Synoptic Meteorology II: Introduction to Isentropic Potential Vorticity

7-9 April 2015

Readings: Sections 4.1, 4.2, and 4.3.1 of *Midlatitude Synoptic Meteorology*.

Introduction

We start by casting the horizontal momentum equation, neglecting friction and presented without derivation, into isentropic coordinates:

$$\frac{D\vec{\mathbf{v}}}{Dt} + f\hat{\mathbf{k}} \times \vec{\mathbf{v}} = -\nabla_\theta M \tag{1}$$

In (1), the subscript of θ implies that the gradient operator is applied on an isentropic surface. M is the Montgomery streamfunction.

On an isentropic surface, the total derivative in (1) takes the form:

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + \vec{\mathbf{v}} \cdot \nabla_\theta(\ ) + \frac{d\theta}{dt}\frac{\partial(\ )}{\partial\theta} \tag{2}$$

where $d\theta/dt = \dot{\theta}$ is the diabatic heating rate, which we referred to as dQ/dt earlier this semester. This term states that cross-isentropic flow occurs only when there is local diabatic heating ($d\theta/dt$ non-zero) contributing to potential temperature not being conserved following the motion. This implies vertical motion *between* isentropic surfaces. (Note, as we did in the last lecture, that vertical motion in isentropic coordinates can also be *along* an isentropic surface, or dry adiabatic in nature.)

Note the similarity of (1) to its form in isobaric coordinates:

$$\frac{D\vec{\mathbf{v}}}{Dt} + f\hat{\mathbf{k}} \times \vec{\mathbf{v}} = -\nabla_p \Phi \tag{3}$$

In our last lecture, we demonstrated how the Montgomery streamfunction on an isentropic surface is an analog to the geopotential height on an isobaric surface. Therefore, it makes sense that the term on the right-hand side of the horizontal momentum equation in isentropic coordinates would be formulated based upon the Montgomery streamfunction rather than the geopotential height.

If we expand the total derivative in (1) by making use of (2), we obtain:

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + \vec{\mathbf{v}} \cdot \nabla_\theta \vec{\mathbf{v}} + \dot{\theta}\frac{\partial\vec{\mathbf{v}}}{\partial\theta} = -f\hat{\mathbf{k}} \times \vec{\mathbf{v}} - \nabla_\theta M \tag{4}$$

We now wish to obtain the vorticity equation applicable on isentropic surfaces. Recall that the vorticity equation on an isobaric surface was obtained by taking $\hat{\mathbf{k}}\cdot\nabla\times$ of the horizontal momentum equation. In component form, this is equivalent to finding $\partial/\partial x$ of the v-momentum equation and subtracting $\partial/\partial y$ of the u-momentum equation from it. Doing so, we obtain:

$$\frac{\partial\zeta}{\partial t}+\vec{\mathbf{v}}\cdot\nabla_{\theta}(\zeta+f)+\dot{\theta}\frac{\partial\zeta}{\partial\theta}+(\zeta+f)\nabla_{\theta}\cdot\vec{\mathbf{v}}=\hat{\mathbf{k}}\cdot\frac{\partial\vec{\mathbf{v}}}{\partial\theta}\times\nabla_{\theta}\dot{\theta} \tag{5}$$

If we assume that potential temperature is conserved (purely dry adiabatic flow), then the diabatic heating rate $\dot{\theta}$ is zero. Thus, the third term on the left-hand side of (5) and the only term on the right-hand side of (5) are both zero. Simplifying,

$$\frac{\partial\zeta}{\partial t}+\vec{\mathbf{v}}\cdot\nabla_{\theta}(\zeta+f)+(\zeta+f)\nabla_{\theta}\cdot\vec{\mathbf{v}}=0 \tag{6}$$

The first term of (6) can alternately be written in terms of $\zeta+f$ because f does not change locally with time. As a result,

$$\frac{\partial(\zeta+f)}{\partial t}+\vec{\mathbf{v}}\cdot\nabla_{\theta}(\zeta+f)+(\zeta+f)\nabla_{\theta}\cdot\vec{\mathbf{v}}=0 \tag{7}$$

We can write the continuity equation in isentropic coordinates as:

$$\frac{\partial}{\partial t}\left(\frac{\partial\theta}{\partial p}\right)-\frac{\partial\theta}{\partial p}\nabla_{\theta}\cdot\vec{\mathbf{v}}+\vec{\mathbf{v}}\cdot\nabla_{\theta}\left(\frac{\partial\theta}{\partial p}\right)=0 \tag{8}$$

We now wish to combine (7) and (8) in such as way so as to eliminate the divergence term $\nabla_{\theta}\cdot\vec{\mathbf{v}}$ from the system of equations. To do so, we multiply (7) by $-g\partial\theta/\partial p$ and add to it (8) multiplied by $-g(\zeta+f)$. Doing so, we obtain:

$$g\left[\left(-\frac{\partial\theta}{\partial p}\right)\frac{\partial(\zeta+f)}{\partial t}+(\zeta+f)\frac{\partial}{\partial t}\left(-\frac{\partial\theta}{\partial p}\right)+(\zeta+f)\vec{\mathbf{v}}\cdot\nabla_{\theta}\left(-\frac{\partial\theta}{\partial p}\right)+\left(-\frac{\partial\theta}{\partial p}\right)\vec{\mathbf{v}}\cdot\nabla_{\theta}(\zeta+f)\right]=0 \tag{9}$$

The "inverse" of the chain rule for partial derivatives may be used to combine the first two terms of (9) as well as the last two terms of (9). This allows us to write:

$$g\left[\frac{\partial}{\partial t}\left((\zeta+f)\left(-\frac{\partial\theta}{\partial p}\right)\right)+\vec{\mathbf{v}}\cdot\nabla_{\theta}\left((\zeta+f)\left(-\frac{\partial\theta}{\partial p}\right)\right)\right]=0 \tag{10}$$

We define P, the *isentropic potential vorticity*, as:

$$P = -g\eta \frac{\partial \theta}{\partial p} \qquad (11)$$

where $\eta = \zeta + f$, the absolute vorticity. Because g is a constant, we can take it into the partial derivatives of (10), allowing us to substitute (11) into (10) and obtain:

$$\frac{\partial P}{\partial t} + \vec{\mathbf{v}} \cdot \nabla_{\theta} P = 0 \Rightarrow \frac{DP}{Dt} = 0 \qquad (12)$$

<u>The Conservation of Potential Vorticity</u>

Equation (12) states that the isentropic potential vorticity is *conserved* following the motion along an isentropic surface (i.e., under dry adiabatic conditions). Given that we neglected friction when we stated the horizontal momentum equation in isentropic coordinates, isentropic potential vorticity is also conserved only under frictionless conditions (i.e., not near the surface). The non-conservation of isentropic potential vorticity following the motion on an isentropic surface thus allows us to infer where diabatic heating is occurring and/or where friction is important, both potentially useful pieces of information.

The isentropic potential vorticity P is a multiplicative function of two factors:

- Absolute vorticity η – a rotational constraint
- Static stability $-\partial\theta/\partial p$ – vertical isentrope packing

Isentropic potential vorticity is a large positive value when cyclonic rotation is strong ($\eta > 0$) and/or where static stability is large, representing isentropes that are tightly packed in the vertical ($-\partial\theta/\partial p >> 0$). ***Normally, $-\partial\theta/\partial p > 0$, such that $P < 0$ only occurs where $\eta < 0$***. Localized maxima in isentropic potential vorticity are known as *positive potential vorticity anomalies*, whereas localized minima in isentropic potential vorticity are known as *negative potential vorticity anomalies*.

Because isentropic potential vorticity is conserved (does not change) following the flow, if static stability or absolute vorticity change in value, the other must change in the inverse in order to keep the value of the isentropic potential vorticity constant.

- If the static stability increases, the absolute vorticity must decrease.
- If the static stability decreases, the absolute vorticity must increase.

Both of these statements hold true in the inverse as well; changes in absolute vorticity must be accompanied by changes in the static stability in order to keep the value of the isentropic potential vorticity constant. We can demonstrate this graphically using a vertical column of absolute vorticity between two isentropic surfaces, as depicted in Figure 1.

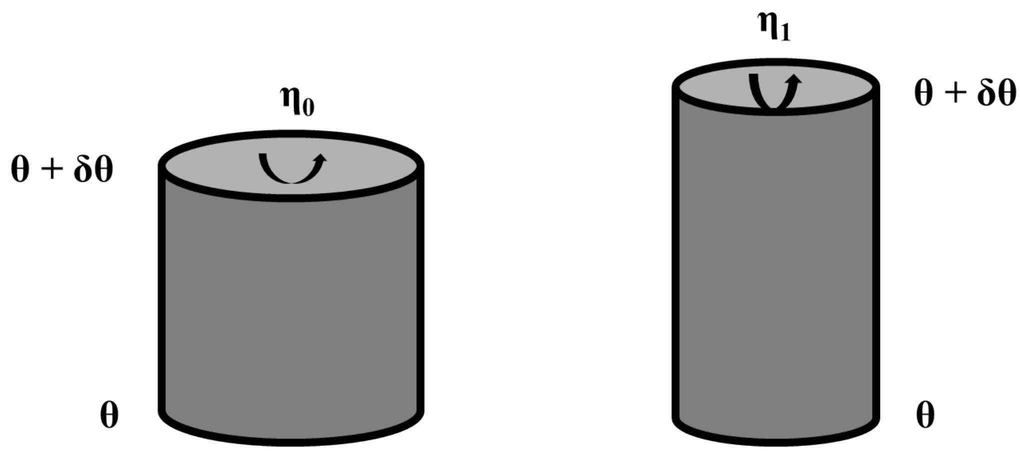

Figure 1. Schematic of a column of absolute vorticity between two isentropic surfaces at an initial (left) and a subsequent (right) time after the vorticity column has been stretched vertically.

The rate of rotation at the initial time, as measured by the absolute vorticity, is given by η_0. The two isentropic surfaces, represented by θ and $\theta + \partial\theta$, are separated by a known increment $\partial\theta$. Likewise, the difference in pressure between these two surfaces, ∂p is also known. If this column of vorticity is stretched vertically, the absolute vorticity increases. This can be shown using the vorticity equation, whether in its full or quasi-geostrophic form.

The new absolute vorticity is given by η_1, where $\eta_1 > \eta_0$. The separation between the two isentropic surfaces, under dry adiabatic conditions, remains $\partial\theta$. However, by virtue of stretching the column vertically, the difference in pressure between these two isentropic surfaces has increased. Thus, since ∂p is larger, $\partial\theta/\partial p$ is smaller. Thus, as absolute vorticity increased, the static stability decreased to conserve isentropic potential vorticity.

On the synoptic-scale, isentropic potential vorticity anomalies evolve through a combination of translation (motion/advection), rotation, and deformation by the synoptic-scale wind field. For these processes, isentropic potential vorticity is conserved following the motion. However, friction and/or diabatic processes can also impact the isentropic potential vorticity. When these processes are important, isentropic potential vorticity is *not* conserved following the motion. We will examine this concept in greater detail in a later lecture.

Typical Values of Isentropic Potential Vorticity and the Dynamic Tropopause

The units of P are relatively complex:

$$P = \left(\frac{m}{s^2}\right)\left(\frac{1}{s}\right)\left(\frac{K}{Pa}\right) = \left(\frac{m}{s^2}\right)\left(\frac{1}{s}\right)\left(\frac{K\ m\ s^2}{kg}\right) = m^2\ K\ s^{-1}\ kg^{-1}$$

For typical mid-latitude, synoptic-scale flow, $g \approx 10$ m s^{-2}, $\eta \approx f \approx 1 \times 10^{-4}$ s^{-1}, and $\partial\theta/\partial p \approx -10$ K per 100 hPa (-10 K per 10000 Pa). It should be noted, however, that the actual value of the static stability is typically less than this scale value except in the presence of strong temperature inversions. If we plug these values into (11), we obtain a characteristic value of $P = 1 \times 10^{-6}$ m^2 K s^{-1} kg^{-1}. For simplicity, we term this value to be equal to 1 PVU, where PVU stands for "potential vorticity unit."

In the troposphere, P is typically less than or equal to 1.5 PVU. Exceptions are typically confined to smaller-scale phenomena associated with strong cyclonic rotation, such as tropical cyclones. In the stratosphere, where the static stability is very large as potential temperature rapidly increases with height, P is typically greater than 2.0 PVU. The tropopause represents the transition region between the lower tropospheric values and higher stratospheric values of isentropic potential vorticity.

This gives rise to the construct of the *dynamic tropopause*, which is most commonly represented by the 1.5 PVU or 2.0 PVU surface. The dynamic tropopause is the representation of the tropopause by a surface of constant potential vorticity. Note that as the isentropic potential vorticity is conserved following the flow on an isentropic surface, so too is potential temperature conserved following the flow on a surface of constant isentropic potential vorticity. Consequently, where potential temperature changes following the flow on the dynamic tropopause, one can infer that diabatic processes (e.g., latent heat release) are ongoing and (potentially) important to the evolution of the synoptic-scale pattern.

Since potential temperature generally increases with height, analyses of potential temperature on the dynamic tropopause can be used to infer the relative height of the tropopause. Where potential temperature is relatively warm on the dynamic tropopause, the tropopause itself is at a relatively high altitude, inferring an upper tropospheric ridge. Conversely, where potential temperature is relatively cold on the dynamic tropopause, the tropopause itself is at a relatively low altitude, inferring an upper tropospheric trough.

This is demonstrated by Figure 2 below. Relatively warm potential temperature on the dynamic tropopause is found across the eastern half of the United States. Conversely, relatively cold potential temperature on the dynamic tropopause is found across the western United States. The horizontal wind field is anticyclonically-curved across the eastern United States whereas it is cyclonically-curved across the western United States, in agreement with what we would expect given the theory above. As one might expect, the strongest winds on the dynamic tropopause are found in conjunction with the strongest horizontal gradient of potential temperature along the dynamic tropopause.

We can confirm these insights by comparing Figure 2 to Figure 3. At 300 hPa, a trough of low pressure is found across the western United States whereas a ridge of high pressure is found across the eastern half of the United States. At any given latitude, air temperatures are relatively cold in the vicinity of the upper-tropospheric trough and relatively warm in the vicinity of the upper-tropospheric ridge. Since pressure is constant on an isobaric surface, this implies relatively cold upper tropospheric potential temperature across the western United States and relatively warm upper tropospheric potential temperature across the eastern United States. The strongest winds are found along the periphery of the trough and ridge, where the horizontal height and potential temperature gradients are at their largest.

Before proceeding, it is worth stating a cautionary note. Air temperature typically increases – or at least remains constant – across the tropopause. As a result, air temperature in the heart of an upper tropospheric trough on an isobaric surface ***intersecting the tropopause*** may appear warm compared to its surroundings, in contrast to the insight drawn above. Therefore, extreme care must be taken when analyzing features on isobaric surfaces intersecting the tropopause. The same does not hold true on the dynamic tropopause, however; there, cold (warm) potential temperature always corresponds to a lower (higher) tropopause height.

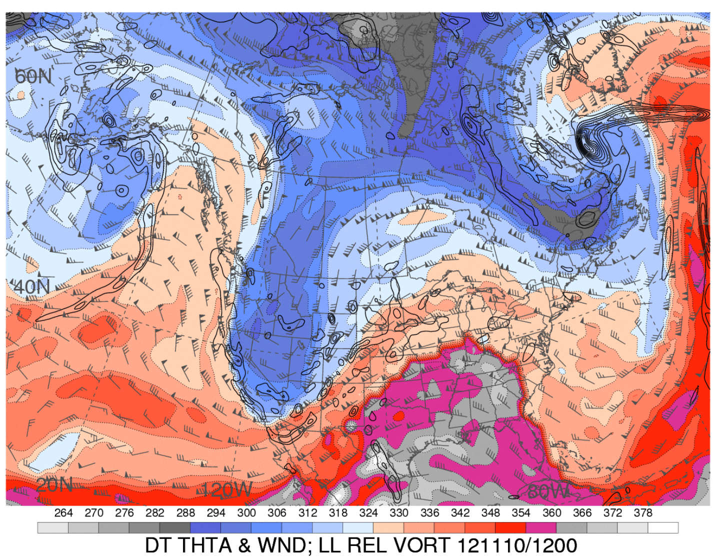

Figure 2. Potential temperature (shaded; units: K) and wind (barbs; half: 5 kt, full: 10 kt; pennant: 50 kt) on the dynamic tropopause, as represented by the 1.5 PVU surface, at 1200 UTC 10 November 2012. Also depicted is the 925-850 hPa layer mean relative vorticity (contours; every 5 x 10^{-5} s^{-1}). Image courtesy Heather Archambault.

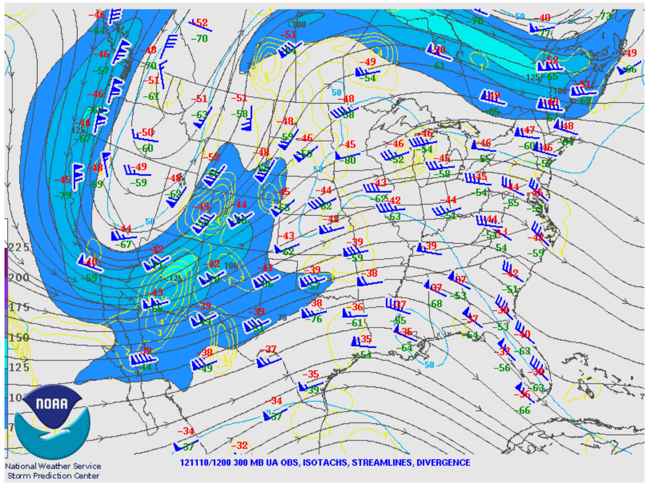

Figure 3. Streamlines (black contours), horizontal winds (shaded; units: kt), and upper air observations (barbs: wind in kt, red numbers: air temperature in °C, green numbers: dewpoint temperature in °C) at 1200 UTC 10 November 2012. Image courtesy Storm Prediction Center.

Application of Potential Vorticity to Isobaric Surfaces

It should be noted that isentropic potential vorticity is conserved following the flow *only* on isentropic surfaces. It is *not* conserved on isobaric surfaces, or even on constant height surfaces. Given that most meteorological data is obtained, displayed, and interpreted on isobaric surfaces, it is fair to ask whether there is a similar quantity that is conserved on isobaric surfaces.

The *Ertel potential vorticity*, or EPV, is conserved following the full three-dimensional flow on isobaric surfaces so long as the flow is dry adiabatic and frictionless. Mathematically, the EPV can be expressed as:

$$EPV = -g\left(f\hat{\mathbf{k}} + \nabla_3 \times \vec{\mathbf{v}}\right) \cdot \nabla_3 \theta \tag{13}$$

In (13), subscripts of 3 refer to the gradient being evaluated in all three dimensions – x, y, and p.

If we complete the vector operations in (13) to expand the EPV into its components, we obtain:

$$EPV = -g\left[\left(\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial p}\right)\frac{\partial \theta}{\partial x} - \left(\frac{\partial \omega}{\partial x} - \frac{\partial u}{\partial p}\right)\frac{\partial \theta}{\partial y} + \eta\frac{\partial \theta}{\partial p}\right] \tag{14}$$

If we presume that the horizontal vorticity – the terms involving ω and the partial derivatives of u and v with respect to p – is relatively small, we can simplify (14). This presumption is equivalent to stating the vertical motion and its horizontal gradients are relatively small on the synoptic-scale and that the troposphere is approximately barotropic such that the horizontal wind speed and direction are nearly constant with height.

This simplified form of (14) is thus given by:

$$EPV \approx -g\eta \frac{\partial \theta}{\partial p} \tag{15}$$

Equation (15) is identical to (11), except applied on an isobaric surface and stated explicitly as an approximation. The basic interpretation of mid-latitude, synoptic-scale weather phenomena is identical whether isentropic or Ertel potential vorticity are utilized. However, care should be taken to remember that the isentropic potential vorticity is only conserved following the motion on isentropic surfaces, while Ertel potential vorticity is only conserved following the motion on isobaric surfaces. (Both are conserved only for frictionless, dry adiabatic flow.)

Likewise, it should be reiterated that the approximate EPV given in (15) above is just that – an approximation – to the full EPV. *For greatest accuracy*, the isentropic potential vorticity should be computed and interpreted exclusively on isentropic surfaces while the full, rather than approximate, Ertel potential vorticity should be computed and interpreted exclusively on isobaric surfaces. Nevertheless, the form of potential vorticity given by the approximate Ertel potential vorticity in (15) is that which is most often considered in synoptic-scale meteorology, largely because it is simpler to compute than both its full form on isobaric surfaces and its isentropic potential vorticity counterpart on isentropic surfaces.

Application of Isentropic Potential Vorticity to Synoptic Analysis

Earlier, we stated that lower values of potential temperature on the dynamic tropopause imply lower tropopause heights and upper tropospheric troughing. Meanwhile, higher values of potential temperature on the dynamic tropopause imply higher tropopause heights and upper tropospheric ridging. This is demonstrated graphically in Figure 1.137 of Bluestein (Vol. II), reproduced below as Figure 4. If we take the dynamic tropopause to be the 1.5 PVU surface, we find that it is found at relatively low altitudes and on relatively cold isentropic surfaces near the poles. Conversely, it is found at relatively high altitudes and on relatively warm isentropic surfaces near the equator.

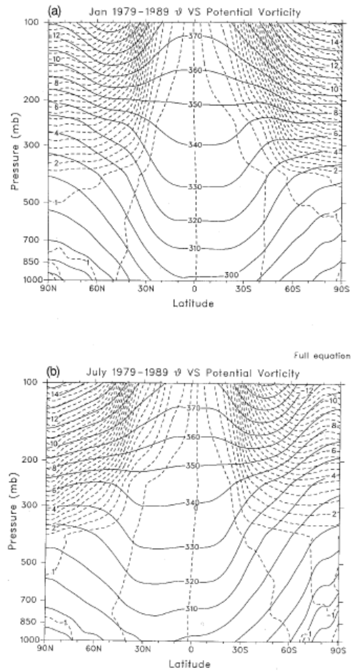

Figure 4. Vertical cross-sections of zonally-averaged, climatological mean potential vorticity (dashed lines; units: PVU) and potential temperature (solid lines; units: K) for (a) January and (b) July. Reproduced from *Synoptic-Dynamic Meteorology in Midlatitudes* (Vol. II) by H. Bluestein, their Figure 1.137.

These concepts can be viewed in light of thickness arguments. Averaged over the course of a year, incident solar radiation is greatest at the equator and lowest at the poles. Correspondingly, the annually-averaged, layer-mean tropospheric temperature is greatest at the equator and lowest at the poles. This implies relatively low thicknesses at higher latitudes and relatively high thicknesses at lower latitudes. Consider the case where the lower surface is taken to be the ground, which is at a constant altitude, and the upper surface is taken to be the tropopause. Lower thicknesses thus imply a lower tropopause height, as seen near the poles, while higher thicknesses imply a higher tropopause height, as seen near the equator.

Figure 1.137 also demonstrates how one isentropic surface can be in the stratosphere at one location and in the troposphere at another. Take, for example, the 350 K isentropic surface. It is found between 200-250 hPa at nearly all latitudes, particularly in panel (a). In the tropics, this is located in the troposphere, beneath the dynamic tropopause. However, the 350 K isentropic surface intersects the dynamic tropopause at around 30°N/30°S latitude, and at higher latitudes it is situated within the stratosphere.

In Figures 2 and 3, we demonstrated the link between potential temperature anomalies on the dynamic tropopause and troughs and ridges on upper tropospheric isobaric charts. Now, we demonstrate the link between potential vorticity on an isentropic surface and troughs and ridges on upper tropospheric isobaric charts. This is illustrated by Figures 5 and 6. Equatorward extensions of the dynamic tropopause reflect upper tropospheric troughs, the presence of cyclonically-curved flow and relatively cold temperatures. Poleward extensions of the dynamic tropopause reflect upper tropospheric ridges, the presence of anticyclonically-curved flow, and relatively warm temperatures.

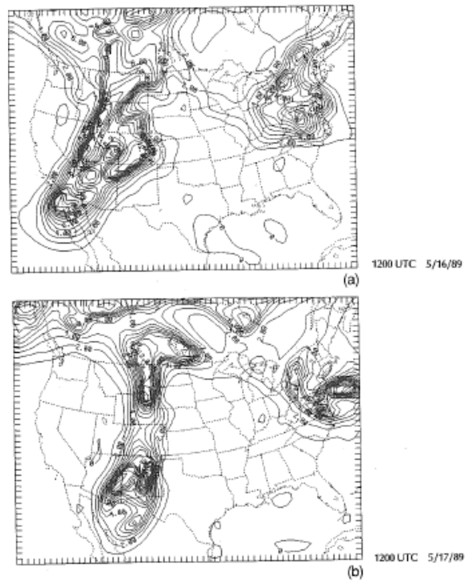

Figure 5. Isentropic potential vorticity (contours; every 1 PVU) on the 325 K isentropic surface valid at (a) 1200 UTC 16 May 1989 and (b) 1200 UTC 17 May 1989. Reproduced from *Synoptic-Dynamic Meteorology in Midlatitudes* (Vol. II) by H. Bluestein, their Figure 1.138.

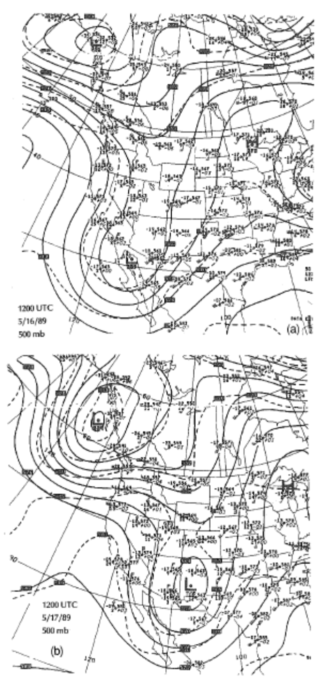

Figure 6. 500 hPa height (solid contours every 60 m), temperature (dashed contours every 5°C), and upper air observations (station plots of temperature in °C, dew point depression in °C, wind in kt, and height in dam) at (a) 1200 UTC 16 May 1989 and (b) 1200 UTC 17 May 1989. Reproduced from *Synoptic-Dynamic Meteorology in Midlatitudes* (Vol. II) by H. Bluestein, their Figure 1.139.