

$$\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{d^3k}{(2\pi)^3 \mathbf{k}^2} = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

by contour integration in conjunction with the residue theorem.
Hint. Use spherical polar coordinates in k-space.

15.5.4 Solve the PDE

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} - a^2 y$$

by Fourier transform, where $y(x, t = 0) = 0,\ x > 0,\ \ y(x = 0, t) = f(t),\ \ t > 0$, and a is a constant.

15.5.5 Show that the three-dimensional Fourier exponential transform of a radially symmetric function may be rewritten as a Fourier sine transform:

$$\frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} f(r) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3x = \frac{1}{k}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} [r f(r)] \sin kr\, dr.$$

15.6 Convolution Theorem

We employ convolutions to solve differential equations and to normalize momentum wave functions.

Let us consider two functions $f(x)$ and $g(x)$ with Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. We define the operation

$$f * g \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x - y)\, dy \tag{15.56}$$

as the **convolution** of the two functions f and g over the interval $(-\infty, \infty)$. This form of an integral appears in probability theory in the determination of the probability density of two random, independent variables. Our solution of Poisson's equation (i.e., the Coulomb potential) may be interpreted as a convolution of a charge distribution, $\rho(\mathbf{r}_2)$, and a weighting function, $(4\pi\varepsilon_0|\mathbf{r}_1 - \mathbf{r}_2|)^{-1}$. In other works this is sometimes referred to as the **Faltung**, the German term for "folding."[5] We now transform the integral in Eq. (15.56) by introducing the Fourier transforms, interchanging the order of integration, and transforming $g(y)$:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(y) f(x - y) dy &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} F(\omega) e^{-i\omega(x-y)}\, d\omega\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \left[\int_{-\infty}^{\infty} g(y) e^{i\omega y}\, dy \right] e^{-i\omega x}\, d\omega \\
&= \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega x} d\omega. \qquad (15.57)
\end{aligned}$$

Comparing with Eq. (15.56), this shows that

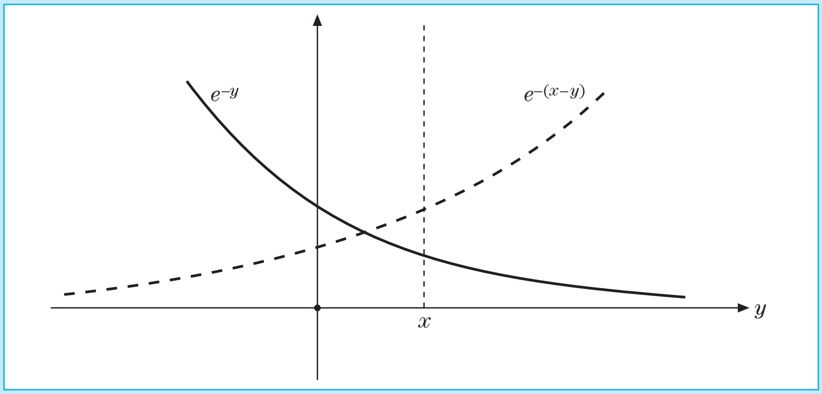
$$f * g = \mathcal{L}^{-1}(FG).$$

In other words, the Fourier inverse transform of a **product** of Fourier transforms is the convolution of the original functions, $f * g$.

[5]For $f(y) = e^{-y}$, $f(y)$ and $f(x - y)$ are plotted in Fig. 15.8. Clearly, $f(y)$ and $f(x - y)$ are mirror images of each other in relation to the vertical line $y = x/2$; that is, we could generate $f(x - y)$ by folding over $f(y)$ on the line $y = x/2$.

Figure 15.8

Convolution–Faltung



EXAMPLE 15.6.1

Convolution Integral Let us apply the convolution Eq. (15.57) with f, F from Example 15.2.2 and g, G from Example 15.4.1 so that

$$f(x) = e^{-a^2x^2}, \qquad F(\omega) = \frac{1}{a\sqrt{2}} \exp\left(-\frac{\omega^2}{4a^2}\right);$$

$$g(y) = \sqrt{\frac{\pi}{2b^2}} e^{-b|y|}, \qquad G(\omega) = \frac{1}{b^2 + \omega^2}.$$

From Example 15.4.1, recall that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega y}\, dt}{b^2 + \omega^2} = 2\int_0^{\infty} \frac{\cos \omega y\, d\omega}{b^2 + \omega^2} = \frac{\pi}{b} e^{-by}, \qquad y > 0,$$

using the Euler identity $e^{i\omega y} = \cos \omega y + i \sin \omega y$ and noticing that the sine integral vanishes because its integrand is odd under reversal of sign of t, whereas the cosine integrand is even.

Now we apply the convolution formula [Eq. (15.57)]

$$\frac{\mathcal{I}}{b}\sqrt{\frac{\pi}{2}} \equiv \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{2}} \frac{1}{b} e^{-b|y|} \exp(-a^2(x-y)^2)\, dy = \frac{1}{a\sqrt{2}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{b^2 + \omega^2} e^{-\omega^2/4a^2}\, d\omega.$$

The integral $\mathcal{I}$ can be manipulated into the error integral erfc (Section 10.4) by splitting the interval and substituting $y \to -y$ in the $(-\infty, 0)$ part, giving

$$\begin{aligned}\mathcal{I} &= \int_{-\infty}^{\infty} e^{-b|y|} \exp(-a^2(x-y)^2)\, dy \\ &= \int_0^{\infty} e^{-by} \exp(-a^2(x-y)^2)\, dy + \int_0^{\infty} e^{-by} \exp(-a^2(x+y)^2)\, dy.\end{aligned}$$

Now we substitute $\xi = y - x$ in the first integral and $\xi = y + x$ in the second, yielding

$$\mathcal{I} = e^{-bx} \int_{-x}^{\infty} e^{-b\xi - a^2\xi^2} d\xi + e^{bx} \int_{x}^{\infty} e^{-b\xi - a^2\xi^2} d\xi.$$

Completing the square in the exponent as in Example 15.2.2 using

$$a^2\xi^2 + b\xi = a^2\left(\xi + \frac{b}{2a^2}\right)^2 - \frac{b^2}{4a^2},$$

we obtain, with the substitution $a\eta = \xi + b/2a^2$,

$$\mathcal{I} = \frac{1}{a} e^{-bx + b^2/4a^2} \int_{-ax+b/2a}^{\infty} e^{-\eta^2} d\eta + \frac{1}{a} e^{bx + b^2/4a^2} \int_{ax+b/2a}^{\infty} e^{-\eta^2} d\eta$$

so that finally

$$\sqrt{\frac{\pi}{2b^2}}\mathcal{I} = \frac{\pi}{2ab\sqrt{2}} e^{b^2/4a^2}\left[e^{-bx}\,\mathrm{erfc}\left(\frac{b}{2a} - ax\right) + e^{bx}\mathrm{erfc}\left(\frac{b}{2a} + ax\right)\right]. \qquad (15.58)$$

Another example is provided by changing $g(y)$ in the previous example to the square pulse $g(y) = 1$, for $|y| < 1$ and zero elsewhere. Its Fourier transform is given in Example 15.2.1 as

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{i\omega y}\, dy = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}.$$

The convolution with $f(x)$ takes the interesting form

$$\int_{-1}^{1} \exp(-a^2(x-y)^2)\, dy = \frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\omega^2/4a^2} \frac{\sin\omega}{\omega} e^{-i\omega x} d\omega,$$

where the left-hand side can again be converted to a difference of error integrals

$$\int_{-1}^{1} \exp(-a^2(x-y)^2)\, dy = \frac{\sqrt{\pi}}{2a}[\mathrm{erfc}(-a(1+x)) - \mathrm{erfc}(a(1-x))]. \quad \blacksquare$$

EXAMPLE 15.6.2 **Coulomb Potential by Convolution** The Coulomb potential for an extended charge distribution ρ of a composite system,

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r',$$

appears to be a three-dimensional case of a convolution integral. If we recall the charge form factor G_E as the Fourier transform of the charge density from Example 15.4.3,

$$\frac{1}{(2\pi)^{3/2}} \int \rho(\mathbf{r}) e^{i\mathbf{p}\cdot\mathbf{r}} d^3r = \frac{G_E(\mathbf{p}^2)}{(2\pi)^{3/2}},$$

and $1/\mathbf{p}^2$ as the Fourier transform of $1/|\mathbf{r}-\mathbf{r}'|$ from Example 15.5.3,

$$\frac{(2\pi)^{3/2}}{4\pi\,|\mathbf{r}-\mathbf{r}'|} = \int \frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{(2\pi)^{3/2}\mathbf{p}^2} d^3p,$$

being careful to include all normalizations, then we can apply the convolution theorem to obtain

$$V(\mathbf{r}) = \int \rho(\mathbf{r}')\frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|} = 4\pi \int \frac{G_E(\mathbf{p}^2)}{\mathbf{p}^2} e^{-i\mathbf{p}\cdot\mathbf{r}} \frac{d^3p}{(2\pi)^3}. \tag{15.59}$$

Let us now evaluate this result for the proton Example 15.4.3. This gives

$$\begin{aligned}
V(\mathbf{r}) &= \frac{4\pi}{R_1^2 - R^2} \int \left(\frac{R_1^2}{1+p^2R_1^2} - \frac{R^2}{1+p^2R^2}\right) \frac{e^{-i\mathbf{p}\cdot\mathbf{r}}}{\mathbf{p}^2} \frac{d^3p}{(2\pi)^3} \\
&= \frac{4\pi}{R_1^2 - R^2} \int \left[\left(-\frac{R_1^2}{1+p^2R_1^2} + \frac{1}{p^2}\right)R_1^2 - R^2\left(\frac{1}{p^2} - \frac{R^2}{1+p^2R^2}\right)\right] \\
&\quad \times e^{-i\mathbf{p}\cdot\mathbf{r}} \frac{d^3p}{(2\pi)^3} \\
&= \frac{R_1^2}{R_1^2 - R^2}\left(\frac{1}{r} - \frac{e^{-r/R_1}}{r}\right) - \frac{R^2}{R_1^2 - R^2}\left(\frac{1}{r} - \frac{e^{-r/R}}{r}\right) \\
&= \frac{1}{r}\left[1 - \frac{1}{R_1^2 - R^2}\left(R_1^2 e^{-r/R_1} - R^2 e^{-r/R}\right)\right]
\end{aligned}$$

for the electrostatic potential, a pointlike Coulomb potential combined with a Yukawa shape which remains finite as $r \to 0$. ■

Parseval's Relation

Results analogous to Eq. (15.57) may be derived for the Fourier sine and cosine transforms (Exercises 15.6.1 and 15.6.2).

For the special case $x = 0$ in Eq. (15.57), we have

$$\int_{-\infty}^{\infty} F(\omega)G(\omega)\,d\omega = \int_{-\infty}^{\infty} f(-y)g(y)\,dy. \tag{15.60}$$

Equation (15.60) and the corresponding sine and cosine convolutions are often called Parseval's relations by analogy with Parseval's theorem for Fourier series (Chapter 14, Exercise 14.4.2). However, the minus sign in $-y$ suggests that modifications be tried. We now do this with g^* instead of g using a different technique.

The Parseval relation[6,7]

$$\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega = \int_{-\infty}^{\infty} f(t)g^*(t)\,dt \tag{15.61}$$

[6]Note that all arguments are positive, in contrast to Eq. (15.60).
[7]Some authors prefer to restrict Parseval's name to series and refer to Eq. (15.61) as Rayleigh's theorem.

may be derived elegantly using the Dirac delta function representation [Eq. (15.26)]. We have

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G^*(x)e^{ixt}dx\,dt, \tag{15.62}$$

with attention to the complex conjugation in the $G^*(x)$ to $g^*(t)$ transform. Integrating over t first and using Eq. (15.26), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)g^*(t)\,dt &= \int_{-\infty}^{\infty} F(\omega)\int_{-\infty}^{\infty} G^*(x)\delta(x-\omega)dx\,d\omega \\ &= \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega, \end{aligned} \tag{15.63}$$

our desired Parseval relation. (The $*$ of complex conjugation can also be applied to f and F instead.) If $f(t) = g(t)$, then the integrals in the Parseval relation are normalization integrals (Section 9.4). Equation (15.63) guarantees that if a function $f(t)$ is normalized to unity, its transform $F(\omega)$ is likewise normalized to unity. This is extremely important in quantum mechanics, as discussed in the next section.

It may be shown that the Fourier transform is a unitary operation (in the Hilbert space L^2 of square integrable functions). The Parseval relation is a reflection of this unitary property—analogous to Exercise 3.4.14 for matrices.

In Fraunhofer diffraction optics, the diffraction pattern (amplitude) appears as the transform of the function describing the aperture (compare Example 15.2.1). With intensity proportional to the square of the amplitude, the Parseval relation implies that the energy passing through the aperture seems to be somewhere in the diffraction pattern—a statement of the conservation of energy.

Parseval's relations may be developed independently of the inverse Fourier transform and then used rigorously to derive the inverse transform. Details are given by Morse and Feshbach,[8] Section 4.8 (see also Exercise 15.6.3).

EXAMPLE 15.6.3

Integral by Parseval's Relation Evaluate the integral $\int_{-\infty}^{\infty}\frac{d\omega}{(a^2+\omega^2)^2}$. We start by recalling from Example 15.4.1 that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\omega x}d\omega}{a^2+\omega^2} = \frac{2}{\sqrt{\pi}}\int_0^{\infty}\frac{\cos\omega x\,d\omega}{a^2+\omega^2} = \sqrt{\frac{\pi}{2a^2}}e^{-ax}, \quad x > 0.$$

Next we apply Parseval's relation to

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\omega}{(a^2+\omega^2)^2} &= \frac{\pi}{2a^2}\int_{-\infty}^{\infty}e^{-2a|x|}dx = \frac{\pi}{a^2}\int_0^{\infty}e^{-2ax}dx \\ &= -\frac{\pi}{2a^3}e^{-2ax}\Big|_0^{\infty} = \frac{\pi}{2a^3}. \end{aligned}$$

■

[8] Morse, P. M., and Feshbach, H. (1953). *Methods of Theoretical Physics*. McGraw-Hill, New York.

EXERCISES

15.6.1 Work out the convolution equation corresponding to Eq. (15.57) for

(a) Fourier sine transforms

$$\frac{1}{2}\int_0^\infty g(y)[f(y+x)+f(y-x)]dy = \int_0^\infty F_s(t)G_s(t)\cos tx\,dt,$$

where f and g are odd functions.

(b) Fourier cosine transforms

$$\frac{1}{2}\int_0^\infty g(y)[f(y+x)+f(|x-y|)]dy = \int_0^\infty F_c(t)G_c(t)\cos tx\,dt,$$

where f and g are even functions.

15.6.2 Show that for both Fourier sine and Fourier cosine transforms Parseval's relation has the form

$$\int_0^\infty F(t)G(t)dt = \int_0^\infty f(y)g(y)dy.$$

15.6.3 Starting from Parseval's relation [Eq. (15.61)], let $g(y) = 1$, $0 \le y \le \alpha$, and zero elsewhere. From this derive the Fourier inverse transform [Eq. (15.28)].

Hint. Differentiate with respect to α.

15.6.4 Solve Poisson's equation $\nabla^2\psi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_0$ by the following sequence of operations:

(a) Take the Fourier transform of both sides of this equation. Solve for the Fourier transform of $\psi(\mathbf{r})$.

(b) Carry out the Fourier inverse transform by using a three-dimensional analog of the convolution theorem [Eq. (15.57)].

15.6.5 With $F(\omega)$ and $G(\omega)$ the Fourier transforms of $f(t)$ and $g(t)$, respectively, show that

$$\int_{-\infty}^\infty |f(t)-g(t)|^2\,dt = \int_{-\infty}^\infty |F(\omega)-G(\omega)|^2\,d\omega.$$

If $g(t)$ is an approximation to $f(t)$, the preceding relation indicates that the mean square deviation in ω-space is equal to the mean square deviation in t-space.

15.6.6 Use the Parseval relation to evaluate $\int_{-\infty}^\infty \frac{\omega^2 d\omega}{(\omega^2+a^2)^2}$.

Hint. Compare Example 15.4.2.

ANS. $\frac{\pi}{2a}$.

15.7 Momentum Representation

In advanced mechanics and in quantum mechanics, linear momentum and spatial position occur on an equal footing. In this section, we start with the usual space distribution and derive the corresponding momentum distribution. For the one-dimensional case, our wave function $\psi(x)$, a solution of the Schrödinger wave equation, has the following properties:

1. $\psi^*(x)\psi(x)dx$ is the probability of finding the quantum particle between x and $x + dx$ and
2. $$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1, \tag{15.64}$$
   corresponding to **one** particle (along the x-axis).
   In addition, we have
3. $$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x)x\psi(x)dx \tag{15.65}$$
   for the **average** position of the particle along the x-axis. This is often called an expectation value.

We want a function $g(p)$ that will give the same information about the momentum.

1. $g^*(p)g(p)dp$ is the probability that our quantum particle has a momentum between p and $p + dp$.
2. $$\int_{-\infty}^{\infty} g^*(p)g(p)dp = 1. \tag{15.66}$$
3. $$\langle p \rangle = \int_{-\infty}^{\infty} g^*(p)p\, g(p)dp. \tag{15.67}$$

As subsequently shown, such a function is given by the Fourier transform of our space function $\psi(x)$. Specifically,[9]

$$g(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)e^{-ipx/\hbar}dx \tag{15.68}$$

$$g^*(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi^*(x)e^{ipx/\hbar}dx. \tag{15.69}$$

The corresponding three-dimensional momentum function is

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint_{-\infty}^{\infty} \psi(\mathbf{r})e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar}d^3r.$$

To verify Eqs. (15.68) and (15.69), let us check properties 2 and 3.

[9] The $\hbar$ may be avoided by using the wave number k, $p = k\hbar$ (and $\mathbf{p} = \mathbf{k}\hbar$) so that

$$\varphi(k) = \frac{1}{(2\pi)^{1/2}} \int \psi(x)e^{-ikx}dx.$$

Property 2, the normalization, is automatically satisfied as a Parseval relation [Eq. (15.61)]. If the space function $\psi(x)$ is normalized to unity, the momentum function $g(p)$ is also normalized to unity.

To check property 3, we must show that

$$\langle p \rangle = \int_{-\infty}^{\infty} g^*(p) p g(p)\, dp = \int_{-\infty}^{\infty} \psi^*(x) \frac{\hbar}{i} \frac{d}{dx} \psi(x) dx, \tag{15.70}$$

where $(\hbar/i)(d/dx)$ is the momentum operator in the space representation. We replace the momentum functions by Fourier transformed space functions, and the first integral becomes

$$\frac{1}{2\pi\hbar} \iiint_{-\infty}^{\infty} p e^{-ip(x-x')/\hbar} \psi^*(x') \psi(x)\, dp\, dx'\, dx. \tag{15.71}$$

Now

$$p e^{-ip(x-x')/\hbar} = \frac{d}{dx}\left[-\frac{\hbar}{i} e^{-ip(x-x')/\hbar} \right]. \tag{15.72}$$

Substituting into Eq. (15.71) and integrating by parts, holding x' and p constant, we obtain

$$\langle p \rangle = \iint_{-\infty}^{\infty} \left[\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-ip(x-x')/\hbar} dp \right] \cdot \psi^*(x') \frac{\hbar}{i} \frac{d}{dx} \psi(x) dx'\, dx. \tag{15.73}$$

Here, we assume $\psi(x)$ vanishes as $x \to \pm\infty$, eliminating the integrated part. Again using the Dirac delta function [Eq. (15.23)], Eq. (15.73) reduces to Eq. (15.70) to verify our momentum representation. Note that technically we have employed the inverse Fourier transform in Eq. (15.68). This was chosen deliberately to yield the proper sign in Eq. (15.73).

EXAMPLE 15.7.1

Hydrogen Atom The hydrogen atom ground state[10] may be described by the spatial wave function

$$\psi(\mathbf{r}) = \left(\frac{1}{\pi a_0^3} \right)^{1/2} e^{-r/a_0}, \tag{15.74}$$

with a_0 being the Bohr radius, $\hbar^2/me^2$. We now have a three-dimensional wave function. The transform corresponding to Eq. (15.68) is

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3 r. \tag{15.75}$$

Substituting Eq. (15.74) into Eq. (15.75) and using

$$\int e^{-ar + i\mathbf{b}\cdot\mathbf{r}} d^3 r = \frac{8\pi a}{(a^2 + b^2)^2}, \tag{15.76}$$

[10] For a momentum representation treatment of the hydrogen atom, $l = 0$ states, see Ivash, E. V. (1972). A momentum representation treatment of the hydrogen atom problem. *Am. J. Phys.* **40**, 1095.

we obtain the hydrogenic momentum wave function

$$g(\mathbf{p}) = \frac{2^{3/2}}{\pi} \frac{a_0^{3/2}\hbar^{5/2}}{\left(a_0^2 p^2 + \hbar^2\right)^2}. \tag{15.77}$$

■

Such momentum functions have been found useful in problems such as Compton scattering from atomic electrons, the wavelength distribution of the scattered radiation, depending on the momentum distribution of the target electrons.

The relation between the ordinary space representation and the momentum representation may be clarified by considering the basic commutation relations of quantum mechanics. We go from a classical Hamiltonian to the Schrödinger equation by requiring that momentum p and position x **not** commute. Instead, we require that

$$[p, x] \equiv px - xp = -i\hbar. \tag{15.78}$$

For the multidimensional case, Eq. (15.78) is replaced by

$$[p_k, x_j] = -i\hbar\delta_{kj}. \tag{15.79}$$

The Schrödinger (space) representation is obtained by using

$$(x): \qquad p_k \to -i\hbar\frac{\partial}{\partial x_k},$$

replacing the momentum by a partial space derivative. We see that

$$[p, x]\psi(x) = -i\hbar\psi(x). \tag{15.80}$$

However, Eq. (15.78) can equally well be satisfied by using

$$(p): \qquad x_j \to i\hbar\frac{\partial}{\partial p_j}.$$

This is the momentum representation. Then

$$[p, x]g(p) = -i\hbar g(p). \tag{15.81}$$

Hence, the representation (x) is not unique; (p) is an alternate possibility.

In general, the Schrödinger representation (x) leading to the Schrödinger equation is more convenient because the potential energy V is generally given as a function of position $V(x, y, z)$. The momentum representation (p) usually leads to an integral equation. For an exception, consider the harmonic oscillator.

EXAMPLE 15.7.2

Simple Harmonic Oscillator The classical Hamiltonian (kinetic energy + potential energy = total energy) is

$$H(p, x) = \frac{p^2}{2m} + \frac{1}{2}kx^2 = E, \tag{15.82}$$

where k is Hooke's law constant.

In the Schrödinger representation we obtain

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{1}{2}kx^2\psi(x)=E\psi(x). \tag{15.83}$$

For total energy E equal to $\sqrt{(k/m)}\hbar/2$, there is a solution (Section 13.1)

$$\psi(x)=e^{-(\sqrt{mk}/(2\hbar))x^2}. \tag{15.84}$$

The momentum representation leads to

$$\frac{p^2}{2m}g(p)-\frac{\hbar^2 k}{2}\frac{d^2g(p)}{dp^2}=Eg(p). \tag{15.85}$$

Again, for

$$E=\sqrt{\frac{k}{m}}\frac{\hbar}{2} \tag{15.86}$$

the momentum wave equation (15.85) is satisfied by

$$g(p)=e^{-p^2/(2\hbar\sqrt{mk})}. \tag{15.87}$$

Either representation, space or momentum (and an infinite number of other possibilities), may be used, depending on which is more convenient for the particular problem under consideration.

The demonstration that $g(p)$ is the momentum wave function corresponding to Eq. (15.83)—that it is the Fourier inverse transform of Eq. (15.83)—is left as Exercise 15.7.3. ■

SUMMARY

Fourier integrals derive their importance from the momentum space representation in quantum mechanics. Fourier transformation of an ODE with constant coefficients leads to a polynomial, and that of a PDE with constant coefficients converts the PDE to an ODE.

EXERCISES

15.7.1 A linear quantum oscillator in its ground state has a wave function

$$\psi(x)=a^{-1/2}\pi^{-1/4}e^{-x^2/2a^2}.$$

Show that the corresponding momentum function is

$$g(p)=a^{1/2}\pi^{-1/4}\hbar^{-1/2}e^{-a^2p^2/2\hbar^2}.$$

15.7.2 The nth excited state of the linear quantum oscillator is described by

$$\psi_n(x)=a^{-1/2}2^{-n/2}\pi^{-1/4}(n!)^{-1/2}e^{-x^2/2a^2}H_n(x/a),$$

where $H_n(x/a)$ is the nth Hermite polynomial (Section 13.1). As an extension of Exercise 15.7.1, find the momentum function corresponding to $\psi_n(x)$.

Hint. $\psi_n(x)$ may be represented by $\mathcal{L}_+^n \psi_0(x)$, where $\mathcal{L}_+$ is the raising operator.

15.7.3 A free particle in quantum mechanics is described by a plane wave

$$\psi_k(x, t) = e^{i[kx-(\hbar k^2/2m)t]}.$$

Combining waves of adjacent momentum with an amplitude weighting factor $\varphi(k)$, we form a wave packet

$$\Psi(x, t) = \int_{-\infty}^{\infty} \varphi(k) e^{i[kx-(\hbar k^2/2m)t]} dk.$$

(a) Solve for $\varphi(k)$ given that

$$\Psi(x, 0) = e^{-x^2/2a^2}.$$

(b) Using the known value of $\varphi(k)$, integrate to get the explicit form of $\Psi(x, t)$. Note that this wave packet diffuses or spreads out with time.

$$ANS. \quad \Psi(x, t) = \frac{e^{-\{x^2/2[a^2+(i\hbar/m)t]\}}}{[1+(i\hbar t/ma^2)]^{1/2}}.$$

Note. An interesting discussion of this problem from the evolution operator point of view is given by S. M. Blinder, Evolution of a Gaussian wave-packet. *Am. J. Phys.* **36**, 525 (1968).

15.7.4 Find the time-dependent momentum wave function $g(k, t)$ corresponding to $\Psi(x, t)$ of Exercise 15.7.3. Show that the momentum wave packet $g^*(k, t)g(k, t)$ is **independent** of time.

15.7.5 The deuteron (Example 9.1.2) may be described reasonably well with a Hulthén wave function

$$\psi(\mathbf{r}) = A[e^{-\alpha r} - e^{-\beta r}]/r,$$

with A, α, and β constants. Find $g(\mathbf{p})$, the corresponding momentum wave function.
Note. The Fourier transform may be rewritten as Fourier sine and cosine transforms or as a Laplace transform (Section 15.8).

15.7.6 The nuclear form factor $F(k)$ and the charge distribution $\rho(r)$ are three-dimensional Fourier transforms of each other:

$$F(k) = \frac{1}{(2\pi)^{3/2}} \int \rho(r) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r.$$

If the measured form factor is

$$F(k) = (2\pi)^{-3/2}\left(1 + \frac{k^2}{a^2}\right)^{-1},$$

find the corresponding charge distribution.

$$ANS. \quad \rho(r) = \frac{a^2}{4\pi}\frac{e^{-ar}}{r}.$$

15.7.7 Check the normalization of the hydrogen momentum wave function

$$g(\mathbf{p}) = \frac{2^{3/2}}{\pi} \frac{a_0^{3/2}\hbar^{5/2}}{\left(a_0^2 p^2 + \hbar^2\right)^2}$$

by direct evaluation of the integral

$$\int g^*(\mathbf{p}) g(\mathbf{p}) d^3 p.$$

15.7.8 With $\psi(\mathbf{r})$ a wave function in ordinary space and $\varphi(\mathbf{p})$ the corresponding momentum function, show that

(a) $$\frac{1}{(2\pi\hbar)^{3/2}} \int \mathbf{r}\psi(\mathbf{r}) e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar} d^3 r = i\hbar \boldsymbol{\nabla}_p \varphi(\mathbf{p}),$$

(b) $$\frac{1}{(2\pi\hbar)^{3/2}} \int \mathbf{r}^2\psi(\mathbf{r}) e^{-i\mathbf{r}\cdot\mathbf{p}/\hbar} d^3 r = (i\hbar \boldsymbol{\nabla}_p)^2 \varphi(\mathbf{p}).$$

Note. $\boldsymbol{\nabla}_p$ is the gradient in momentum space:

$$\hat{\mathbf{x}}\frac{\partial}{\partial p_x} + \hat{\mathbf{y}}\frac{\partial}{\partial p_y} + \hat{\mathbf{z}}\frac{\partial}{\partial p_z}.$$

These results may be extended to any positive integer power of r and therefore to any (analytic) function that may be expanded as a Maclaurin series in r.

15.7.9 The ordinary space wave function $\psi(\mathbf{r}, t)$ satisfies the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 \psi + V(\mathbf{r})\psi.$$

Show that the corresponding time-dependent momentum wave function satisfies the analogous equation

$$i\hbar \frac{\partial \varphi(\mathbf{p}, t)}{\partial t} = \frac{p^2}{2m}\varphi + V(i\hbar\boldsymbol{\nabla}_p)\varphi.$$

Note. Assume that $V(\mathbf{r})$ may be expressed by a Maclaurin series and use Exercise 15.7.10. $V(i\hbar\boldsymbol{\nabla}_p)$ is the same function of the variable $i\hbar\boldsymbol{\nabla}_p$ as $V(\mathbf{r})$ is of the variable $\mathbf{r}$.

15.7.10 The one-dimensional, time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

For the special case of $V(x)$ an analytic function of x, show that the corresponding momentum wave equation is

$$V\left(i\hbar\frac{d}{dp}\right) g(p) + \frac{p^2}{2m} g(p) = E g(p).$$

Derive this momentum wave equation from the Fourier transform [Eq. (15.68)] and its inverse. Do not use the substitution $x \to i\hbar(d/dp)$ directly.

15.8 Laplace Transforms

Definition

The Laplace transform $f(s)$ or $\mathcal{L}$ of a function $F(t)$ is defined by[11]

$$f(s) = \mathcal{L}\{F(t)\} = \lim_{a\to\infty}\int_0^a e^{-st}F(t)dt = \int_0^\infty e^{-st}F(t)dt. \tag{15.88}$$

A few comments on the existence of the integral are in order. The infinite integral of $F(t)$,

$$\int_0^\infty F(t)dt,$$

need not exist. For instance, $F(t)$ may diverge exponentially for large t. However, if there is some constant such that

$$|e^{-s_0 t}F(t)| \le M, \tag{15.89}$$

a positive constant for sufficiently large t, $t > t_0$, the Laplace transform [Eq. (15.88)] will exist for $s > s_0$; $F(t)$ is said to be of exponential order. As a counterexample, $F(t) = e^{t^2}$ does not satisfy the condition given by Eq. (15.89) and is **not** of exponential order. $\mathcal{L}\{e^{t^2}\}$ does **not** exist.

The Laplace transform may also fail to exist because of a sufficiently strong singularity in the function $F(t)$ as $t \to 0$; that is,

$$\int_0^\infty e^{-st}t^n\,dt$$

diverges at the origin for $n \le -1$. The Laplace transform $\mathcal{L}\{t^n\}$ does not exist for $n \le -1$.

Since for two functions $F(t)$ and $G(t)$, for which the integrals exist,

$$\mathcal{L}\{aF(t) + bG(t)\} = a\mathcal{L}\{F(t)\} + b\mathcal{L}\{G(t)\}, \tag{15.90}$$

the operation denoted by $\mathcal{L}$ is **linear**.

EXAMPLE 15.8.1

Elementary Functions To illustrate the Laplace transform, let us apply the operation to some of the elementary functions. If

$$F(t) = 1, \quad t > 0,$$

then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \frac{1}{s}, \quad \text{for} \quad s > 0. \tag{15.91}$$

[11] This is sometimes called a one-sided Laplace transform; the integral from $-\infty$ to $+\infty$ is referred to as a two-sided Laplace transform. Some authors introduce an additional factor of s. This extra s appears to have little advantage and continually gets in the way (see Additional Reading, Jeffreys and Jeffreys, Section 14.13). Generally, we take s to be real and positive. It is possible to have s complex, provided $\Re(s) > 0$.

Again, let

$$F(t) = e^{kt}, \quad t > 0.$$

The Laplace transform becomes

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st} e^{kt} dt = \frac{1}{s-k}, \quad \text{for } s > k, \tag{15.92}$$

where the integral is finite. Using this relation, we obtain the Laplace transform of certain other functions. Since

$$\cosh kt = \frac{1}{2}(e^{kt} + e^{-kt}), \qquad \sinh kt = \frac{1}{2}(e^{kt} - e^{-kt}), \tag{15.93}$$

we have

$$\begin{aligned} \mathcal{L}\{\cosh kt\} &= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \frac{s}{s^2 - k^2}, \\ \mathcal{L}\{\sinh kt\} &= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{k}{s^2 - k^2}, \end{aligned} \tag{15.94}$$

both valid for $s > k$, where the integrals are finite. Because the results are analytic functions of s, they may be continued analytically over the complex s-plane. This will prove useful for the inverse Laplace transform in Section 15.12. We use the relations

$$\cos kt = \cosh ikt, \qquad \sin kt = -i \sinh ikt$$

in Eq. (15.94), with k replaced by ik, to find that the Laplace transforms

$$\begin{aligned} \mathcal{L}\{\cos kt\} &= \frac{s}{s^2 + k^2}, \\ \mathcal{L}\{\sin kt\} &= \frac{k}{s^2 + k^2}, \end{aligned} \tag{15.95}$$

both valid for $s > 0$, where the integrals are finite. Another derivation of this last transform is given in the next section. Note that $\lim_{s\to 0} \mathcal{L}\{\sin kt\} = 1/k$. This suggests we assign a value of $1/k$ to the Laplace transform $\lim_{s\to 0} \int_0^\infty e^{-st} \sin kt\, dt$.

Finally, for $F(t) = t^n$, we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt,$$

which is the factorial function. Hence,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0, n > -1. \tag{15.96}$$

Note that in all these transforms we have the variable s in the denominator-negative powers of s. In particular, $\lim_{s\to\infty} f(s) = 0$. The significance of this point is that if $f(s)$ involves positive powers of s, then $\lim_{s\to\infty} f(s) \to \infty$ and no inverse transform exists. ■

Inverse Transform

There is little importance to these operations, unless we can carry out the inverse transform as in Fourier transforms. That is, with

$$\mathcal{L}\{F(t)\} = f(s),$$

then

$$\mathcal{L}^{-1}\{f(s)\} = F(t).$$

Taken literally, this inverse transform is **not** unique. However, to the physicist and engineer the inverse operation may almost always be taken as unique in practical problems.

The inverse transform can be determined in various ways. A table of transforms can be built up and used to carry out the inverse transformation exactly as a table of logarithms can be used to look them up. The preceding transforms constitute the beginnings of such a table. For a more complete set of Laplace transforms, see AMS-55, Chapter 29. Employing partial fraction expansions and various operational theorems, which are considered in succeeding sections, facilitates use of the tables. There is some justification for suspecting that these tables are probably of more value in solving textbook exercises than in solving real-world problems.

A general technique for $\mathcal{L}^{-1}$ will be developed in Section 15.12 by using the calculus of residues. The difficulties and the possibilities of a numerical approach—numerical inversion—are considered at the end of this section.

Partial Fraction Expansion

Utilization of a table of transforms (or inverse transforms) is facilitated by expanding $f(s)$ in **partial fractions**.

Frequently, $f(s)$, our transform, occurs in the form $g(s)/h(s)$, where $g(s)$ and $h(s)$ are polynomials with no common factors, $g(s)$ being of lower degree than $h(s)$. If the factors of $h(s)$ are all linear and distinct, then by the theory of partial fractions, we may write

$$f(s) = \frac{c_1}{s-a_1} + \frac{c_2}{s-a_2} + \cdots + \frac{c_n}{s-a_n}, \tag{15.97}$$

where the c_i are independent of s. The a_i are the roots of $h(s)$. If any one of the roots (e.g., a_1) is multiple (occurring m times), then $f(s)$ has the form

$$f(s) = \frac{c_{1,m}}{(s-a_1)^m} + \frac{c_{1,m-1}}{(s-a_1)^{m-1}} + \cdots + \frac{c_{1,1}}{s-a_1} + \sum_{i=2}^{n} \frac{c_i}{s-a_i}. \tag{15.98}$$

Finally, if one of the factors is quadratic, (s^2+ps+q), the numerator, instead of being a simple constant, will have the form

$$\frac{as+b}{s^2+ps+q}.$$

There are various ways of determining the constants introduced. For instance, in Eq. (15.97) we may multiply through by $(s - a_i)$ and obtain

$$c_i = \lim_{s \to a_i} (s - a_i) f(s). \tag{15.99}$$

In elementary cases a direct solution is often the easiest.

EXAMPLE 15.8.2

Partial Fraction Expansion Let

$$f(s) = \frac{k^2}{s(s^2 + k^2)}. \tag{15.100}$$

We want to bring $f(s)$ to the form

$$f(s) = \frac{c}{s} + \frac{as + b}{s^2 + k^2}.$$

Putting the right side of this equation over a common denominator and equating like powers of s in the numerator, we obtain

$$\frac{k^2}{s(s^2 + k^2)} = \frac{c(s^2 + k^2) + s(as + b)}{s(s^2 + k^2)}, \tag{15.101}$$

$$c + a = 0, \quad s^2; \qquad b = 0, \quad s^1; \qquad ck^2 = k^2, \quad s^0.$$

Solving these $(s \neq 0)$, we have

$$c = 1, \quad b = 0, \quad a = -1,$$

giving

$$f(s) = \frac{1}{s} - \frac{s}{s^2 + k^2} \tag{15.102}$$

and

$$\mathcal{L}^{-1}\{f(s)\} = 1 - \cos kt \tag{15.103}$$

by Eqs. (15.91) and (15.95). ■

EXAMPLE 15.8.3

A Step Function As one application of Laplace transforms, consider the evaluation of

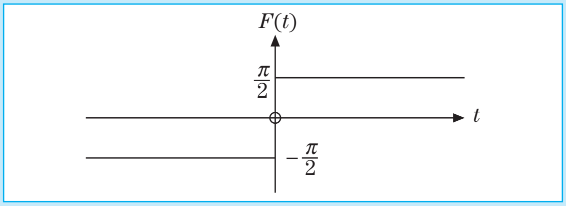
$$F(t) = \int_0^\infty \frac{\sin tx}{x} dx. \tag{15.104}$$

Suppose we take the Laplace transform of this definite integral, which is finite by virtue of the sign changes of the sine:

$$\mathcal{L}\left\{\int_0^\infty \frac{\sin tx}{x} dx\right\} = \int_0^\infty e^{-st} \int_0^\infty \frac{\sin tx}{x} dx\, dt. \tag{15.105}$$

Figure 15.9

$F(t) = \int_0^\infty \frac{\sin tx}{x} dx$, a Step Function



Now interchanging the order of integration (which is justified),[12] we get

$$\int_0^\infty \frac{1}{x}\left[\int_0^\infty e^{-st}\sin tx\,dt\right]dx = \int_0^\infty \frac{dx}{s^2+x^2} \qquad (15.106)$$

by integrating by parts as in Example 15.4.1. The factor in square brackets is the Laplace transform of $\sin tx$ from Eq. (15.95). Hence,

$$\int_0^\infty \frac{dx}{s^2+x^2} = \frac{1}{s}\tan^{-1}\left(\frac{x}{s}\right)\Big|_0^\infty = \frac{\pi}{2s} = f(s). \qquad (15.107)$$

By Eq. (15.91) we carry out the inverse transformation to obtain

$$F(t) = \frac{\pi}{2}, \quad t > 0, \qquad (15.108)$$

in agreement with an evaluation by the calculus of residues (Section 7.2). It has been assumed that $t > 0$ in $F(t)$. For $F(-t)$ we need note only that $\sin(-tx) = -\sin tx$, giving $F(-t) = -F(t)$. Finally, if $t = 0$, $F(0)$ is clearly zero. Therefore,

$$\int_0^\infty \frac{\sin tx}{x}dx = \frac{\pi}{2}[2u(t)-1] = \begin{cases} \frac{\pi}{2}, & t > 0 \\ 0, & t = 0 \\ -\frac{\pi}{2}, & t < 0, \end{cases} \qquad (15.109)$$

where $u(t)$ is the Heaviside unit step function of Example 15.3.1. Note that $\int_0^\infty(\sin tx/x)dx$, taken as a function of t, describes a step function (Fig. 15.9), a step of height π at $t = 0$. ■

The technique in the preceding example was to

- introduce a second integration—the Laplace transform;
- reverse the order of integration and integrate; and
- take the inverse Laplace transform.

There are many opportunities in which this technique of reversing the order of integration can be applied and proved very useful. Exercise 15.8.6 is a variation of this.

[12]See Jeffreys and Jeffreys (1966), Chapter 1 (uniform convergence of integrals).

Numerical Inversion

As an integration, the Laplace transform is a highly stable operation—stable in the sense that small fluctuations (or errors) in $F(t)$ are averaged out in the determination of the area under a curve. Also, the weighting factor, e^{-st}, means that the behavior of $F(t)$ at large t is effectively ignored—unless s is small. As a result of these two effects, a large change in $F(t)$ at large t indicates a very small, perhaps insignificant, change in $f(s)$. In contrast to the Laplace transform operation, going from $f(s)$ to $F(t)$ is highly unstable. A minor change in $f(s)$ may result in a wild variation of $F(t)$. All significant figures may disappear. In a matrix formulation, the matrix is ill conditioned with respect to inversion.

There is no general, completely satisfactory numerical method for inverting Laplace transforms. However, if we are willing to restrict attention to relatively smooth functions, various possibilities open up. Bellman, Kalaba, and Lockett[13] convert the Laplace transform to a Mellin transform ($x = e^{-t}$) and use numerical quadrature based on shifted Legendre polynomials, $P_n^*(x) = P_n(1-2x)$. The key step is analytic inversion of the resulting matrix. Krylov and Skoblya[14] focus on the evaluation of the Bromwich integral (Section 15.12). As one technique, they replace the integrand with an interpolating polynomial of negative powers and integrate analytically.

EXERCISES

15.8.1 Prove that

$$\lim_{s\to\infty} sf(s) = \lim_{t\to+0} F(t).$$

Hint. Assume that $F(t)$ can be expressed as $F(t) = \sum_{n=0}^{\infty} a_n t^n$.

15.8.2 Show that

$$\frac{1}{\pi}\lim_{s\to 0} \mathcal{L}\{\cos xt\} = \delta(x).$$

15.8.3 Verify that

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{b^2 - a^2}\right\} = \frac{s}{(s^2+a^2)(s^2+b^2)}, \qquad a^2 \neq b^2.$$

15.8.4 Using partial fraction expansions, show that

(a) $$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \frac{e^{-at} - e^{-bt}}{b-a}, \qquad a \neq b.$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s}{(s+a)(s+b)}\right\} = \frac{ae^{-at} - be^{-bt}}{a-b}, \qquad a \neq b.$$

[13] Bellman, R., Kalaba, R. E., and Lockett, J. A. (1966). *Numerical Inversion of the Laplace Transforms*. Elsevier, New York.

[14] Krylov, V. I., and Skoblya, N. S. (1969). *Handbook of Numerical Inversion of Laplace Transforms* (D. Louvish, Trans.). Israel Program for Scientific Translations, Jerusalem.

15.8.5 Using partial fraction expansions, show that for $a^2 \neq b^2$,

(a) $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = -\frac{1}{a^2-b^2}\left\{\frac{\sin at}{a} - \frac{\sin bt}{b}\right\},$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{a^2-b^2}\{a\sin at - b\sin bt\}.$$

15.8.6 The electrostatic potential of a charged conducting disk is known to have the general form (circular cylindrical coordinates)

$$\Phi(\rho, z) = \int_0^\infty e^{-k|z|} J_0(k\rho) f(k)\, dk,$$

with $f(k)$ unknown. At large distances ($z \to \infty$) the potential must approach the Coulomb potential $Q/4\pi\varepsilon_0 z$. Show that

$$\lim_{k\to 0} f(k) = \frac{q}{4\pi\varepsilon_0}.$$

Hint. You may set $\rho = 0$ and assume a Maclaurin expansion of $f(k)$ or, using e^{-kz}, construct a delta sequence.

15.8.7 A function $F(t)$ can be expanded in a power series (Maclaurin); that is,

$$F(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Then

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} \sum_{n=0}^{\infty} a_n t^n\, dt = \sum_{n=0}^{\infty} a_n \int_0^\infty e^{-st} t^n\, dt.$$

Show that $f(s)$, the Laplace transform of $F(t)$, contains no powers of s greater than s^{-1}. Check your result by calculating $\mathcal{L}\{\delta(t)\}$ and comment on this fiasco.

15.9 Laplace Transform of Derivatives

Perhaps the main application of Laplace transforms is in converting differential equations into simpler forms that may be solved more easily. It will be seen, for instance, that **coupled differential equations with constant coefficients transform to simultaneous linear algebraic equations**.

Let us transform the first derivative of $F(t)$:

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st}\frac{dF(t)}{dt}dt.$$

Integrating by parts, we obtain

$$\begin{aligned}\mathcal{L}\{F'(t)\} &= e^{-st}F(t)\Big|_0^\infty + s\int_0^\infty e^{-st}F(t)dt \\ &= s\mathcal{L}\{F(t)\} - F(0). \qquad (15.110)\end{aligned}$$

Strictly speaking, $F(0) = F(+0)$,[15] and dF/dt is required to be at least piecewise continuous for $0 \le t < \infty$. Naturally, both $F(t)$ and its derivative must be such that the integrals do not diverge. Incidentally, Eq. (15.110) provides another proof of Exercise 15.8.7. An extension gives

$$\mathcal{L}\{F^{(2)}(t)\} = s^2\mathcal{L}\{F(t)\} - sF(+0) - F'(+0), \tag{15.111}$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n\mathcal{L}\{F(t)\} - s^{n-1}F(+0) - \cdots - F^{(n-1)}(+0). \tag{15.112}$$

The Laplace transform, like the Fourier transform, replaces differentiation with multiplication. In the following examples, ODEs become algebraic equations. Here is the power and the utility of the Laplace transform. When the coefficients of the derivatives are not constant, Laplace transforms do not simplify the ODE, as a rule.

Note how the initial conditions, $F(+0)$, $F'(+0)$, and so on, are incorporated into the transform. Equation (15.111) may be used to derive $\mathcal{L}\{\sin kt\}$. We use the identity

$$-k^2 \sin kt = \frac{d^2}{dt^2}\sin kt. \tag{15.113}$$

Then, applying the Laplace transform operation, we have

$$\begin{aligned} -k^2\mathcal{L}\{\sin kt\} &= \mathcal{L}\left\{\frac{d^2}{dt^2}\sin kt\right\} \\ &= s^2\mathcal{L}\{\sin kt\} - s\sin(0) - \frac{d}{dt}\sin kt|_{t=0}. \end{aligned} \tag{15.114}$$

Since $\sin(0) = 0$ and $d/dt \sin kt|_{t=0} = k$,

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}, \tag{15.115}$$

verifying Eq. (15.95).

EXAMPLE 15.9.1

Classical Harmonic Oscillator As a physical example, consider a mass m oscillating under the influence of an ideal spring, spring constant k. Friction is neglected. Then Newton's second law becomes

$$m\frac{d^2X(t)}{dt^2} + kX(t) = 0. \tag{15.116}$$

The initial conditions are taken to be

$$X(0) = X_0, \qquad X'(0) = 0.$$

Applying the Laplace transform, we obtain

$$m\mathcal{L}\left\{\frac{d^2X}{dt^2}\right\} + k\mathcal{L}\{X(t)\} = 0, \tag{15.117}$$

[15] Zero is approached from the positive side.

and by use of Eq. (15.111) this becomes

$$ms^2x(s) - msX_0 + kx(s) = 0, \tag{15.118}$$

$$x(s) = X_0\frac{s}{s^2+\omega_0^2}, \quad \text{with } \omega_0^2 \equiv \frac{k}{m}. \tag{15.119}$$

From Eq. (15.95) this is seen to be the transform of $\cos\omega_0 t$, which gives

$$X(t) = X_0 \cos\omega_0 t, \tag{15.120}$$

as expected. ■

Dirac Delta Function

For use with differential equations, one further transform is helpful—the Dirac delta function:[16]

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-st_0}, \quad \text{for } t_0 \geq 0, \tag{15.121}$$

and for $t_0 = 0$,

$$\mathcal{L}\{\delta(t)\} = 1, \tag{15.122}$$

where, for Laplace transforms, $\delta(0)$ is interpreted as

$$\delta(0) = \lim_{t_0\to 0^+} \delta(t-t_0). \tag{15.123}$$

As an alternate method, $\delta(t)$ may be considered the limit as $\varepsilon \to 0$ of $F(t)$, where

$$F(t) = \begin{cases} 0, & t < 0, \\ \varepsilon^{-1}, & 0 < t < \varepsilon, \\ 0, & t > \varepsilon. \end{cases} \tag{15.124}$$

By direct calculation,

$$\mathcal{L}\{F(t)\} = \frac{1-e^{-\varepsilon s}}{\varepsilon s}. \tag{15.125}$$

Taking the limit of the integral (instead of the integral of the limit), we have

$$\lim_{\varepsilon\to 0}\mathcal{L}\{F(t)\} = 1$$

or Eq. (15.122)

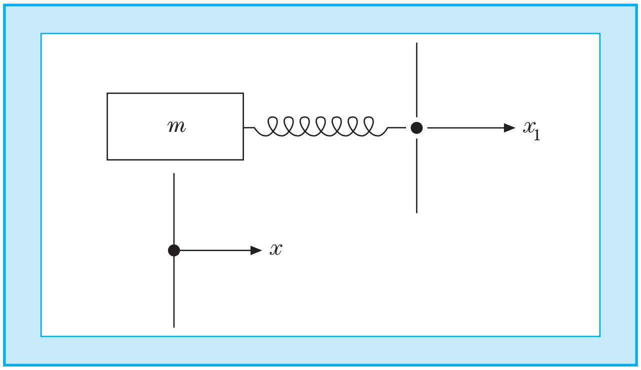
$$\mathcal{L}\{\delta(t)\} = 1.$$

This delta function is frequently called the impulse function because it is so useful in describing impulsive forces, that is, forces lasting only a short time.

[16]Strictly speaking, the Dirac delta function is undefined. However, the integral over it is well defined. This approach is developed in Section 1.14 using delta sequences.

Figure 15.10

Spring



EXAMPLE 15.9.2

Impulsive Force Newton's second law for impulsive force acting on a particle of mass m becomes

$$m\frac{d^2X}{dt^2} = P\delta(t), \tag{15.126}$$

where P is a constant. Transforming, we obtain

$$ms^2x(s) - msX(0) - mX'(0) = P. \tag{15.127}$$

For a particle starting from rest, $X'(0) = 0$.[17] We shall also take $X(0) = 0$. Then

$$x(s) = \frac{P}{ms^2}, \tag{15.128}$$

and

$$X(t) = \frac{P}{m}t, \tag{15.129}$$

$$\frac{dX(t)}{dt} = \frac{P}{m}, \quad \text{a constant.} \tag{15.130}$$

The effect of the impulse $P\delta(t)$ is to transfer (instantaneously) P units of linear momentum to the particle. ■

EXERCISES

15.9.1 Use the expression for the transform of a second derivative to obtain the transform of $\cos kt$.

15.9.2 A mass m is attached to one end of an unstretched spring, spring constant k (Fig. 15.10). At time $t = 0$ the free end of the spring experiences a constant acceleration, a, away from the mass. Using Laplace transforms,

[17] This really should be $X'(+0)$. To include the effect of the impulse, consider that the impulse will occur at $t = \varepsilon$ and let $\varepsilon \to 0$.

(a) find the position x of m as a function of time; and
(b) determine the limiting form of $x(t)$ for small t.

$$\text{ANS.} \quad (a)\ x = \frac{1}{2}at^2 - \frac{a}{\omega^2}(1 - \cos\omega t), \quad \omega^2 = \frac{k}{m},$$
$$(b)\ x = \frac{a\omega^2}{4!}t^4, \quad \omega t \ll 1.$$

15.10 Other Properties

Substitution

If we replace the parameter s by $s - a$ in the definition of the Laplace transform [Eq. (15.88)], we have

$$f(s-a) = \int_0^\infty e^{-(s-a)t}F(t)dt = \int_0^\infty e^{-st}e^{at}F(t)dt$$
$$= \mathcal{L}\{e^{at}F(t)\}. \tag{15.131}$$

Hence, the replacement of s with $s - a$ corresponds to multiplying $F(t)$ by e^{at} and conversely. This result can be used to good advantage in extending our table of transforms. From Eq. (15.95) we find immediately that

$$\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2 + k^2}; \tag{15.132}$$

also,

$$\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2 + k^2}, \quad s > a. \tag{15.133}$$

EXAMPLE 15.10.1

Damped Oscillator These expressions are useful when we consider an oscillating mass with damping proportional to the velocity. Equation (15.116), with such damping added, becomes

$$mX''(t) + bX'(t) + kX(t) = 0, \tag{15.134}$$

where b is a proportionality constant. Let us assume that the particle starts from rest at $X(0) = X_0$, $X'(0) = 0$. The transformed equation is

$$m[s^2x(s) - sX_0] + b[sx(s) - X_0] + kx(s) = 0 \tag{15.135}$$

so that

$$x(s) = X_0\frac{ms + b}{ms^2 + bs + k}. \tag{15.136}$$

This may be handled by completing the square of the denominator,

$$s^2 + \frac{b}{m}s + \frac{k}{m} = \left(s + \frac{b}{2m}\right)^2 + \left(\frac{k}{m} - \frac{b^2}{4m^2}\right). \tag{15.137}$$

If the damping is small, $b^2 < 4\,k\,m$, the last term is positive and will be denoted by ω_1^2. Splitting the numerator of $x(s)$ into $(s + b/2m) + b/2m$ gives

$$x(s) = X_0 \frac{s + b/m}{(s + b/2m)^2 + \omega_1^2}$$

$$= X_0 \frac{s + b/2m}{(s + b/2m)^2 + \omega_1^2} + X_0 \frac{(b/2m\omega_1)\omega_1}{(s + b/2m)^2 + \omega_1^2}. \qquad (15.138)$$

By Eqs. (15.132) and (15.133),

$$X(t) = X_0 e^{-(b/2m)t} \left(\cos \omega_1 t + \frac{b}{2m\omega_1} \sin \omega_1 t \right)$$

$$= X_0 \frac{\omega_0}{\omega_1} e^{-(b/2m)t} \cos(\omega_1 t - \varphi), \qquad (15.139)$$

where

$$\tan \varphi = \frac{b}{2m\omega_1}, \qquad \omega_0^2 = \frac{k}{m}. \qquad (15.140)$$

Of course, as $b \to 0$, this solution goes over to the undamped solution (Section 15.9). ■

RLC Analog

It is worth noting the similarity between this damped simple harmonic oscillation of a mass on a spring and a resistance, inductance, and capacitance (RLC) circuit (Fig. 15.11). At any instant the sum of the potential differences around the loop must be zero (Kirchhoff's law, conservation of energy). This gives

$$L \frac{dI}{dt} + RI + \frac{1}{C} \int_0^t I\, dt = 0. \qquad (15.141)$$

Differentiating the current I with respect to time (to eliminate the integral), we have

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0. \qquad (15.142)$$

Figure 15.11

RLC Circuit

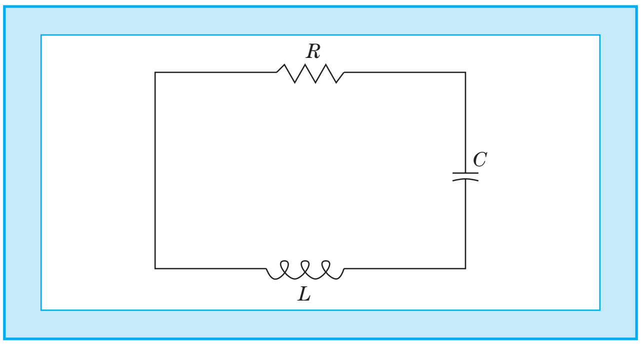
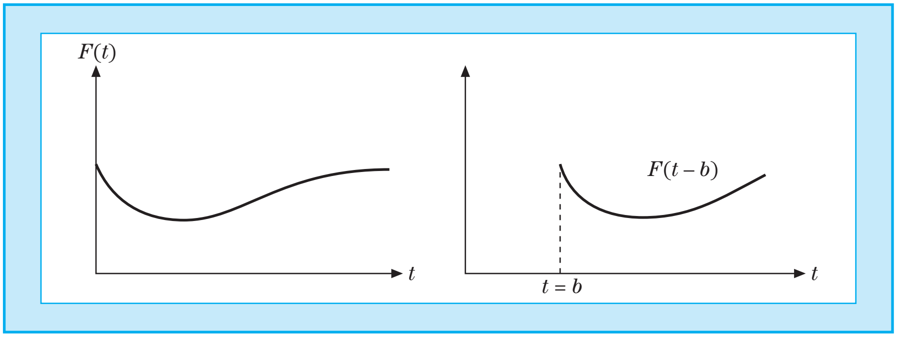


Figure 15.12

Translation



If we replace $I(t)$ with $X(t)$, L with m, R with b, and C^{-1} with k, Eq. (15.142) is identical to the mechanical problem. It is but one example of the unification of diverse branches of physics by mathematics. A more complete discussion is provided by Olson.[18]

Translation

This time, let $f(s)$ be multiplied by e^{-bs}, $b > 0$:

$$e^{-bs} f(s) = e^{-bs} \int_0^\infty e^{-st} F(t) dt$$
$$= \int_0^\infty e^{-s(t+b)} F(t) dt. \quad (15.143)$$

Now let $t + b = \tau$; then Eq. (15.143) becomes

$$e^{-bs} f(s) = \int_b^\infty e^{-s\tau} F(\tau - b) d\tau$$
$$= \int_0^\infty e^{-s\tau} F(\tau - b) d\tau, \quad (15.144)$$

if we assume that $F(t) = 0$ for $t < 0$, so that $F(\tau - b) = 0$ for $0 \le \tau < b$. In that case, we can extend the lower limit to zero without changing the value of the integral. This relation is often called the Heaviside shifting theorem (Fig. 15.12). We obtain

$$e^{-bs} f(s) = \mathcal{L}\{F(t - b)\}, \quad F(t) = 0, \ t < 0. \quad (15.145)$$

EXAMPLE 15.10.2

Electromagnetic Waves The electromagnetic wave equation with $E = E_y$ or E_z, a transverse wave propagating along the x-axis, is

$$\frac{\partial^2 E(x, t)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 E(x, t)}{\partial t^2} = 0. \quad (15.146)$$

[18] Olson, H. F. (1943). *Dynamical Analogies*. Van Nostrand, New York.

Transforming this equation with respect to t, we get

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\{E(x,t)\} - \frac{s^2}{v^2}\mathcal{L}\{E(x,t)\} + \frac{s}{v^2}E(x,0) + \frac{1}{v^2}\left.\frac{\partial E(x,t)}{\partial t}\right|_{t=0} = 0. \quad (15.147)$$

If we have the initial conditions $E(x, 0) = 0$ and

$$\left.\frac{\partial E(x,t)}{\partial t}\right|_{t=0} = 0,$$

then

$$\frac{\partial^2}{\partial x^2}\mathcal{L}\{E(x,t)\} = \frac{s^2}{v^2}\mathcal{L}\{E(x,t)\}. \quad (15.148)$$

The solution (of this **ODE**) is

$$\mathcal{L}\{E(x,t)\} = c_1 e^{-(s/v)x} + c_2 e^{+(s/v)x}. \quad (15.149)$$

The "constants" c_1 and c_2 are obtained by additional boundary conditions. They are constant with respect to x but may depend on s. If our wave remains finite as $x \to \infty$, $\mathcal{L}\{E(x,t)\}$ will also remain finite. Hence, $c_2 = 0$.

If $E(0, t)$ is denoted by $F(t)$, then $c_1 = f(s)$ and

$$\mathcal{L}\{E(x,t)\} = e^{-(s/v)x} f(s). \quad (15.150)$$

From the translation property [Eq. (15.144)] we find immediately that

$$E(x,t) = \begin{cases} F\left(t - \frac{x}{v}\right), & t \geq \frac{x}{v}, \\ 0, & t < \frac{x}{v}, \end{cases} \quad (15.151)$$

which is consistent with our initial condition. Differentiation and substitution into Eq. (15.146) verifies Eq. (15.151). Our solution represents a wave (or pulse) moving in the positive x-direction with velocity v. Note that for $x > vt$ the region remains undisturbed; the pulse has not had time to get there. The other independent solution has $c_1 = 0$ and corresponds to a signal propagated along the negative x-axis. ■

Derivative of a Transform

When $F(t)$, which is at least piecewise continuous, and s are chosen so that $e^{-st}F(t)$ converges exponentially for large s, the integral

$$f(s) = \int_0^\infty e^{-st} F(t)dt$$

is uniformly convergent and may be differentiated (under the integral sign) with respect to s. Then

$$f'(s) = \int_0^\infty (-t)e^{-st} F(t)dt = \mathcal{L}\{-tF(t)\}. \quad (15.152)$$

Continuing this process, we obtain

$$f^{(n)}(s) = \mathcal{L}\{(-t)^n F(t)\}. \tag{15.153}$$

All the integrals so obtained will be uniformly convergent because of the decreasing exponential behavior of $e^{-st}F(t)$.

This same technique may be applied to generate more transforms. For example,

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st}e^{kt}dt = \frac{1}{s-k}, \quad s > k. \tag{15.154}$$

Differentiating with respect to s (or with respect to k), we obtain

$$\mathcal{L}\{te^{kt}\} = \frac{1}{(s-k)^2}, \quad s > k. \tag{15.155}$$

Integration of Transforms

Again, with $F(t)$ at least piecewise continuous and x large enough, so that $e^{-xt}F(t)$ decreases exponentially (as $x \to \infty$), the integral

$$f(x) = \int_0^\infty e^{-xt}F(t)dt \tag{15.156}$$

is uniformly convergent with respect to x. This justifies reversing the order of integration in the following equation:

$$\begin{aligned}\int_s^b f(x)dx &= \int_s^b \int_0^\infty e^{-xt}F(t)dt\,dx \\ &= \int_0^\infty \frac{F(t)}{t}(e^{-st} - e^{-bt})dt,\end{aligned} \tag{15.157}$$

on integrating with respect to x. The lower limit s is chosen large enough so that $f(s)$ is within the region of uniform convergence. Now letting $b \to \infty$, we have

$$\int_s^\infty f(x)dx = \int_0^\infty \frac{F(t)}{t}e^{-st}dt = \mathcal{L}\left\{\frac{F(t)}{t}\right\}, \tag{15.158}$$

provided that $F(t)/t$ is finite at $t = 0$ or diverges less strongly than t^{-1} [so that $\mathcal{L}\{F(t)/t\}$ will exist].

Limits of Integration—Unit Step Function

The actual limits of integration for the Laplace transform may be specified with the (Heaviside) unit step function

$$u(t-k) = \begin{cases} 0, & t < k \\ 1, & t \geq k. \end{cases}$$

For instance,

$$\mathcal{L}\{u(t-k)\} = \int_k^\infty e^{-st} dt = \frac{1}{s}e^{-ks}.$$

A rectangular pulse of width k and unit height is described by $F(t) = u(t) - u(t-k)$. Taking the Laplace transform, we obtain

$$\mathcal{L}\{u(t) - u(t-k)\} = \int_0^k e^{-st}\, dt = \frac{1}{s}(1 - e^{-ks}).$$

The unit step function is implicit in Eq. (15.144) and could also be invoked in Exercise 15.10.13.

EXERCISES

15.10.1 Solve Eq. (15.134), which describes a damped simple harmonic oscillator for $X(0) = X_0$, $X'(0) = 0$, and
(a) $b^2 = 4\,\mathrm{k\,m}$ (critically damped),
(b) $b^2 > 4\,\mathrm{k\,m}$ (overdamped).

ANS. (a) $X(t) = X_0 e^{-(b/2m)t}\left(1 + \frac{b}{2m}t\right)$.

15.10.2 Solve Eq. (15.134), which describes a damped simple harmonic oscillator for $X(0) = 0$, $X'(0) = v_0$, and
(a) $b^2 < 4\,\mathrm{k\,m}$ (underdamped),
(b) $b^2 = 4\,\mathrm{k\,m}$ (critically damped),

ANS. (a) $X(t) = \frac{v_0}{\omega_1} e^{-(b/2m)t} \sin\omega_1 t$,
(b) $X(t) = v_0 t e^{-(b/2m)t}$.

(c) $b^2 > 4\,\mathrm{k\,m}$ (overdamped).

15.10.3 The motion of a body falling in a resisting medium may be described by

$$m\frac{d^2X(t)}{dt^2} = mg - b\frac{dX(t)}{dt}$$

when the retarding force is proportional to the velocity. Find $X(t)$ and $dX(t)/dt$ for the initial conditions

$$X(0) = \left.\frac{dX}{dt}\right|_{t=0} = 0.$$

15.10.4 With $J_0(t)$ expressed as a contour integral, apply the Laplace transform operation, reverse the order of integration, and thus show that

$$\mathcal{L}\{J_0(t)\} = (s^2 + 1)^{-1/2}, \quad \text{for } s > 0.$$

15.10.5 Develop the Laplace transform of $J_n(t)$ from $\mathcal{L}\{J_0(t)\}$ by using the Bessel function recurrence relations.
Hint. Here is a chance to use mathematical induction.

15.10.6 A calculation of the magnetic field of a circular current loop in circular cylindrical coordinates leads to the integral

$$\int_0^\infty e^{-kz} k J_1(ka)dk, \quad \Re(z) \geq 0.$$

Show that this integral is equal to $a/(z^2 + a^2)^{3/2}$.

15.10.7 Verify the following Laplace transforms:

(a) $\mathcal{L}\{j_0(at)\} = \mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\cot^{-1}\left(\frac{s}{a}\right),$

(b) $\mathcal{L}\{y_0(at)\}$ does not exist.

15.10.8 Develop a Laplace transform solution of Laguerre's ODE

$$tF''(t) + (1 - t)F'(t) + nF(t) = 0.$$

Note that you need a derivative of a transform and a transform of derivatives. Go as far as you can with n; then (and only then) set $n = 0$.

15.10.9 Show that the Laplace transform of the Laguerre polynomial $L_n(at)$ is given by

$$\mathcal{L}\left\{L_n(at)\right\} = \frac{(s-a)^n}{s^{n+1}}, \quad s > 0.$$

15.10.10 Show that

$$\mathcal{L}\left\{E_1(t)\right\} = \frac{1}{s}\ln(s+1), \quad s > 0,$$

where

$$E_1(t) = \int_t^\infty \frac{e^{-\tau}}{\tau}d\tau = \int_1^\infty \frac{e^{-xt}}{x}dx.$$

$E_1(t)$ is the exponential–integral function.

15.10.11 (a) From Eq. (15.158) show that

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{F(t)}{t}dt,$$

provided the integrals exist.

(b) From the preceding result show that

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2},$$

in agreement with Eqs. (15.109) and (7.41).

15.10.12 (a) Show that

$$\mathcal{L}\left\{\frac{\sin kt}{t}\right\} = \cot^{-1}\left(\frac{s}{k}\right).$$

(b) Using this result (with $k = 1$), prove that

$$\mathcal{L}\{\mathrm{si}(t)\} = -\frac{1}{s}\tan^{-1} s,$$

where

$$\mathrm{si}(t) = -\int_t^\infty \frac{\sin x}{x}\,dx, \qquad \text{the sine integral.}$$

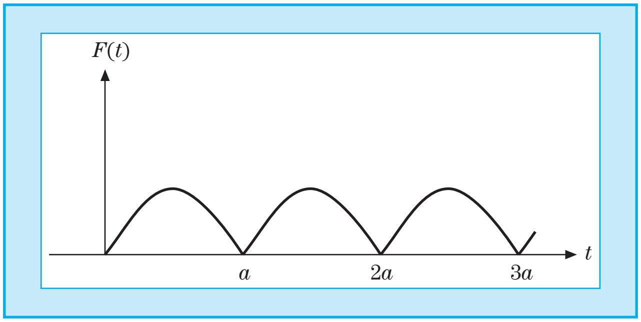
15.10.13 If $F(t)$ is periodic (Fig. 15.13) with a period a so that $F(t+a) = F(t)$ for all $t \geq 0$, show that

$$\mathcal{L}\left\{F(t)\right\} = \frac{\int_0^a e^{-st}F(t)\,dt}{1 - e^{-as}},$$

with the integration now over only the **first period** of $F(t)$.

Figure 15.13

Periodic Function



15.10.14 Find the Laplace transform of the square wave (period a) defined by

$$F(t) = \begin{cases} 1, & 0 < t < a/2 \\ 0, & a/2 < t < a. \end{cases}$$

ANS. $f(s) = \dfrac{1}{s} \cdot \dfrac{1 - e^{-as/2}}{1 - e^{-as}}$.

15.10.15 Show that

(a) $\mathcal{L}\{\cosh at \cos at\} = \dfrac{s^3}{s^4 + 4a^4}$, (c) $\mathcal{L}\{\sinh at \cos at\} = \dfrac{as^2 - 2a^3}{s^4 + 4a^4}$,

(b) $\mathcal{L}\{\cosh at \sin at\} = \dfrac{as^2 + 2a^3}{s^4 + 4a^4}$, (d) $\mathcal{L}\{\sinh at \sin at\} = \dfrac{2a^2 s}{s^4 + 4a^4}$.

15.10.16 Show that

(a) $\mathcal{L}^{-1}\{(s^2 + a^2)^{-2}\} = \dfrac{1}{2a^3}\sin at - \dfrac{1}{2a^2} t\cos at$,

(b) $\mathcal{L}^{-1}\{s(s^2 + a^2)^{-2}\} = \dfrac{1}{2a} t\sin at$,

(c) $\mathcal{L}^{-1}\{s^2(s^2 + a^2)^{-2}\} = \dfrac{1}{2a}\sin at + \dfrac{1}{2} t\cos at$,

(d) $\mathcal{L}^{-1}\{s^3(s^2 + a^2)^{-2}\} = \cos at - \dfrac{a}{2} t\sin at$.

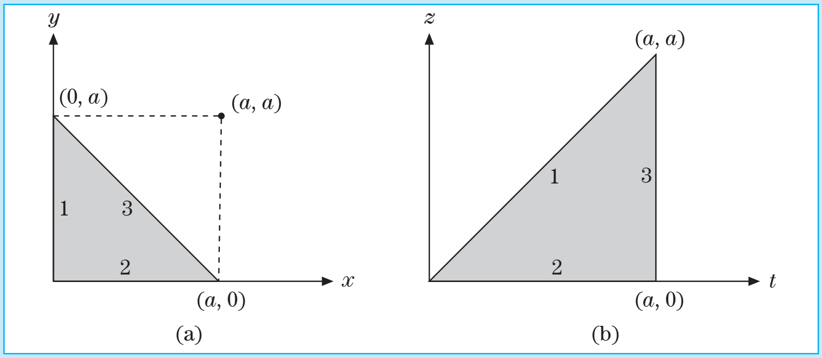


Figure 15.14

Change of Variables: (a) *xy*-Plane and (b) *zt*-Plane

15.11 Convolution or Faltungs Theorem

One of the most important properties of the Laplace transform is that given by the convolution or Faltungs theorem.[19] We take two transforms

$$f_1(s) = \mathcal{L}\{F_1(t)\} \quad \text{and} \quad f_2(s) = \mathcal{L}\{F_2(t)\} \tag{15.159}$$

and multiply them together. To avoid complications, when changing variables, we hold the upper limits finite:

$$f_1(s) \cdot f_2(s) = \lim_{a\to\infty} \int_0^a e^{-sx} F_1(x)dx \int_0^{a-x} e^{-sy} F_2(y)dy. \tag{15.160}$$

The upper limits are chosen so that the area of integration, shown in Fig. 15.14a, is the shaded triangle, not the square. Substituting $x = t - z$, $y = z$, the region of integration is mapped into the triangle, shown in Fig. 15.14b. If we integrate over a square in the xy-plane, we have a parallelogram in the tz-plane, which simply adds complications. This modification is permissible because the two integrands are assumed to decrease exponentially. In the limit $a \to \infty$, the integral over the unshaded triangle will give zero contribution. To verify the mapping, map the vertices: $t = x + y$, $z = y$. Using Jacobians to transform the element of area (Chapter 2), we have

$$dx\,dy = \begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \\ \dfrac{\partial x}{\partial z} & \dfrac{\partial y}{\partial z} \end{vmatrix} dt\,dz = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} dt\,dz \tag{15.161}$$

[19] An alternate derivation employs the Bromwich integral (Section 15.12). This is Exercise 15.12.3.

or $dx\,dy = dt\,dz$. With this substitution, Eq. (15.160) becomes

$$f_1(s) \cdot f_2(s) = \lim_{a\to\infty} \int_0^a e^{-st} \int_0^t F_1(t-z)F_2(z)dz\,dt$$
$$= \mathcal{L}\left\{ \int_0^t F_1(t-z)F_2(z)dz \right\}. \tag{15.162}$$

For convenience this integral is represented by the symbol

$$\int_0^t F_1(t-z)F_2(z)dz \equiv F_1 * F_2, \tag{15.163}$$

and referred to as the **convolution**, closely analogous to the Fourier convolution (Section 15.6). If we interchange $f_1 \to f_2$ and $F_1 \to F_2$ (or replace $z \to t - z$), we find

$$F_1 * F_2 = F_2 * F_1, \tag{15.164}$$

showing that the relation is symmetric.

Carrying out the inverse transform, we also find

$$\mathcal{L}^{-1}\{f_1(s) \cdot f_2(s)\} = \int_0^t F_1(t-z)F_2(z)dz. \tag{15.165}$$

This can be useful in the development of new transforms or as an alternative to a partial fraction expansion. One immediate application is in the solution of integral equations. Since the upper limit t is variable, this Laplace convolution is useful in treating Volterra integral equations. The Fourier convolution with fixed (infinite) limits would apply to Fredholm integral equations.

EXAMPLE 15.11.1

Driven Oscillator with Damping As an illustration of the use of the convolution theorem, let us return to the mass m on a spring, with damping and a driving force $F(t)$. The equation of motion [Eq. (15.134)] now becomes

$$mX''(t) + bX'(t) + kX(t) = F(t). \tag{15.166}$$

Initial conditions $X(0) = 0$, $X'(0) = 0$ are used to simplify this illustration, and the transformed equation is

$$ms^2x(s) + bs\,x(s) + kx(s) = f(s) \tag{15.167}$$

or

$$x(s) = \frac{f(s)}{m} \cdot \frac{1}{(s + b/2m)^2 + \omega_1^2}, \tag{15.168}$$

where $\omega_1^2 \equiv k/m - b^2/4m^2$, as before.

By the convolution theorem [Eq. (15.160) or Eq. (15.165)], and noting that

$$\frac{\omega_1}{(s + b/2m)^2 + \omega_1^2} = \mathcal{L}(e^{-bt/2m} \sin \omega_1 t),$$

we have

$$X(t) = \frac{1}{m\omega_1} \int_0^t F(t - z) e^{-(b/2m)z} \sin \omega_1 z \, dz. \tag{15.169}$$

If the force is impulsive, $F(t) = P\delta(t)$,[20]

$$X(t) = \frac{P}{m\omega_1} e^{-(b/2m)t} \sin \omega_1 t. \tag{15.170}$$

P represents the momentum transferred by the impulse, and the constant P/m takes the place of an initial velocity $X'(0)$.

If $F(t) = F_0 \sin \omega t$, Eq. (15.168) may be used, but a partial fraction expansion is perhaps more convenient. With

$$f(s) = \frac{F_0 \omega}{s^2 + \omega^2},$$

Eq. (15.168) becomes

$$\begin{aligned} x(s) &= \frac{F_0 \omega}{m} \cdot \frac{1}{s^2 + \omega^2} \cdot \frac{1}{(s + b/2m)^2 + \omega_1^2} \\ &= \frac{F_0 \omega}{m} \left[\frac{a's + b'}{s^2 + \omega^2} + \frac{c's + d'}{(s + b/2m)^2 + \omega_1^2} \right]. \end{aligned} \tag{15.171}$$

The coefficients a', b', c', and d' are independent of s. Direct calculation shows

$$\begin{aligned} -\frac{1}{a'} &= \frac{b}{m}\omega^2 + \frac{m}{b}\left(\omega_0^2 - \omega^2\right)^2, \\ -\frac{1}{b'} &= -\frac{m}{b}\left(\omega_0^2 - \omega^2\right)\left[\frac{b}{m}\omega^2 + \frac{m}{b}\left(\omega_0^2 - \omega^2\right)^2\right]. \end{aligned}$$

Since the $c's + d'$ term will lead to exponentially decreasing terms (transients, as shown above the denominator is the Laplace transform of $e^{-bt/2m} \sin \omega_1 t$), they will be discarded here. Carrying out the inverse operation, we find for the steady-state solution

$$X(t) = \frac{F_0}{\left[b^2\omega^2 + m^2\left(\omega_0^2 - \omega^2\right)^2\right]^{1/2}} \sin(\omega t - \varphi), \tag{15.172}$$

where

$$\tan \varphi = \frac{b\omega}{m\left(\omega_0^2 - \omega^2\right)}.$$

Differentiating the denominator, we find that the amplitude has a maximum when

$$\omega^2 = \omega_0^2 - \frac{b^2}{2m^2} = \omega_1^2 - \frac{b^2}{4m^2}. \tag{15.173}$$

[20] Note that $\delta(t)$ lies **inside** the interval $[0, t]$.

This is the resonance condition.[21] At resonance the amplitude becomes $F_0/b\omega_1$, showing that the mass m goes into infinite oscillation at resonance if damping is neglected ($b = 0$). It is worth noting that we have had three different characteristic frequencies: resonance for forced oscillations, with damping,

$$\omega_2^2 = \omega_0^2 - \frac{b^2}{2m^2};$$

free oscillation frequency, with damping,

$$\omega_1^2 = \omega_0^2 - \frac{b^2}{4m^2};$$

and free oscillation frequency, no damping,

$$\omega_0^2 = \frac{k}{m}.$$

They coincide only if the damping is zero ($b = 0$).

Returning to Eqs. (15.166) and (15.167), Eq. (15.167) is our ODE for the response of a dynamical system to an arbitrary driving force. The final response clearly depends on both the driving force and the characteristics of our system. This dual dependence is separated in the transform space. In Eq. (15.168) the transform of the response (output) appears as the product of two factors, one describing the driving force (input) and the other describing the dynamical system. This latter part, which modifies the input and yields the output, is often called a **transfer function**. Specifically, $[(s + b/2m)^2 + \omega_1^2]^{-1}$ is the transfer function corresponding to this damped oscillator. The concept of a transfer function is of great use in the field of servomechanisms. Often, the characteristics of a particular servomechanism are described by giving its transfer function. The convolution theorem then yields the output signal for a particular input signal. ■

EXERCISES

15.11.1 From the convolution theorem, show that

$$\frac{1}{s} f(s) = \mathcal{L}\left\{\int_0^t F(x)dx\right\},$$

where $f(s) = \mathcal{L}\{F(t)\}$.

15.11.2 Using the convolution integral, calculate

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + a^2)(s^2 + b^2)}\right\}, \quad a^2 \neq b^2.$$

15.11.3 An undamped oscillator is driven by a force $F_0 \sin \omega t$. Find the displacement as a function of time. Notice that it is a linear combination of two simple harmonic motions, one with the frequency of the driving

[21] The amplitude (squared) has the typical resonance denominator, the Lorentz line shape.

force and one with the frequency ω_0 of the free oscillator. (Assume $X(0) = X'(0) = 0$.)

$$ANS.\quad X(t) = \frac{F_0/m}{\omega^2 - \omega_0^2}\left(\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t\right).$$

15.12 Inverse Laplace Transform

Bromwich Integral

We now develop an expression for the inverse Laplace transform, $\mathcal{L}^{-1}$, appearing in the equation

$$F(t) = \mathcal{L}^{-1}\{f(s)\}. \tag{15.174}$$

One approach lies in the Fourier transform for which we know the inverse relation. There is a difficulty, however. Our Fourier transformable function had to satisfy the Dirichlet conditions. In particular, we required that

$$\lim_{\omega\to\infty} G(\omega) = 0 \tag{15.175}$$

so that the infinite integral would be well defined.[22] Now we wish to treat functions, $F(t)$, that may diverge exponentially. To surmount this difficulty, we extract an exponential factor, $e^{\gamma t}$, from our (possibly) divergent Laplace function and write

$$F(t) = e^{\gamma t}G(t). \tag{15.176}$$

If $F(t)$ diverges as $e^{\alpha t}$, we require γ to be greater than α so **that** $G(t)$ **will be convergent**. Now, with $G(t) = 0$ for $t < 0$ and otherwise suitably restricted so that it may be represented by a Fourier transform [Eq. (15.20)],

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iut}du \int_0^{\infty} G(v)e^{-iuv}dv. \tag{15.177}$$

Using Eq. (15.177), we may rewrite Eq. (15.176) as

$$F(t) = \frac{e^{\gamma t}}{2\pi}\int_{-\infty}^{\infty} e^{iut}du \int_0^{\infty} F(v)e^{-\gamma v}e^{-iuv}dv. \tag{15.178}$$

Now with the change of variable,

$$s = \gamma + iu, \tag{15.179}$$

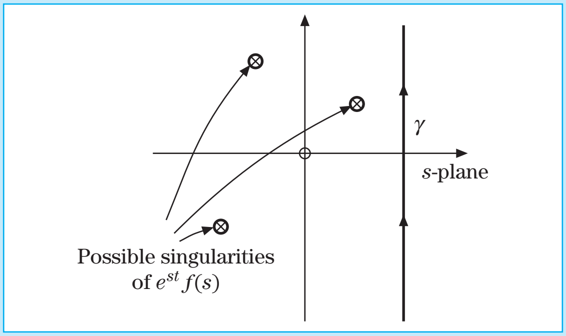
the integral over v is cast into the form of a Laplace transform

$$\int_0^{\infty} F(v)e^{-sv}dv = f(s); \tag{15.180}$$

[22] If delta functions are included, $G(\omega)$ may be a cosine. Although this does not satisfy Eq. (15.175), $G(\omega)$ is still bounded.

Figure 15.15

Singularities of $e^{st} f(s)$



s is now a complex variable and $\Re(s) \geq \gamma$ to guarantee convergence. Notice that the Laplace transform has mapped a function specified on the positive real axis onto the complex plane, $\Re(s) \geq \gamma$.[23]

Because γ is a constant, $ds = idu$. Substituting Eq. (15.180) into Eq. (15.178), we obtain

$$F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s) ds. \qquad (15.181)$$

Here is our **inverse transform**. We have rotated the line of integration through 90° (by using $ds = idu$). The path has become an infinite vertical line in the complex plane, the constant γ having been chosen so that all the singularities of $f(s)$ are to the left of the line $\gamma + is$ (Fig. 15.15).

Equation (15.181), our inverse transformation, is usually known as the Bromwich integral, although sometimes it is referred to as the Fourier–Mellin theorem or Fourier–Mellin integral. This integral may now be evaluated by the regular methods of contour integration (Chapter 7), if there are no branch cuts—that is, f is a single-valued analytic function. If $t > 0$, the contour may be closed by an infinite semicircle in the left half-plane, provided the integral over this semicircle is negligible. Then, by the residue theorem (Section 7.2),

$$F(t) = \Sigma(\text{residues included for } \Re(s) < \gamma). \qquad (15.182)$$

Possibly, this means of evaluation, with $\Re(s)$ ranging through negative values, seems paradoxical in view of our previous requirement that $\Re(s) \geq \gamma$. The paradox disappears when we recall that the requirement $\Re(s) \geq \gamma$ was imposed to guarantee convergence of the Laplace transform integral that defined $f(s)$. Once $f(s)$ is obtained, we may proceed to exploit its properties as

[23]For a derivation of the inverse Laplace transform using only real variables, see C. L. Bohn and R. W. Flynn, Real variable inversion of Laplace transforms: An application in plasma physics. *Am. J. Phys.* **46**, 1250 (1978).

an analytical function in the complex plane wherever we choose.[24] In effect, we are employing analytic continuation to get $\mathcal{L}\{F(t)\}$ in the left half-plane exactly as the recurrence relation for the factorial function was used to extend the Euler integral definition [Eq. (10.5)] to the left half-plane.

Perhaps two examples may clarify the evaluation of Eq. (15.182).

EXAMPLE 15.12.1

Inversion via Calculus of Residues If $f(s) = a/(s^2 - a^2)$, then

$$e^{st} f(s) = \frac{ae^{st}}{s^2 - a^2} = \frac{ae^{st}}{(s+a)(s-a)}. \tag{15.183}$$

The residues may be found by using Exercise 7.1.1 or various other means. The first step is to identify the singularities, the poles. Here, we have one simple pole at $s = a$ and another simple pole at $s = -a$. By Exercise 7.1.1, the residue at $s = a$ is $(\frac{1}{2})e^{at}$ and the residue at $s = -a$ is $(-\frac{1}{2})e^{-at}$. Then

$$\text{Residues} = \left(\frac{1}{2}\right)(e^{at} - e^{-at}) = \sinh at = F(t), \tag{15.184}$$

in agreement with Eq. (15.94). ■

EXAMPLE 15.12.2

Another Inversion If

$$f(s) = \frac{1 - e^{-as}}{s}, \quad a > 0,$$

then $e^{s(t-a)}$ grows exponentially for $t < a$ on the semicircle in the left-hand s-plane so that contour integration and the residue theorem are not applicable. However, we can evaluate the integral explicitly as follows. We let $\gamma \to 0$ and substitute $s = iy$ so that

$$F(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [e^{iyt} - e^{it(t-a)}]\frac{dy}{y}. \tag{15.185}$$

Using the Euler identity, only the sines survive that are odd in y, and we obtain

$$F(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin ty}{y} - \frac{\sin(t-a)y}{y}\right]. \tag{15.186}$$

If $k > 0$, $\int_0^\infty \frac{\sin ky}{y} dy$ gives $\pi/2$ and $-\pi/2$ if $k < 0$. As a consequence, $F(t) = 0$ if $t > a > 0$ and if $t < 0$. If $0 < t < a$, then $F(t) = 1$. This can be written compactly in terms of the Heaviside unit step function $u(t)$ as follows:

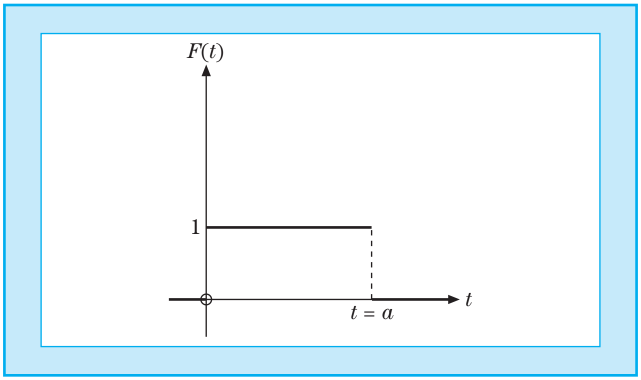
$$F(t) = u(t) - u(t-a) = \begin{cases} 0, & t < 0, \\ 1, & 0 < t < a, \\ 0, & t > a. \end{cases} \tag{15.187}$$

Thus, $F(t)$ is a step function of unit height and length a (Fig. 15.16). ■

[24] In numerical work, $f(s)$ may well be available only for discrete real, positive values of s. Then numerical procedures are indicated. See Section 15.8 and the Krylov and Skoblya reference in Additional Reading.

Figure 15.16

Finite-Length Step Function $u(t) - u(t-a)$



Two general comments are in order. First, these two examples hardly begin to show the usefulness and power of the Bromwich integral. It is always available for inverting a complicated transform, when the tables prove inadequate.

Second, this derivation is not presented as a rigorous one. Rather, it is given more as a plausibility argument, although it can be made rigorous. The determination of the inverse transform is similar to the solution of a differential equation. It makes little difference how you get the solution. Guess at it if you want. The solution can always be checked by substitution back into the original differential equation. Similarly, $F(t)$ can (and, to check for careless errors, should) be checked by determining whether by Eq. (15.88)

$$\mathcal{L}\{F(t)\} = f(s).$$

Two alternate derivations of the Bromwich integral are the subjects of Exercises 15.12.1 and 15.12.2.

As a final illustration of the use of the Laplace inverse transform, we discuss some results from the work of Brillouin and Sommerfeld (1914) in electromagnetic theory.

EXAMPLE 15.12.3

Velocity of Electromagnetic Waves in a Dispersive Medium The **group velocity** u of traveling waves is related to the phase velocity v by the equation

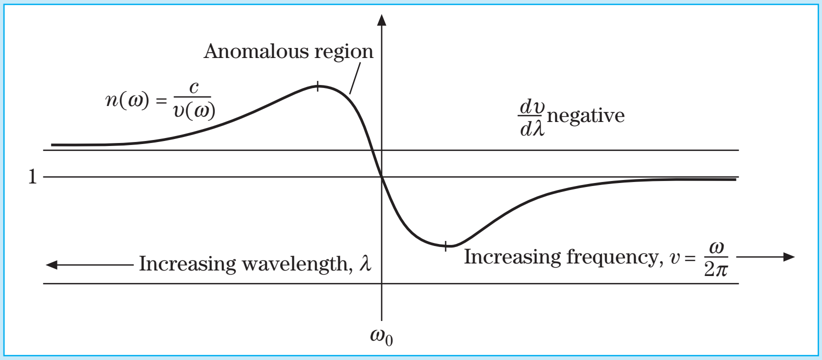
$$u = v - \lambda \frac{dv}{d\lambda}, \tag{15.188}$$

where λ is the wavelength. In the vicinity of an absorption line (resonance) $dv/d\lambda$ may be sufficiently negative so that $u > c$ (Fig. 15.17). The question immediately arises whether a signal can be transmitted faster than c, the velocity of light in vacuum. This question, which assumes that such a group velocity is meaningful, is of fundamental importance to the theory of special relativity. We need a solution to the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}, \tag{15.189}$$

Figure 15.17

Optical Dispersion



corresponding to a harmonic vibration starting at the origin at time zero. Our medium is **dispersive**, meaning that v is a function of the angular frequency. Imagine, for instance, a plane wave, angular frequency ω, incident on a shutter at the origin. At $t = 0$ the shutter is (instantaneously) opened, and the wave is permitted to advance along the positive x-axis.

Let us then build up a solution starting at $x = 0$. It is convenient to use the Cauchy integral formula [Eq. (6.37)],

$$\psi(0, t) = \frac{1}{2\pi i}\oint \frac{e^{-izt}}{z - z_0} dz = e^{-iz_0 t}$$

(for a contour encircling $z = z_0$ in the positive sense). Using $s = -iz$ and $z_0 = \omega$, we obtain

$$\psi(0, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s + i\omega} ds = \begin{cases} 0, & t < 0, \\ e^{-i\omega t}, & t > 0. \end{cases} \tag{15.190}$$

To be complete, the loop integral is along the vertical line $\Re(s) = \gamma$ **and** an infinite semicircle as shown in Fig. 15.18. The location of the infinite semicircle is chosen so that the integral over it vanishes. This means a semicircle in the left half-plane for $t > 0$ and the residue is enclosed. For $t < 0$ we choose the right half-plane and no singularity is enclosed. The fact that this is the Bromwich integral may be verified by noting that

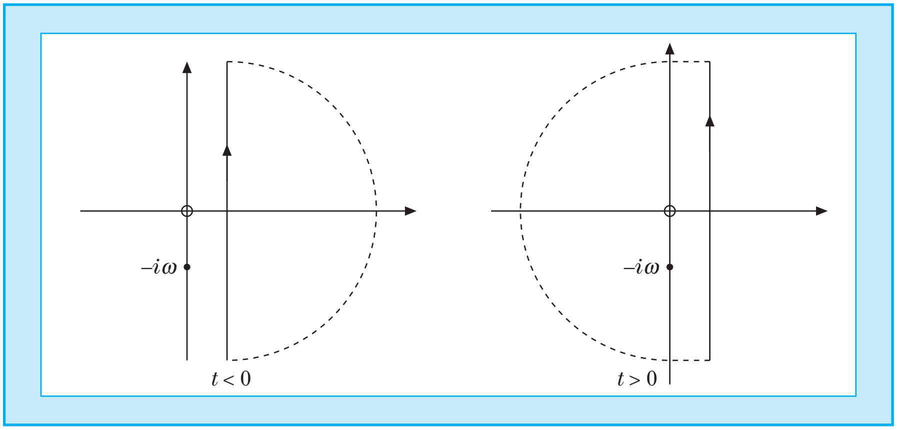
$$F(t) = \begin{cases} 0, & t < 0, \\ e^{-i\omega t}, & t > 0 \end{cases} \tag{15.191}$$

and applying the Laplace transform. The transformed function $f(s)$ becomes

$$f(s) = \frac{1}{s + i\omega}. \tag{15.192}$$

Figure 15.18

Possible Closed Contours



Our Cauchy–Bromwich integral provides us with the time dependence of a signal leaving the origin at $t = 0$. To include the space dependence, we note that

$$e^{s(t-x/v)}$$

satisfies the wave equation. With this as a clue, we replace t by $t - x/v$ and write a solution

$$\psi(x, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{s(t-x/v)}}{s + i\omega} ds. \tag{15.193}$$

It was seen in the derivation of the Bromwich integral that our variable s replaces the ω of the Fourier transformation. Hence, the wave velocity v becomes a function of s, that is, $v(s)$. Its particular form need not concern us here. We need only the property

$$\lim_{|s| \to \infty} v(s) = c, \text{ constant.} \tag{15.194}$$

This is suggested by the asymptotic behavior of the curve on the right side of Fig. 15.17.[25]

Evaluating Eq. (15.193) by the calculus of residues, we may close the path of integration by a semicircle in the right half-plane, provided

$$t - \frac{x}{c} < 0.$$

Hence,

$$\psi(x, t) = 0, \qquad t - \frac{x}{c} < 0, \tag{15.195}$$

[25]Equation (15.193) follows rigorously from the theory of anomalous dispersion and the Kronig–Kramers optical dispersion relations.

Table 15.1 Laplace Transform Operations

	Operations	Equation No.
Laplace transform	$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$	(15.88)
Transform of derivative	$sf(s) - F(+0) = \mathcal{L}\{F'(t)\}$	(15.110)
	$s^2 f(s) - sF(+0) - F'(+0) = \mathcal{L}\{F''(t)\}$	(15.111)
Transform of integral	$\frac{1}{s} f(s) = \mathcal{L}\left\{ \int_0^t F(x) dx \right\}$	Exercise 15.11.1
Substitution	$f(s-a) = \mathcal{L}\{e^{at} F(t)\}$	(15.131)
Translation	$e^{-bs} f(s) = \mathcal{L}\{F(t-b)u(t-b)\}$	(15.145)
Derivative of transform	$f^{(n)}(s) = \mathcal{L}\{(-t)^n F(t)\}$	(15.153)
Integral of transform	$\int_s^\infty f(x)\, dx = \mathcal{L}\left\{ \frac{F(t)}{t} \right\}$	(15.158)
Convolution	$f_1(s) f_2(s) = \mathcal{L}\left\{ \int_0^t F_1(t-z) F_2(z)\, dz \right\}$	(15.162)
Inverse transform, Bromwich integral	$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s) ds = F(t)$	(15.181)

which means that the velocity of our signal cannot exceed the velocity of light in vacuum c. This simple but very significant result was extended by Sommerfeld and Brillouin to show just how the wave advanced in the dispersive medium. ■

Summary: Inversion of Laplace Transform

- Direct use of tables and references; use of partial fractions (Section 15.8) and the operational theorems of Table 15.1.
- Bromwich integral [Eq. (15.181)] and the calculus of residues.
- Numerical inversion (Section 15.8) and references.

EXERCISES

15.12.1 Derive the Bromwich integral from Cauchy's integral formula.
Hint. Apply the inverse transform $\mathcal{L}^{-1}$ to

$$f(s) = \frac{1}{2\pi i} \lim_{\alpha \to \infty} \int_{\gamma - i\alpha}^{\gamma + i\alpha} \frac{f(z)}{s - z} dz,$$

where $f(z)$ is analytic for $\Re(z) \geq \gamma$.

15.12.2 Starting with

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s) ds,$$

show that by introducing

$$f(s) = \int_0^\infty e^{-sz} F(z) dz,$$

we can convert one integral into the Fourier representation of a Dirac delta function. From this, derive the inverse Laplace transform.

15.12.3 Derive the Laplace transformation convolution theorem by use of the Bromwich integral.

15.12.4 Find

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

(a) by a partial fraction expansion,
(b) repeat using the Bromwich integral.

15.12.5 Find

$$\mathcal{L}^{-1}\left\{\frac{k^2}{s(s^2+k^2)}\right\}$$

(a) by using a partial fraction expansion,
(b) repeat using the convolution theorem,
(c) repeat using the Bromwich integral.

ANS. $F(t) = 1 - \cos kt.$

15.12.6 Use the Bromwich integral to find the function whose transform is $f(s) = s^{-1/2}$. Note that $f(s)$ has a branch point at $s = 0$. The negative x-axis may be taken as a cut line.

ANS. $F(t) = (\pi t)^{-1/2}.$

15.12.7 Evaluate the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{(s^2-a^2)^{-1/2}\right\}$$

by each of the following methods:
(a) Expansion in a series and term-by-term inversion.
(b) Direct evaluation of the Bromwich integral.
(c) Change of variable in the Bromwich integral: $s = (a/2)(z + z^{-1})$.

15.12.8 Show that

$$\mathcal{L}^{-1}\left\{\frac{\ln s}{s}\right\} = -\ln t - \gamma,$$

where $\gamma = 0.5772\ldots$, the Euler–Mascheroni constant.

15.12.9 Evaluate the Bromwich integral for

$$f(s) = \frac{s}{(s^2+a^2)^2}.$$

15.12.10 Heaviside expansion theorem. If the transform $f(s)$ may be written as a ratio

$$f(s) = \frac{g(s)}{h(s)},$$

where $g(s)$ and $h(s)$ are analytic functions, with $h(s)$ having simple, isolated zeros at $s = s_i$, show that

$$F(t) = \mathcal{L}^{-1}\left\{\frac{g(s)}{h(s)}\right\} = \sum_i \frac{g(s_i)}{h'(s_i)} e^{s_i t}.$$

Hint. See Exercise 7.1.2.

15.12.11 Using the Bromwich integral, invert $f(s) = s^{-2}e^{-ks}$. Express $F(t) = \mathcal{L}^{-1}\{f(s)\}$ in terms of the (shifted) unit step function $u(t-k)$.

ANS. $F(t) = (t-k)u(t-k)$.

15.12.12 You have the following Laplace transform:

$$f(s) = \frac{1}{(s+a)(s+b)}, \quad a \neq b.$$

Invert this transform by each of three methods:
(a) partial fractions and use of tables;
(b) convolution theorem; and
(c) Bromwich integral.

ANS. $F(t) = \dfrac{e^{-bt} - e^{-at}}{a-b}, \quad a \neq b.$

Additional Reading

Champeney, D. C. (1973). *Fourier Transforms and Their Physical Applications*. Academic Press, New York. Fourier transforms are developed in a careful, easy to follow manner. Approximately 60% of the book is devoted to applications of interest in physics and engineering.

Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1954). *Tables of Integral Transforms*. McGraw-Hill, New York. This text contains extensive tables of Fourier sine, cosine, and exponential transforms, Laplace and inverse Laplace transforms, Mellin and inverse Mellin transforms, Hankel transforms, and other more specialized integral transforms.

Hanna, J. R. (1990). *Fourier Series and Integrals of Boundary Value Problems*. Wiley, Somerset, NJ. This book is a broad treatment of the Fourier solution of boundary value problems. The concepts of convergence and completeness are given careful attention.

Jeffreys, H., and Jeffreys, B. S. (1972). *Methods of Mathematical Physics*, 3rd ed. Cambridge Univ. Press, Cambridge, UK.

Krylov, V. I., and Skoblya, N. S. (1969). *Handbook of Numerical Inversion of Laplace Transform*. Israel Program for Scientific Translations, Jerusalem.

Lepage, W. R. (1961). *Complex Variables and the Laplace Transform for Engineers*. McGraw-Hill, New York. Reprinted, Dover, New York (1980). A complex variable analysis which is carefully developed and then applied to Fourier and Laplace transforms. It is written to be read by students, but it is intended for the serious student.

McCollum, P. A., and Brown, B. F. (1965). *Laplace Transform Tables and Theorems*. Holt, Rinehart and Winston, New York.

Miles, J. W. (1971). *Integral Transforms in Applied Mathematics*. Cambridge Univ. Press, Cambridge, UK. This is a brief but interesting and useful treatment for the advanced undergraduate. It emphasizes applications rather than abstract mathematical theory.

Papoulis, A. (1962). *The Fourier Integral and Its Applications*. McGraw-Hill, New York. This is a rigorous development of Fourier and Laplace transforms and has extensive applications in science and engineering.

Roberts, G. E., and Kaufman, H. (1966). *Table of Laplace Transforms*. Saunders, Philadelphia.

Sneddon, I. N. (1951). *Fourier Transforms*. McGraw-Hill, New York. Reprinted, Dover, New York (1995). A detailed comprehensive treatment, this book is loaded with applications to a wide variety of fields of modern and classical physics.

Sneddon, I. H. (1972). *The Use of Integral Transforms*. McGraw-Hill, New York. Written for students in science and engineering in terms they can understand, this book covers all the integral transforms mentioned in this chapter as well as in several others. Many applications are included.

Van der Pol, B., and Bremmer, H. (1987). *Operational Calculus Based on the Two-Sided Laplace Integral*, 3rd ed. Cambridge Univ. Press, Cambridge, UK. Here is a development based on the integral range $-\infty$ to $+\infty$, rather than the useful 0 to ∞. Chapter 5 contains a detailed study of the Dirac delta function (impulse function).

Wolf, K. B. (1979). *Integral Transforms in Science and Engineering*. Plenum, New York. This book is a very comprehensive treatment of integral transforms and their applications.

Titchmarsh, E. C. (1937). *Introduction to the Theory of Fourier Integrals*, 2nd ed. Oxford Univ. Press, New York.