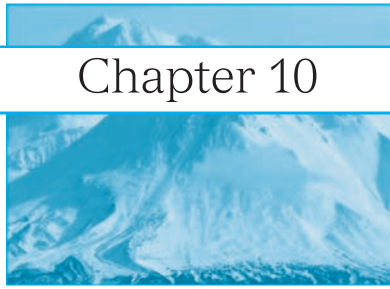


Chapter 10

The Gamma Function (Factorial Function)

The gamma function appears in physical problems of all kinds, such as the normalization of Coulomb wave functions and the computation of probabilities in statistical mechanics. Its importance stems from its usefulness in developing other functions that have direct physical application. The gamma function, therefore, is included here. A discussion of the numerical evaluation of the gamma function appears in Section 10.3. Closely related functions, such as the error integral, are presented in Section 10.4.

10.1 Definitions and Simple Properties

At least three different, convenient definitions of the gamma function are in common use. Our first task is to state these definitions, to develop some simple, direct consequences, and to show the equivalence of the three forms.

Infinite Limit (Euler)

The first definition, due to Euler, is

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \quad z \neq 0, -1, -2, -3, \ldots. \qquad (10.1)$$

This definition of $\Gamma(z)$ is useful in developing the Weierstrass infinite-product form of $\Gamma(z)$ [Eq. (10.17)] and in obtaining the derivative of $\ln\Gamma(z)$ (Section 10.2). Here and elsewhere in this chapter, z may be either real or complex.

Replacing z with $z+1$, we have

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)(z+3)\cdots(z+n+1)}n^{z+1}\\ &= \lim_{n\to\infty}\frac{nz}{z+n+1}\cdot\frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)}n^{z}\\ &= z\Gamma(z).\end{aligned}\tag{10.2}$$

This is the basic functional relation for the gamma function. It should be noted that it is a **difference** equation. The gamma function is one of a general class of functions that do not satisfy any differential equation with rational coefficients. Specifically, the gamma function is one of the very few functions of mathematical physics that does not satisfy any of the ordinary differential equations (ODEs) common to physics. In fact, it does not satisfy any useful or practical differential equation.

Also, from the definition

$$\Gamma(1) = \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n(n+1)}n = 1. \tag{10.3}$$

Now, application of Eq. (10.2) gives

$$\begin{aligned}&\Gamma(2) = 1,\\ &\Gamma(3) = 2\Gamma(2) = 2, \ldots\\ &\Gamma(n) = 1\cdot 2\cdot 3\cdots(n-1) = (n-1)!\end{aligned}\tag{10.4}$$

We see that the gamma function interpolates the factorials by a continuous function that returns the factorials at integer arguments.

Definite Integral (Euler)

A second definition, also frequently called the Euler integral, is

$$\Gamma(z) \equiv \int_0^\infty e^{-t}t^{z-1}dt, \quad \Re(z) > 0. \tag{10.5}$$

The restriction on z is necessary to avoid divergence of the integral at $t = 0$. When the gamma function does appear in physical problems, it is often in this form or some variation, such as

$$\Gamma(z) = 2\int_0^\infty e^{-t^2}t^{2z-1}dt, \quad \Re(z) > 0 \tag{10.6}$$

or

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1}dt, \quad \Re(z) > 0.$$

EXAMPLE 10.1.1

The Euler Integral Interpolates the Factorials The Euler integral for positive integer argument, $z = n + 1$ with $n > 0$, yields the factorials. Using integration by parts repeatedly we find

$$\begin{aligned}\int_0^\infty e^{-t} t^n dt &= -t^n e^{-t}\Big|_0^\infty + n\int_0^\infty e^{-t} t^{n-1} dt = n\int_0^\infty e^{-t} t^{n-1} dt \\ &= n\left[-t^{n-1} e^{-t}\Big|_0^\infty + (n-1)\int_0^\infty e^{-t} t^{n-2} dt\right] \\ &= n(n-1)\int_0^\infty e^{-t} t^{n-2} dt = \cdots = n!\end{aligned}$$

because $\int_0^\infty e^{-t} dt = 1$. Thus, the Euler integral also interpolates the factorials. ■

When $z = \frac{1}{2}$, Eq. (10.6) contains the Gauss error integral, and we have the interesting result

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}. \tag{10.7}$$

The value $\Gamma(\frac{1}{2})$ can be derived directly from the square of Eq. (10.6) for $z = \frac{1}{2}$ by introducing plane polar coordinates ($x^2 + y^2 = \rho^2$, $dx\,dy = \rho\,d\rho\,d\varphi$) in the product of integrals

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right)^2 &= 4\int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\,dy \\ &= 4\int_{\varphi=0}^{\pi/2} \int_{\rho=0}^\infty e^{-\rho^2} \rho\,d\rho\,d\varphi = -\pi e^{-\rho^2}\Big|_0^\infty = \pi.\end{aligned}$$

Generalizations of Eq. (10.7), the Gaussian integrals, are

$$\int_0^\infty x^{2s+1} \exp(-ax^2)\,dx = \frac{s!}{2a^{s+1}}, \tag{10.8}$$

$$\int_0^\infty x^{2s} \exp(-ax^2)\,dx = \frac{(s-\frac{1}{2})!}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1} a^s}\sqrt{\frac{\pi}{a}}, \tag{10.9}$$

which are of major importance in statistical mechanics. A proof is left to the reader in Exercise 10.1.11. The double factorial notation is explained in Exercise 5.2.12 and Eq. (10.33b).

To show the equivalence of the two definitions, Eqs. (10.1) and (10.5), consider the function of two variables

$$F(z, n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\,dt, \quad \Re(z) > 0, \tag{10.10}$$

with n a positive integer. Since

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n \equiv e^{-t}, \tag{10.11}$$

from the definition of the exponential

$$\lim_{n\to\infty} F(z, n) = F(z, \infty) = \int_0^{\infty} e^{-t} t^{z-1} dt \equiv \Gamma(z) \tag{10.12}$$

by Eq. (10.5).

Returning to $F(z, n)$, we evaluate it in successive integrations by parts. For convenience let $u = t/n$. Then

$$F(z, n) = n^z \int_0^1 (1-u)^n u^{z-1} du. \tag{10.13}$$

Integrating by parts, we obtain for $\Re(z) > 0$,

$$\frac{F(z, n)}{n^z} = (1-u)^n \frac{u^z}{z}\bigg|_0^1 + \frac{n}{z}\int_0^1 (1-u)^{n-1} u^z du. \tag{10.14}$$

Repeating this, with the integrated part vanishing at both end points each time, we finally get

$$\begin{aligned} F(z, n) &= n^z \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} \int_0^1 u^{z+n-1} du \\ &= \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z. \end{aligned} \tag{10.15}$$

This is identical to the expression on the right side of Eq. (10.1). Hence,

$$\lim_{n\to\infty} F(z, n) = F(z, \infty) \equiv \Gamma(z) \tag{10.16}$$

by Eq. (10.1), completing the proof.

Using the functional equation (10.2) we can extend $\Gamma(z)$ from positive to negative arguments. For example, starting with Eq. (10.2) we define

$$\Gamma\left(-\frac{1}{2}\right) = -\frac{1}{0.5}\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi},$$

and Eq. (10.2) for $z \to 0$ implies that $\Gamma(z \to 0) \to \infty$. Moreover, $z\Gamma(z) \to 1$ for $z \to 0$ [Eq. (10.2)] shows that $\Gamma(z)$ has a simple pole at the origin. Similarly, we find simple poles of $\Gamma(z)$ at all negative integers.

Infinite Product (Weierstrass)

The third definition (Weierstrass's form) is

$$\frac{1}{\Gamma(z)} \equiv z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \tag{10.17}$$

where γ is the Euler–Mascheroni constant [Eq. (5.27)]

$$\gamma = \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \ln n\right) = 0.5772156\cdots. \tag{10.18}$$

This infinite product form may be used to develop the reflection identity, Eq. (10.24a), and applied in the exercises, such as Exercise 10.1.17. This form can be derived from the original definition [Eq. (10.1)] by rewriting it as

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)\cdots(z+n)} n^z = \lim_{n\to\infty} \frac{1}{z}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right)^{-1} n^z. \quad (10.19)$$

Inverting Eq. (10.19) and using

$$n^{-z} = e^{-z\ln n}, \quad (10.20)$$

we obtain

$$\frac{1}{\Gamma(z)} = z\lim_{n\to\infty} e^{(-\ln n)z}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right). \quad (10.21)$$

Multiplying and dividing by

$$\exp\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)z\right] = \prod_{m=1}^{n} e^{z/m}, \quad (10.22)$$

we get

$$\frac{1}{\Gamma(z)} = z\left\{\lim_{n\to\infty}\exp\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)z\right]\right\}$$
$$\times\left[\lim_{n\to\infty}\prod_{m=1}^{n}\left(1+\frac{z}{m}\right)e^{-z/m}\right]. \quad (10.23)$$

As shown in Section 5.2, the infinite series in the exponent converges and defines γ, the Euler–Mascheroni constant. Hence, Eq. (10.17) follows.

The Weierstrass infinite product definition of $\Gamma(z)$ leads directly to an important identity,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}, \quad (10.24a)$$

using the infinite product formulas for $\Gamma(z)$, $\Gamma(1-z)$, and $\sin z$ [Eq. (7.60)].

Alternatively, we can start from the product of Euler integrals

$$\Gamma(z+1)\Gamma(1-z) = \int_0^\infty s^z e^{-s}ds\int_0^\infty t^{-z}e^{-t}dt$$
$$= \int_0^\infty v^z\frac{dv}{(v+1)^2}\int_0^\infty e^{-u}u\,du = \frac{\pi z}{\sin\pi z},$$

transforming from the variables s, t to $u = s+t$, $v = s/t$, as suggested by combining the exponentials and the powers in the integrands. The Jacobian is

$$J = -\begin{vmatrix} 1 & 1 \\ \frac{1}{t} & -\frac{s}{t^2}\end{vmatrix} = \frac{s+t}{t^2} = \frac{(v+1)^2}{u},$$

where $(v+1)t = u$. The integral $\int_0^\infty e^{-u}u\,du = 1$, whereas that over v may be derived by contour integration, giving $\frac{\pi z}{\sin\pi z}$. Similarly, one can establish

Legendre's duplication formula

$$\Gamma(1+z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\,\Gamma(2z+1).$$

Setting $z = \frac{1}{2}$ in Eq. (10.24a), we obtain

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \tag{10.24b}$$

(taking the positive square root) in agreement with Eqs. (10.7) and (10.9).

The Weierstrass definition shows immediately that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, -3, \ldots$, and that $[\Gamma(z)]^{-1}$ has no poles in the finite complex plane, which means that $\Gamma(z)$ has no zeros. This behavior may also be seen in Eq. (10.24a), in which we note that $\pi/(\sin \pi z)$ is never equal to zero.

Actually, the infinite product definition of $\Gamma(z)$ may be derived from the Weierstrass factorization theorem with the specification that $[\Gamma(z)]^{-1}$ have simple zeros at $z = 0, -1, -2, -3, \ldots$. The Euler–Mascheroni constant is fixed by requiring $\Gamma(1) = 1$. See also the product expansions of entire functions in Chapter 7.

In mathematical probability theory the gamma distribution (probability density) is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \leq 0. \end{cases} \tag{10.25a}$$

The constant $[\beta^{\alpha}\Gamma(\alpha)]^{-1}$ is included so that the total (integrated) probability will be unity. For $x \to E$, kinetic energy, $\alpha \to \frac{3}{2}$ and $\beta \to kT$, Eq. (10.25a) yields the classical Maxwell–Boltzmann statistics.

> ***Biographical Data***
> **Weierstrass, Karl Theodor Wilhelm.** Weierstrass, a German mathematician, was born in 1815 in Ostenfelde, Germany, and died in 1897 in Berlin. He obtained a degree in mathematics in 1841 and studied Abel's and Jacobi's work on elliptical functions and extended their work on analytic functions while living as a schoolteacher. After being recognized as the father of modern analysis, he became a professor at the University of Berlin and a member of the Academy of Sciences in 1856.

Factorial Notation

So far, this discussion has been presented in terms of the classical notation. As pointed out by Jeffreys and others, the -1 of the $z-1$ exponent in our second definition [Eq. (10.5)] is a continual nuisance. Accordingly, Eq. (10.5) is rewritten as

$$\int_0^{\infty} e^{-t} t^{z} dt \equiv z!, \quad \Re(z) > 1, \tag{10.25b}$$

to **define** a factorial function $z!$. Occasionally, we may even encounter Gauss's notation, $\prod(z)$, for the factorial function

$$\prod(z) = z!. \tag{10.26}$$

The Γ notation is due to Legendre. The factorial function of Eq. (10.25a) is, of course, related to the gamma function by

$$\Gamma(z) = (z-1)!, \qquad \text{or} \qquad \Gamma(z+1) = z!. \tag{10.27}$$

If $z = n$, a positive integer [Eq. (10.4)] shows that

$$z! = n! = 1 \cdot 2 \cdot 3 \cdots n, \tag{10.28}$$

the familiar factorial. However, it should be noted that since $z!$ is now defined by Eq. (10.25b) [or equivalently by Eq. (10.27)] **the factorial function is no longer limited to positive integral values of the argument** (Fig. 10.1). The difference relation [Eq. (10.2)] becomes

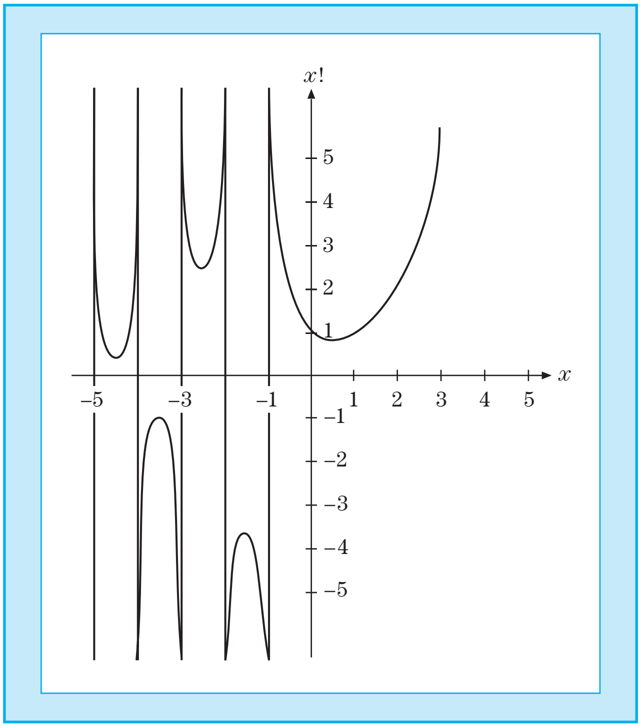
$$(z-1)! = \frac{z!}{z}. \tag{10.29}$$

This shows immediately that

$$0! = 1 \tag{10.30}$$

Figure 10.1

The Factorial Function—Extension to Negative Arguments



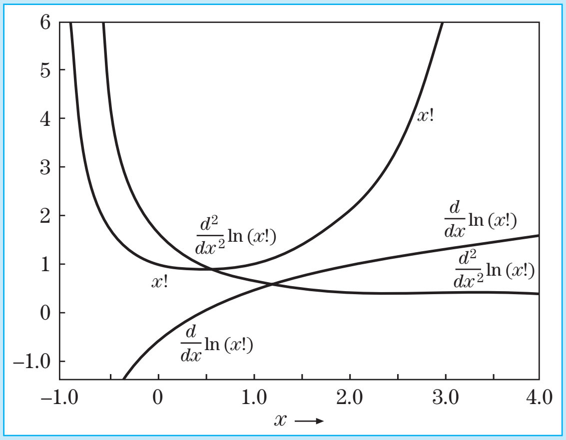


Figure 10.2

The Factorial Function and the First Two Derivatives of ln($x!$)

and

$$n! = \pm\infty \quad \text{for } n\text{, a } \textbf{negative} \text{ integer.} \tag{10.31}$$

In terms of the factorial, Eq. (10.24a) becomes

$$z!(-z)! = \frac{\pi z}{\sin \pi z} \tag{10.32}$$

because

$$\Gamma(z)\Gamma(1-z) = (z-1)!(1-z-1)! = (z-1)!(-z)! = \frac{1}{z}z!(-z)!.$$

By restricting ourselves to the real values of the argument, we find that $x!$ defines the curve shown in Fig. 10.2. The minimum of the curve in Fig. 10.1 is

$$x! = (0.46163\cdots)! = 0.88560\cdots. \tag{10.33a}$$

Double Factorial Notation

In many problems of mathematical physics, particularly in connection with Legendre polynomials (Chapter 11), we encounter products of the odd positive integers and products of the even positive integers. For convenience, these are given special labels as double factorials:

$$\begin{aligned} 1\cdot 3\cdot 5\cdots(2n+1) &= (2n+1)!! \\ 2\cdot 4\cdot 6\cdots(2n) &= (2n)!!. \end{aligned} \tag{10.33b}$$

Clearly, these are related to the regular factorial functions by

$$(2n)!! = 2^n n! \quad \text{and} \quad (2n+1)!! = \frac{(2n+1)!}{2^n n!}. \tag{10.33c}$$

Integral Representation

An integral representation that is useful in developing asymptotic series for the Bessel functions is

$$\int_C e^{-z} z^\nu dz = (e^{2\pi i\nu} - 1)\nu!, \tag{10.34}$$

where C is the contour shown in Fig. 10.3. This contour integral representation is only useful when ν is not an integer, $z = 0$ then being a **branch point**. Equation (10.34) may be verified for $\nu > -1$ by deforming the contour as shown in Fig. 10.4. The integral from ∞ to $\epsilon > 0$ in Fig. 10.4 yields $-(\nu!)$, placing the phase of z at 0. The integral from ϵ to ∞ (in the fourth quadrant) then yields $e^{2\pi i\nu}\nu!$, the phase of z having increased to 2π. Since the circle of radius ϵ around the origin contributes nothing as $\epsilon \to 0$, when $\nu > -1$, Eq. (10.34) follows.

It is often convenient to put this result into a more symmetrical form

$$\int_C e^{-z}(-z)^\nu dz = 2i\sin(\nu\pi)\nu!, \tag{10.35}$$

multiplying both sides of Eq. (10.34) by $(-1)^\nu = e^{-\pi i\nu}$.

This analysis establishes Eqs. (10.34) and (10.35) for $\nu > -1$. It is relatively simple to extend the range to include all nonintegral ν. First, we note that the integral exists for $\nu < -1$, as long as we stay away from the origin. Second, integrating by parts we find that Eq. (10.35) yields the familiar difference relation [Eq. (10.29)]. If we take the difference relation to define the factorial

Figure 10.3

Factorial Function Contour

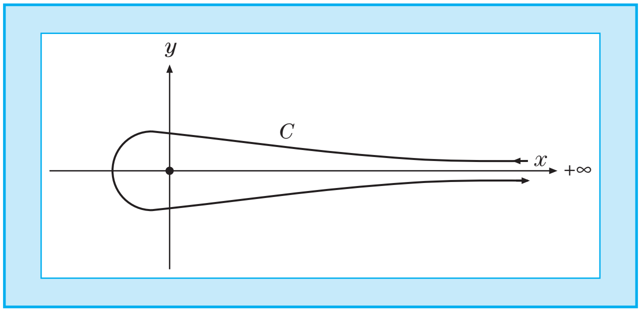
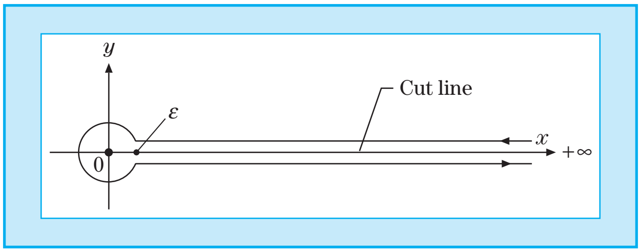


Figure 10.4

The Contour of Fig. 10.3 Deformed



function of $\nu < -1$, then Eqs. (10.34) and (10.35) are verified for all ν (except negative integers).

EXERCISES

10.1.1 Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z)$$

from the Euler integral [Eq. (10.5)]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

10.1.2 In a power series solution for the Legendre functions of the second kind, we encounter the expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)\cdot(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)},$$

in which s is a positive integer. Rewrite this expression in terms of factorials.

10.1.3 Show that

$$\frac{(s-n)!}{(2s-2n)!} = \frac{(-1)^{n-s}(2n-2s)!}{(n-s)!},$$

where s and n are integers with $s < n$. This result can be used to avoid negative factorials such as in the series representations of the spherical Neumann functions and the Legendre functions of the second kind.

10.1.4 Show that $\Gamma(z)$ may be written

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dz, \quad \Re(z) > 0,$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \quad \Re(z) > 0.$$

10.1.5 In a Maxwellian distribution the fraction of particles with speed between v and $v + dv$ is

$$\frac{dN}{N} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right)v^2 dv,$$

where N is the total number of particles. The average or expectation value of v^n is defined as $\langle v^n\rangle = N^{-1}\int v^n dN$. Show that

$$\langle v^n\rangle = \left(\frac{2kT}{m}\right)^{n/2}\left(\frac{n+1}{2}\right)!\Big/\frac{1}{2}!.$$

10.1.6 By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k \ln x\,dx = \frac{1}{(k+1)^2}, \qquad k > -1.$$

10.1.7 Show that

$$\int_0^\infty e^{-x^4}\,dx = \left(\frac{1}{4}\right)!.$$

10.1.8 Show that

$$\lim_{x\to 0} \frac{(ax-1)!}{(x-1)!} = \frac{1}{a}.$$

10.1.9 Locate the poles of $\Gamma(z)$. Show that they are simple poles and determine the residues.

10.1.10 Show that the equation $x! = k$, $k \neq 0$, has an infinite number of real roots.

10.1.11 Show that

(a) $\displaystyle\int_0^\infty x^{2s+1}\exp(-ax^2)\,dx = \frac{s!}{2a^{s+1}}.$

(b) $\displaystyle\int_0^\infty x^{2s}\exp(-ax^2)\,dx = \frac{(s-\frac{1}{2})!}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}.$

These Gaussian integrals are of major importance in statistical mechanics.

10.1.12 (a) Develop recurrence relations for $(2n)!!$ and for $(2n+1)!!$.
(b) Use these recurrence relations to calculate (or define) $0!!$ and $(-1)!!$.

ANS. $0!! = 1, \quad (-1)!! = 1.$

10.1.13 For s a nonnegative integer, show that

$$(-2s-1)!! = \frac{(-1)^s}{(2s-1)!!} = \frac{(-1)^s 2^s s!}{(2s)!}.$$

10.1.14 Express the coefficient of the nth term of the expansion of $(1+x)^{1/2}$
(a) in terms of factorials of integers; and
(b) in terms of the double factorial (!!) functions.

ANS. $a_n = (-1)^{n+1}\dfrac{(2n-3)!}{2^{2n-2}n!(n-2)!} = (-1)^{n+1}\dfrac{(2n-3)!!}{(2n)!!},\ n = 2, 3, \cdots.$

10.1.15 Express the coefficient of the nth term of the expansion of $(1+x)^{1/2}$
(a) in terms of the factorials of integers; and
(b) in terms of the double factorial (!!) functions.

ANS. $\quad a_n = (-1)^n\dfrac{(2n)!}{2^{2n}(n!)^2} = (-1)^n\dfrac{(2n-1)!!}{(2n)!!}, \quad n = 1, 2, 3\cdots.$

10.1.16 The Legendre polynomial may be written as

$$P_n(\cos\theta) = 2\frac{(2n-1)!!}{(2n)!!}\left\{\cos n\theta + \frac{1}{1}\cdot\frac{n}{2n-1}\cos(n-2)\theta + \frac{1\cdot 3}{1\cdot 2}\frac{n(n-1)}{(2n-1)(2n-3)}\cos(n-4)\theta + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\cos(n-6)\theta + \cdots\right\}.$$

Let $n = 2s+1$. Then

$$P_n(\cos\theta) = P_{2s+1}(\cos\theta) = \sum_{m=0}^{s} a_m \cos(2m+1)\theta.$$

Find a_m in terms of factorials and double factorials.

10.1.17 (a) Show that

$$\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right) = (-1)^n\pi,$$

where n is an integer.

(b) Express $\Gamma(\frac{1}{2}+n)$ and $\Gamma(\frac{1}{2}-n)$ separately in terms of $\pi^{1/2}$ and a !! function.

$$\textit{ANS. } \Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\pi^{1/2}.$$

10.1.18 From one of the definitions of the factorial or gamma function, show that

$$|(ix)!|^2 = \frac{\pi x}{\sinh \pi x}.$$

10.1.19 Prove that

$$|\Gamma(\alpha+i\beta)| = |\Gamma(\alpha)|\prod_{n=0}^{\infty}\left[1+\frac{\beta^2}{(\alpha+n)^2}\right]^{-1/2}.$$

This equation has been useful in calculations of beta decay theory.

10.1.20 Show that

$$|(n+ib)!| = \left(\frac{\pi b}{\sinh \pi b}\right)^{1/2}\prod_{s=1}^{n}(s^2+b^2)^{1/2}$$

for n, a positive integer.

10.1.21 Show that

$$|x!| \geq |(x+iy)!|$$

for all x. The variables x and y are real.

10.1.22 Show that

$$\left|\left(-\frac{1}{2}+iy\right)!\right|^2 = \frac{\pi}{\cosh \pi y}.$$

10.1.23 The probability density associated with the normal distribution of statistics is given by

$$f(x) = \frac{1}{\sigma(2\pi)^{1/2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

with $(-\infty, \infty)$ for the range of x. Show that
(a) the mean value of x, $\langle x \rangle$ is equal to μ; and
(b) the standard deviation $(\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$ is given by σ.

10.1.24 From the gamma distribution

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0, & x \leq 0, \end{cases}$$

show that
(a) $\langle x \rangle$ (mean) $= \alpha\beta$, (b) σ^2 (variance) $\equiv \langle x^2 \rangle - \langle x \rangle^2 = \alpha\beta^2$.

10.1.25 The wave function of a particle scattered by a Coulomb potential is $\psi(r, \theta)$. At the origin the wave function becomes

$$\psi(0) = e^{-\pi\gamma/2}\Gamma(1 + i\gamma),$$

where $\gamma = Z_1 Z_2 e^2/\hbar v$. Show that

$$|\psi(0)|^2 = \frac{2\pi\gamma}{e^{2\pi\gamma} - 1}.$$

10.1.26 Derive the contour integral representation of Eq. (10.34)

$$(2i)\nu! \sin \nu\pi = \int_C e^{-z}(-z)^\nu dz.$$

10.2 Digamma and Polygamma Functions

Digamma Function

As may be noted from the definitions in Section 10.1, it is inconvenient to deal with the derivatives of the gamma or factorial function directly. Instead, it is customary to take the natural logarithm of the factorial function [Eq. (10.1)], convert the product to a sum, and then differentiate; that is, we start from

$$z! = z\Gamma(z) = \lim_{n\to\infty} \frac{n!}{(z+1)(z+2)\cdots(z+n)} n^z. \tag{10.36}$$

Then, because the logarithm of the limit is equal to the limit of the logarithm, we have

$$\begin{aligned} \ln(z!) = \lim_{n\to\infty} [&\ln(n!) + z \ln n - \ln(z+1) \\ &- \ln(z+2) - \cdots - \ln(z+n)]. \end{aligned} \tag{10.37}$$

Differentiating with respect to z, we obtain and define

$$\frac{d}{dz}\ln(z!) \equiv \psi(z+1) = \lim_{n\to\infty}\left(\ln n - \frac{1}{z+1} - \frac{1}{z+2} - \cdots - \frac{1}{z+n}\right), \quad (10.38)$$

which defines $\psi(z+1)$, the digamma function. From the definition of the Euler–Mascheroni constant,[1] Eq. (10.38) may be rewritten as

$$\begin{aligned}\psi(z+1) &= -\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\\ &= -\gamma + \sum_{n=1}^{\infty}\frac{z}{n(n+z)}.\end{aligned} \quad (10.39)$$

Clearly,

$$\psi(1) = -\gamma = -0.577\,215\,664\,901\cdots.^{2} \quad (10.40)$$

Another, even more useful, expression for $\psi(z)$ is derived in Section 10.3.

Polygamma Function

The digamma function may be differentiated repeatedly, giving rise to the polygamma function:

$$\begin{aligned}\psi^{(m)}(z+1) &\equiv \frac{d^{m+1}}{dz^{m+1}}\ln(z!)\\ &= (-1)^{m+1}m!\sum_{n=1}^{\infty}\frac{1}{(z+n)^{m+1}}, \quad m = 1, 2, 3, \ldots.\end{aligned} \quad (10.41)$$

A plot of $\psi(x+1)$ and $\psi'(x+1)$ is included in Fig. 10.2. Since the series in Eq. (10.41) defines the Riemann zeta function,[3] when z is set to zero,

$$\zeta(m) \equiv \sum_{n=1}^{\infty}\frac{1}{n^m}, \quad (10.42)$$

we have

$$\psi^{(m)}(1) = (-1)^{m+1}m!\zeta(m+1), \quad m = 1, 2, 3, \ldots. \quad (10.43)$$

The values of the polygamma functions of positive integral argument, $\psi^{(m)}(n)$, may be calculated using Exercise 10.2.6.

In terms of the more common Γ notation,

$$\frac{d^{n+1}}{dz^{n+1}}\ln\Gamma(z) = \frac{d^n}{dz^n}\psi(z) = \psi^{(n)}(z). \quad (10.44a)$$

[1] Compare Section 5.2, Eq. (5.27). We add and subtract $\sum_{s=1}^{n} s^{-1}$.
[2] γ has been computed to 1271 places by D. E. Knuth, *Math. Comput.* **16**, 275 (1962) and to 3566 decimal places by D. W. Sweeney, *Math. Comput.* **17**, 170 (1963). It may be of interest that the fraction 228/395 gives γ accurate to six places.
[3] See Chapter 5. For $z \neq 0$ this series may be used to define a generalized zeta function.

Maclaurin Expansion, Computation

It is now possible to write a Maclaurin expansion for $\ln(z!)$:

$$\ln(z!) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \psi^{(n-1)}(1) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n} \zeta(n) \qquad (10.44b)$$

convergent for $|z| < 1$; for $z = x$, the range is $-1 < x \le 1$. Equation (10.44b) is a possible means of computing $z!$ for real or complex z, but Stirling's series (Section 10.3) is usually better. In addition, an excellent table of values of the gamma function for complex arguments based on the use of Stirling's series and the recurrence relation [Eq. (10.29)] is available[4] and can be accessed by symbolic software, such as Mathematica, Maple, Mathcad, and Reduce.

Series Summation

The digamma and polygamma functions may also be used in summing series. If the general term of the series has the form of a rational fraction (with the highest power of the index in the numerator at least two less than the highest power of the index in the denominator), it may be transformed by the method of partial fractions. The infinite series may then be expressed as a finite sum of digamma and polygamma functions. The usefulness of this method depends on the availability of tables of digamma and polygamma functions. Such tables and examples of series summation are given in AMS-55, Chapter 6.

EXAMPLE 10.2.1

Catalan's Constant Catalan's constant, Exercise 5.2.13, or $\beta(2)$, is given by

$$K = \beta(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}. \qquad (10.44c)$$

Grouping the positive and negative terms separately and starting with unit index [to match the form of $\psi^{(2)}$; Eq. (10.41)], we obtain

$$K = 1 + \sum_{n=1}^{\infty} \frac{1}{(4n+1)^2} - \frac{1}{9} - \sum_{n=1}^{\infty} \frac{1}{(4n+3)^2}.$$

Now, quoting Eq. (10.41), we get

$$K = \tfrac{8}{9} + \tfrac{1}{16}\psi^{(1)}\left(1+\tfrac{1}{4}\right) - \tfrac{1}{16}\psi^{(1)}\left(1+\tfrac{3}{4}\right). \qquad (10.44d)$$

Using the values of $\psi^{(1)}$ from Table 6.1 of AMS-55, we obtain

$$K = 0.91596559\ldots.$$

Compare this calculation of Catalan's constant with the calculations of Chapter 5, using either direct summation by computer or a modification using Riemann zeta functions and then a (shorter) computer code. ■

[4] *Table of the Gamma Function for Complex Arguments*, Applied Mathematics Series No. 34. National Bureau of Standards, Washington, DC (1954).

EXERCISES

10.2.1 Verify that the following two forms of the digamma function,

$$\psi(x+1) = \sum_{r=1}^{x} \frac{1}{r} - \gamma$$

and

$$\psi(x+1) = \sum_{r=1}^{\infty} \frac{x}{r(r+x)} - \gamma,$$

are equal to each other (for x a positive integer).

10.2.2 Show that $\psi(z+1)$ has the series expansion

$$\psi(z+1) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}.$$

10.2.3 For a power series expansion of $\ln(z!)$, AMS-55 lists

$$\ln(z!) = -\ln(1+z) + z(1-\gamma) + \sum_{n=2}^{\infty} (-1)^n [\zeta(n) - 1] z^n / n.$$

(a) Show that this agrees with Eq. (10.44b) for $|z| < 1$.
(b) What is the range of convergence of this new expression?

10.2.4 Show that

$$\frac{1}{2} \ln\left(\frac{\pi z}{\sin \pi z}\right) = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} z^{2n}, \qquad |z| < 1.$$

Hint. Try Eq. (10.32).

10.2.5 Write out a Weierstrass infinite product definition of $z!$. Without differentiating, show that this leads directly to the Maclaurin expansion of $\ln(z!)$ [Eq. (10.44b)].

10.2.6 Derive the difference relation for the polygamma function

$$\psi^{(m)}(z+2) = \psi^{(m)}(z+1) + (-1)^m \frac{m!}{(z+1)^{m+1}}, \qquad m = 0, 1, 2, \ldots.$$

10.2.7 Show that if

$$\Gamma(x+iy) = u + iv$$

then

$$\Gamma(x-iy) = u - iv.$$

This is a special case of the Schwarz reflection principle (Section 6.5).

10.2.8 The Pochhammer symbol $(a)_n$ is defined as

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1$$

(for integral n).

(a) Express $(a)_n$ in terms of factorials.
(b) Find $(d/da)(a)_n$ in terms of $(a)_n$ and digamma functions.

$$\textit{ANS.}\quad \frac{d}{da}(a)_n = (a)_n[\psi(a+n) - \psi(a)].$$

(c) Show that

$$(a)_{n+k} = (a+n)_k \cdot (a)_n.$$

10.2.9 Verify the following special values of the ψ form of the di- and polygamma functions:

$$\psi(1) = -\gamma, \qquad \psi^{(1)}(1) = \zeta(2), \qquad \psi^{(2)}(1) = -2\zeta(3).$$

10.2.10 Derive the polygamma function recurrence relation

$$\psi^{(m)}(1+z) = \psi^{(m)}(z) + (-1)^m m!/z^{m+1}, \quad m = 0, 1, 2, \dots.$$

10.2.11 Verify

(a) $\displaystyle\int_0^\infty e^{-r} \ln r\, dr = -\gamma.$

(b) $\displaystyle\int_0^\infty re^{-r} \ln r\, dr = 1 - \gamma.$

(c) $\displaystyle\int_0^\infty r^n e^{-r} \ln r\, dr = (n-1)! + n\int_0^\infty r^{n-1} e^{-r} \ln r\, dr, \quad n = 1, 2, 3, \dots.$

Hint. These may be verified by integration by parts, three parts, or differentiating the integral form of $n!$ with respect to n.

10.2.12 Dirac relativistic wave functions for hydrogen involve factors such as $[2(1-\alpha^2 Z^2)^{1/2}]!$, where α, the fine structure constant, is $\frac{1}{137}$, and Z is the atomic number. Expand $[2(1-\alpha^2 Z^2)^{1/2}]!$ in a series of powers of $\alpha^2 Z^2$.

10.2.13 The quantum mechanical description of a particle in a Coulomb field requires a knowledge of the phase of the complex factorial function. Determine the phase of $(1+ib)!$ for small b.

10.2.14 The total energy radiated by a black body is given by

$$u = \frac{8\pi k^4 T^4}{c^3 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx.$$

Show that the integral in this expression is equal to $3!\zeta(4)$ [$\zeta(4) = \pi^4/90 = 1.0823\dots$]. The final result is the Stefan–Boltzmann law.

10.2.15 As a generalization of the result in Exercise 10.2.14, show that

$$\int_0^\infty \frac{x^s dx}{e^x - 1} = s!\zeta(s+1), \quad \Re(s) > 0.$$

10.2.16 The neutrino energy density (Fermi distribution) in the early history of the universe is given by

$$\rho_\nu = \frac{4\pi}{h^3} \int_0^\infty \frac{x^3}{\exp(x/kT) + 1} dx.$$

Show that

$$\rho_\nu = \frac{7\pi^5}{30h^3}(kT)^4.$$

10.2.17 Prove that

$$\int_0^\infty \frac{x^s dx}{e^x + 1} = s!(1 - 2^{-s})\zeta(s+1), \quad \Re(s) > 0.$$

Exercises 10.2.15 and 10.2.17 actually constitute Mellin integral transforms.

10.2.18 Prove that

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1 - e^{-t}} dt, \quad \Re(z) > 0.$$

10.2.19 Using di- and polygamma functions sum the series

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \qquad \text{(b)} \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

Note. You can use Exercise 10.2.6 to calculate the needed digamma functions.

10.2.20 Show that

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{(b-a)}\{\psi(a+1) - \psi(a+1)\},$$

$a \neq b$, and neither a nor b is a negative integer. It is of interest to compare this summation with the corresponding integral

$$\int_1^\infty \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\{\ln(1+b) - \ln(1+a)\}.$$

10.3 Stirling's Series

For computation of $\ln(z!)$ for very large z (statistical mechanics) and for numerical computations at nonintegral values of z, a series expansion of $\ln(z!)$ in negative powers of z is desirable. Perhaps the most elegant way of deriving such an expansion is by the method of steepest descents (Section 7.3). The following method, starting with a numerical integration formula, does not require knowledge of contour integration and is particularly direct.

Derivation from Euler–Maclaurin Integration Formula

The Euler–Maclaurin formula (Section 5.9) for evaluating a definite integral[5] is

$$\int_0^n f(x)\,dx = \tfrac{1}{2}f(0) + f(1) + f(2) + \cdots + \tfrac{1}{2}f(n)$$
$$- b_2[f'(n) - f'(0)] - b_4[f'''(n) - f'''(0)] - \cdots, \quad (10.45)$$

[5]This is obtained by repeated integration by parts.

in which the b_{2n} are related to the Bernoulli numbers B_{2n} by

$$(2n)!b_{2n} = B_{2n}, \tag{10.46}$$

$$\begin{aligned} B_0 &= 1, & B_6 &= \tfrac{1}{42}, \\ B_2 &= \tfrac{1}{6}, & B_8 &= -\tfrac{1}{30}, \\ B_4 &= -\tfrac{1}{30}, & B_{10} &= \tfrac{5}{66}, \quad \text{and so on.} \end{aligned} \tag{10.47}$$

By applying Eq. (10.45) to the elementary definite integral

$$\int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{z}, \qquad f(x) = \frac{1}{(z+x)^2}, \tag{10.48}$$

(for z not on the negative real axis), we obtain for $n \to \infty$,

$$\frac{1}{z} = \frac{1}{2z^2} + \psi^{(1)}(z+1) - \frac{2!b_2}{z^3} - \frac{4!b_4}{z^5} - \cdots. \tag{10.49}$$

This is the reason for using Eq. (10.48). The Euler–Maclaurin evaluation yields $\psi'(z+1)$, which is $d^2 \ln(z!)/dz^2 = \sum_{n=1}^\infty \frac{1}{(z+n)^2}$ from Eq. (10.41).

Using Eq. (10.46) and solving for $\psi^{(1)}(z+1)$, we have

$$\begin{aligned} \psi'(z+1) = \frac{d}{dz}\psi(z+1) &= \frac{1}{z} - \frac{1}{2z^2} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \cdots \\ &= \frac{1}{z} - \frac{1}{2z^2} + \sum_{n=1}^\infty \frac{B_{2n}}{z^{2n+1}}. \end{aligned} \tag{10.50}$$

Since the Bernoulli numbers diverge strongly, this series does not converge. It is a semiconvergent or asymptotic series (Section 5.10), in which the sum always has a finite number of terms (compare Section 5.10).

Integrating once, we get the digamma function

$$\begin{aligned} \psi(z+1) &= C_1 + \ln z + \frac{1}{2z} - \frac{B_2}{2z^2} - \frac{B_4}{4z^4} - \cdots \\ &= C_1 + \ln z + \frac{1}{2z} - \sum_{n=1}^\infty \frac{B_{2n}}{2nz^{2n}}. \end{aligned} \tag{10.51}$$

Integrating Eq. (10.51) with respect to z from $z-1$ to z and then letting z approach infinity, C_1, the constant of integration may be shown to vanish. This gives us a second expression for the digamma function, often more useful than Eq. (10.39).

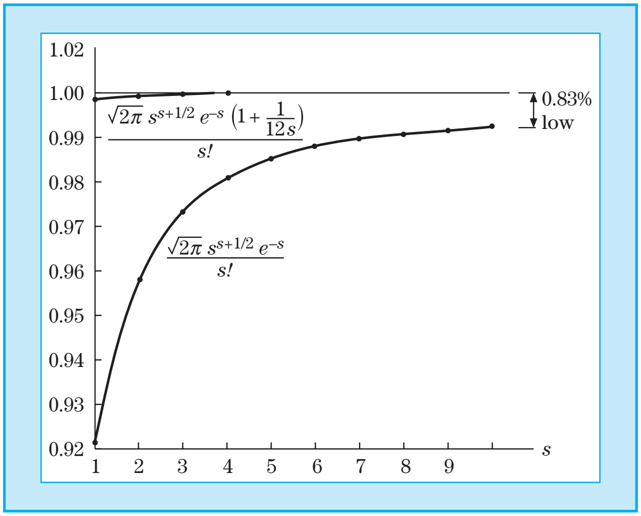
Stirling's Series

The indefinite integral of the digamma function [Eq. (10.51)] is

$$\ln(z!) = C_2 + \left(z + \frac{1}{2}\right)\ln z - z + \frac{B_2}{2z} + \cdots + \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \cdots, \tag{10.52}$$

Figure 10.5

Accuracy of Stirling's Formula



in which C_2 is another constant of integration. To fix C_2, we start from the asymptotic formula [Eq. (7.89) from Example 7.3.2]

$$z! \sim \sqrt{2\pi}\, z^{z+1/2} e^{-z}.$$

This gives for large enough $|z|$

$$\ln(z!) \sim \tfrac{1}{2}\ln 2\pi + (z+1/2)\ln z - z, \tag{10.53}$$

and comparing with Eq. (10.52), we find that C_2 is

$$C_2 = \tfrac{1}{2}\ln 2\pi, \tag{10.54}$$

giving for $|z| \to \infty$

$$\ln(z!) = \frac{1}{2}\ln 2\pi + \left(z+\frac{1}{2}\right)\ln z - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \cdots. \tag{10.55}$$

This is Stirling's series, an asymptotic expansion. The absolute value of the error is less than the absolute value of the first term neglected. For large enough $|z|$, the simplest approximation $\ln(z!) \sim z\ln z - z$ may be sufficient.

To help convey a sense of the remarkable precision of Stirling's series for $s!$, the ratio of the first term of Stirling's approximation to $s!$ is plotted in Fig. 10.5. A tabulation gives the ratio of the first term in the expansion to $s!$ and the ratio of the first two terms in the expansion to $s!$ (Table 10.1). The derivation of these forms is Exercise 10.3.1.

Numerical Computation

The possibility of using the Maclaurin expansion [Eq. (10.44b)] for the numerical evaluation of the factorial function is mentioned in Section 10.2. However, for large x, Stirling's series [Eq. (10.55)] gives much more accuracy. The

Table 10.1

Stirling's Formula Compared with Stirling's Series for $n = 2$

s	$\frac{1}{s!}\sqrt{2\pi}s^{s+1/2}e^{-s}$	$\frac{1}{s!}\sqrt{2\pi}s^{s+1/2}e^{-s}\left(1+\frac{1}{12s}\right)$
1	0.92213	0.99898
2	0.95950	0.99949
3	0.97270	0.99972
4	0.97942	0.99983
5	0.98349	0.99988
6	0.98621	0.99992
7	0.98817	0.99994
8	0.98964	0.99995
9	0.99078	0.99996
10	0.99170	0.99998

Table of the Gamma Function for Complex Arguments, Applied Mathematics Series No. 34, National Bureau of Standards, is based on the use of Stirling's series for $z = x + iy$, $9 \le x \le 10$. Lower values of x are reached with the recurrence relation [Eq. (10.29)]. Now suppose the numerical value of $x!$ is needed for some particular value of x in a computer code. How shall we instruct the computer to do $x!$? Stirling's series followed by the recurrence relation is a good possibility. An even better possibility is to fit $x!$, $0 \le x \le 1$, by a short power series (polynomial) and then calculate $x!$ directly from this empirical fit. Presumably, the computer has been told the values of the coefficients of the polynomial. Such polynomial fits have been made by Hastings[6] for various accuracy requirements. For example,

$$x! = 1 + \sum_{n=1}^{8} b_n x^n + \varepsilon(x), \tag{10.56a}$$

with

$$\begin{aligned} b_1 &= -0.57719\,1652 & b_5 &= -0.75670\,4078 \\ b_2 &= 0.98820\,5891 & b_6 &= 0.48219\,9394 \\ b_3 &= -0.89705\,6937 & b_7 &= -0.19352\,7818 \\ b_4 &= 0.91820\,6857 & b_8 &= 0.03586\,8343 \end{aligned} \tag{10.56b}$$

with the magnitude of the error $|\varepsilon(x)| < 3 \times 10^{-7}$, $0 \le x \le 1$.

This is **not** a least-squares fit. Hastings employed a Chebyshev polynomial technique to minimize the maximum value of $|\varepsilon(x)|$ in Eq. (10.56a).

SUMMARY

The Euler integral

$$n! = \int_0^\infty e^{-t} t^n dt = \Gamma(n+1), \quad n = 0, 1, \ldots$$

gives the most direct entry to the gamma function. The functional equation $\Gamma(z+1) = z\Gamma(z)$ is its characteristic property that leads to the infinite product representation of its inverse, an entire function, and to the pole expansion

[6]Hastings, C., Jr. (1955). *Approximations for Digital Computers*. Princeton Univ. Press, Princeton, NJ.

of its logarithmic derivative, a meromorphic function. The asymptotic expansion, known as Stirling's series, is widely applied in statistical mechanics and leads to similar asymptotic expansions for the error integral and other related functions.

EXERCISES

10.3.1 Rewrite Stirling's series to give $z!$ instead of $\ln(z!)$.

$$ANS.\ z! = \sqrt{2\pi} z^{z+1/2} e^{-z}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} + \cdots\right).$$

10.3.2 Use Stirling's formula to estimate 52!, the number of possible rearrangements of cards in a standard deck of playing cards.

10.3.3 By integrating Eq. (10.51) from $z-1$ to z and then letting $z \to \infty$, evaluate the constant C_1 in the asymptotic series for the digamma function $\psi(z+1)$.

10.3.4 Show that the constant C_2 in Stirling's formula Eq. (10.52) equals $\frac{1}{2}\ln 2\pi$ by using the logarithm of the doubling formula.

10.3.5 By direct expansion verify the doubling formula for $z = n + \frac{1}{2}$; n is an integer.

10.3.6 Without using Stirling's series show that

(a) $\ln(n!) < \int_1^{n+1} \ln x\, dx$, (b) $\ln(n!) > \int_1^{n} \ln x\, dx$; n is an integer ≥ 2.

Notice that the arithmetic mean of these two integrals gives a good approximation for Stirling's series.

10.3.7 Test for convergence

$$\sum_{p=0}^{\infty}\left[\frac{(p-\frac{1}{2})!}{p!}\right]^2 \cdot \frac{2p+1}{2p+2} = \pi \sum_{p=0}^{\infty} \frac{(2p-1)!!(2p+1)!!}{(2p)!!(2p+2)!!}.$$

This series arises in an attempt to describe the magnetic field created by and enclosed by a current loop.

10.3.8 Show that

$$\lim_{x\to\infty} x^{b-a}\frac{(x+a)!}{(x+b)!} = 1.$$

10.3.9 Show that

$$\lim_{n\to\infty} \frac{(2n-1)!!}{(2n)!!} n^{1/2} = \pi^{-1/2}.$$

10.3.10 Calculate the binomial coefficient $\binom{2n}{n} = \frac{(2n)!}{n!n!}$ (see Chapter 5) to six significant figures for $n = 10$, 20, and 30. Check your values by
(a) a Stirling series approximation through terms in n^{-1}; and
(b) a double precision calculation.

ANS. $\binom{20}{10} = 1.84756 \times 10^5$, $\binom{40}{20} = 1.37846 \times 10^{11}$, $\binom{60}{30} = 1.18264 \times 10^{17}$.

10.3.11 Truncate the Stirling formula for $\ln n!$ so that the error is less than 10% for $n > 1$, $<1\%$ for $n > 10$, and $<0.1\%$ for $n > 100$.

10.3.12 Derive $\frac{d}{dz} \ln \Gamma(z) \sim \ln z - \frac{1}{2z}$ from Stirling's formula.

10.4 The Incomplete Gamma Functions and Related Functions

Generalizing the Euler definition of the gamma function [Eq. (10.5)], we define the incomplete gamma functions by the variable limit integrals

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt, \quad \Re(a) > 0 \tag{10.57}$$

and

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt. \tag{10.58}$$

Clearly, the two functions are related because

$$\gamma(a, x) + \Gamma(a, x) = \Gamma(a). \tag{10.59}$$

These functions are useful for the error integrals discussed later. The choice of employing $\gamma(a, x)$ or $\Gamma(a, x)$ is purely a matter of convenience. If the parameter a is a positive integer, Eq. (10.58) may be integrated completely to yield

$$\begin{aligned} \gamma(n, x) &= (n-1)! \left(1 - e^{-x} \sum_{s=0}^{n-1} \frac{x^s}{s!}\right) \\ \Gamma(n, x) &= (n-1)! e^{-x} \sum_{s=0}^{n-1} \frac{x^s}{s!}, \qquad n = 1, 2, \ldots. \end{aligned} \tag{10.60}$$

For nonintegral a, a power series expansion of $\gamma(a, x)$ for small x and an asymptotic expansion of $\Gamma(a, x)$ are developed in terms of the error function in Section 5.10 [see also Eq. (10.70b)]:

$$\begin{aligned} \gamma(a, x) &= x^a \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!(a+n)}, \\ \Gamma(a, x) &= x^{a-1} e^{-x} \sum_{n=0}^{\infty} \frac{(a-1)!}{(a-1-n)!} \cdot \frac{1}{x^n} \\ &= x^{a-1} e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{(n-a)!}{(-a)!} \cdot \frac{1}{x^n}. \end{aligned} \tag{10.61}$$

These incomplete gamma functions may also be expressed elegantly in terms of confluent hypergeometric functions.

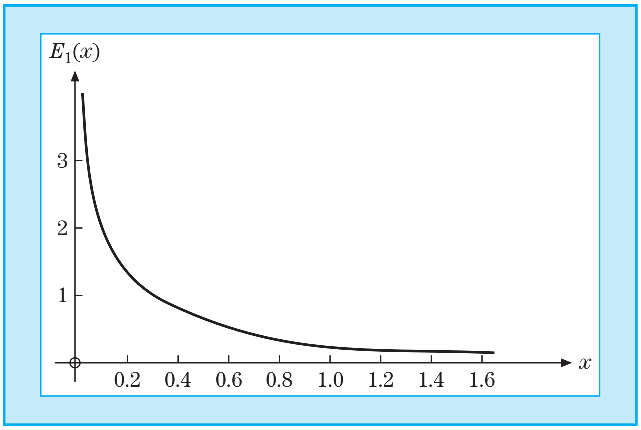


Figure 10.6

The Exponential Integral, $E_1(x) = -\mathrm{Ei}(-x)$

Exponential Integral

Although the incomplete gamma function $\Gamma(a, x)$ in its general form [Eq. (10.61)] is only infrequently encountered in physical problems, a special case is very useful. We define the exponential integral by[7]

$$-\mathrm{Ei}(-x) \equiv \int_x^\infty \frac{e^{-t}}{t} dt = E_1(x) \tag{10.62}$$

(Fig. 10.6). To obtain a series expansion for small x, we start from

$$E_1(x) = \Gamma(0, x) = \lim_{a\to 0}[\Gamma(a) - \gamma(a, x)]. \tag{10.63}$$

Caution is needed here because the integral in Eq. (10.62) diverges logarithmically as $x \to 0$. We may split the divergent term in the series expansion for $\gamma(a, x)$,

$$E_1(x) = \lim_{a\to 0}\left[\frac{a\Gamma(a) - x^a}{a}\right] - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}. \tag{10.64a}$$

Using l'Hôpital's rule (Exercise 5.6.7) and

$$\frac{d}{da}\{a\Gamma(a)\} = \frac{d}{da}a! = \frac{d}{da}e^{\ln(a!)} = a!\psi(a+1), \tag{10.64b}$$

and then Eq. (10.39),[8] we obtain the rapidly converging series

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!} \tag{10.65}$$

(Fig. 10.6). An asymptotic expansion is given in Section 5.10.

[7]The appearance of the two minus signs in $-\mathrm{Ei}(-x)$ is an historical monstrosity. This integral is generally referred to as $E_1(x)$.
[8]$dx^a/da = x^a \ln x$.

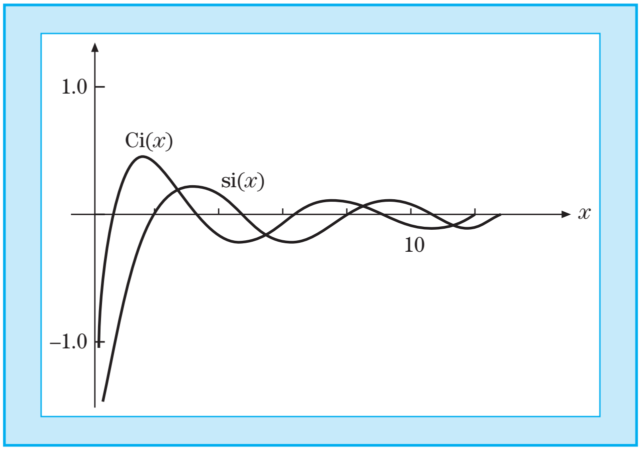


Figure 10.7

Sine and Cosine Integrals

Further special forms related to the exponential integral are the sine integral, cosine integral (Fig. 10.7), and logarithmic integral defined by[9]

$$\begin{aligned} \mathrm{si}(x) &= -\int_x^\infty \frac{\sin t}{t} dt \\ \mathrm{Ci}(x) &= -\int_x^\infty \frac{\cos t}{t} dt \qquad (10.66) \\ \mathrm{li}(x) &= \int_0^x \frac{du}{\ln u} = \mathrm{Ei}(\ln x). \end{aligned}$$

By transforming from real to imaginary argument, we can show that

$$\mathrm{si}(x) = \frac{1}{2i}[\mathrm{Ei}(ix) - \mathrm{Ei}(-ix)] = \frac{1}{2i}[E_1(ix) - E_1(-ix)], \tag{10.67}$$

whereas

$$\mathrm{Ci}(x) = \frac{1}{2}[\mathrm{Ei}(ix) + \mathrm{Ei}(-ix)] = -\frac{1}{2}[E_1(ix) + E_1(-ix)], \quad |\arg x| < \frac{\pi}{2}. \tag{10.68}$$

Adding these two relations, we obtain

$$\mathrm{Ei}(ix) = \mathrm{Ci}(x) + i\,\mathrm{si}(x) \tag{10.69}$$

to show that the relation among these integrals is exactly analogous to that among e^{ix}, $\cos x$, and $\sin x$. In terms of E_1,

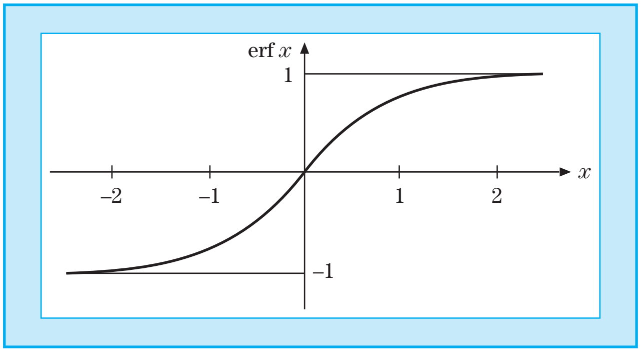
$$E_1(ix) = -\mathrm{Ci}(x) + i\,\mathrm{si}(x).$$

Asymptotic expansions of $\mathrm{Ci}(x)$ and $\mathrm{si}(x)$ may be developed similar to those for the error functions in Section 5.10. Power series expansions about the origin for $\mathrm{Ci}(x)$, $\mathrm{si}(x)$, and $\mathrm{li}(x)$ may be obtained from those for the exponential

[9] Another sine integral is given by $\mathrm{Si}(x) = \mathrm{si}(x) + \pi/2$.

Figure 10.8

Error Function erf x



integral, $E_1(x)$, or by direct integration. The exponential, sine, and cosine integrals are tabulated in AMS-55, Chapter 5, and can also be accessed by symbolic packages such as Mathematica, Maple, Mathcad, and Reduce.

Error Integrals

The error integrals

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \qquad \operatorname{erfc} z = 1 - \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \tag{10.70a}$$

(normalized so that erf $\infty = 1$) are introduced in Section 5.10 (Fig. 10.8). Asymptotic forms are developed there. From the general form of the integrands and Eq. (10.6) we expect that erf z and erfc z may be written as incomplete gamma functions with $a = \frac{1}{2}$. The relations are

$$\operatorname{erf} z = \pi^{-1/2} \gamma\left(\tfrac{1}{2}, z^2\right), \qquad \operatorname{erfc} z = \pi^{-1/2} \Gamma\left(\tfrac{1}{2}, z^2\right). \tag{10.70b}$$

The power series expansion of erf z follows directly from Eq. (10.61).

EXERCISES

10.4.1 Show that

$$\gamma(a, x) = e^{-x} \sum_{n=0}^{\infty} \frac{(a-1)!}{(a+n)!} x^{a+n},$$

(a) by repeatedly integrating by parts; and
(b) demonstrate this relation by transforming it into Eq. (10.61).

10.4.2 Show that

(a) $$\frac{d^m}{dx^m}[x^{-a}\gamma(a, x)] = (-1)^m x^{-a-m} \gamma(a+m, x),$$

(b) $$\frac{d^m}{dx^m}[e^x \gamma(a, x)] = e^x \frac{\Gamma(a)}{\Gamma(a-m)} \gamma(a-m, x).$$

10.4.3 Show that $\gamma(a, x)$ and $\Gamma(a, x)$ satisfy the recurrence relations
(a) $\gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x}$,
(b) $\Gamma(a-1, x) = a\Gamma(a, x) + x^a e^{-x}$.

10.4.4 The potential produced by a 1s hydrogen electron (Exercise 10.4.11) is given by

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0}\left\{\frac{1}{2r}\gamma(3, 2r) + \Gamma(2, 2r)\right\}.$$

(a) For $r \ll 1$, show that

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0}\left\{1 - \frac{2}{3}r^2 + \cdots\right\}.$$

(b) For $r \gg 1$, show that

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0} \cdot \frac{1}{r}.$$

Here, r is a pure number, the number of Bohr radii, a_0.
Note. For computation at intermediate values of r, Eqs. (10.60) are convenient.

10.4.5 The potential of a 2p hydrogen electron is found to be

$$V(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{24a_0}\left\{\frac{1}{r}\gamma(5, r) + \Gamma(4, r)\right\} - \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{120a_0}\left\{\frac{1}{r^3}\gamma(7, r) + r^2\Gamma(2, r)\right\} P_2(\cos\theta).$$

Here, r is expressed in units of a_0, the Bohr radius. $P_2(\cos\theta)$ is a Legendre polynomial (Section 11.1).
(a) For $r \ll 1$, show that

$$V(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{a_0}\left\{\frac{1}{4} - \frac{1}{120}r^2 P_2(\cos\theta) + \cdots\right\}.$$

(b) For $r \gg 1$, show that

$$V(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{a_0 r}\left\{1 - \frac{6}{r^2}P_2(\cos\theta) + \cdots\right\}.$$

10.4.6 Prove the expansion

$$\int_x^\infty \frac{e^{-t}}{t}dt = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}$$

for the exponential integral. Here, γ is the Euler–Mascheroni constant.

10.4.7 Show that $E_1(z)$ may be written as

$$E_1(z) = e^{-z}\int_0^\infty \frac{e^{-zt}}{1+t}dt.$$

Show also that we must impose the condition $|\arg z| \le \pi/2$.

10.4.8 Related to the exponential integral [Eq. (10.62)] is the function

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt.$$

Show that $E_n(x)$ satisfies the recurrence relation

$$E_{n+1}(x) = \frac{1}{n} e^{-x} - \frac{x}{n} E_n(x), \quad n = 1, 2, 3, \ldots .$$

10.4.9 With $E_n(x)$ defined in Exercise 10.4.8, show that $E_n(0) = 1/(n-1)$, $n > 1$.

10.4.10 Using the relation

$$\Gamma(a) = \gamma(a, x) + \Gamma(a, x),$$

show that if $\gamma(a, x)$ satisfies the relations of Exercise 10.4.2, then $\Gamma(a, x)$ must satisfy the same relations.

10.4.11 Calculate the potential produced by a 1s hydrogen electron (Exercise 10.4.4) (Fig. 10.9). Tabulate $V(r)/(q/4\pi\varepsilon_0 a_0)$ for $0.0 \le x \le 4.0$, x in steps of 0.1. Check your calculations for $r \ll 1$ and for $r \gg 1$ by calculating the limiting forms given in Exercise 10.4.4.

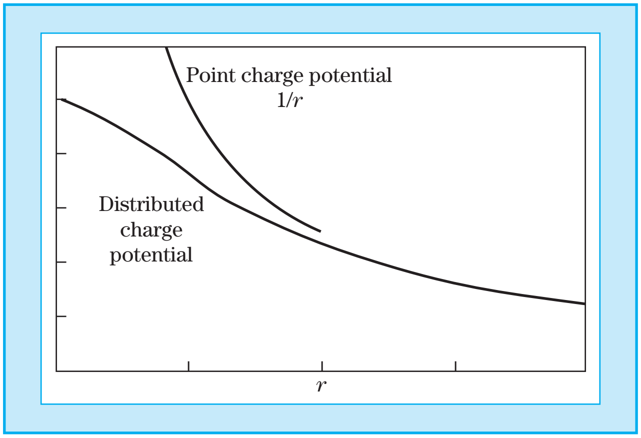


Figure 10.9

Distributed Charge Potential Produced by a 1*s* Hydrogen Electron (Exercise 10.4.11)

Additional Reading

Abramowitz, M., and Stegun, I. A. (Eds.) (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (AMS-55). National Bureau of Standards, Washington, DC. Reprinted, Dover (1974). Contains a wealth of information about gamma functions, incomplete gamma functions, exponential integrals, error functions, and related functions (Chapters 4–6).

Artin, E. (1964). *The Gamma Function* (M. Butler, Trans.). Holt, Rinehart & Winston, New York. Demonstrates that if a function $f(x)$ is smooth (log convex) and equal to $(n-1)!$ when $x = n =$ integer, it is the gamma function.

Davis, H. T. (1933). *Tables of the Higher Mathematical Functions*. Principia, Bloomington, IN. Volume 1 contains extensive information on the gamma function and the polygamma functions.

Gradshteyn, I. S., and Ryzhik, I. M. (2000). *Table of Integrals, Series, and Products*, 6th ed. Academic Press, New York.

Luke, Y. L. (1969). *The Special Functions and Their Approximations*, Vol. 1. Academic Press, New York.

Luke, Y. L. (1975). *Mathematical Functions and Their Approximations*. Academic Press, New York. This is an updated supplement to *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (AMS-55). Chapter 1 deals with the gamma function. Chapter 4 treats the incomplete gamma function and a host of related functions.