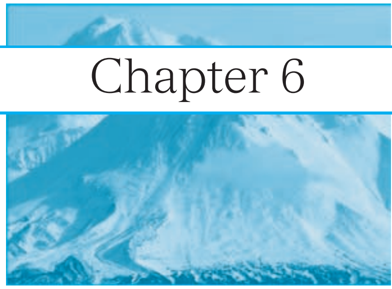


Chapter 6

Functions of a Complex Variable I

Analytic Properties Mapping

> The imaginary numbers are a wonderful flight of God's spirit; they are almost an amphibian between being and not being.
>
> —Gottfried Wilhelm von Leibniz, 1702

The theory of functions of one complex variable contains some of the most powerful and widely useful tools in all of mathematical analysis. To indicate why complex variables are important, we mention briefly several areas of application.

First, for many pairs of functions u and v, both u and v satisfy Laplace's equation in two real dimensions

$$\nabla^2 u = \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0.$$

For example, either u or v may be used to describe a two-dimensional electrostatic potential. The other function then gives a family of curves orthogonal to the equipotential curves of the first function and may be used to describe the electric field $\mathbf{E}$. A similar situation holds for the hydrodynamics of an ideal fluid in irrotational motion. The function u might describe the velocity potential, whereas the function v would then be the stream function.

In some cases in which the functions u and v are unknown, mapping or transforming complex variables permits us to create a (curved) coordinate system tailored to the particular problem.

Second, complex numbers are constructed (in Section 6.1) from pairs of real numbers so that the real number field is embedded naturally in the complex number field. In mathematical terms, the complex number field is an extension of the real number field, and the latter is complete in the sense that any polynomial of order n has n (in general) complex zeros. This fact was first proved by Gauss and is called the fundamental theorem of algebra (see Sections 6.4 and 7.2). As a consequence, real functions, infinite real series, and integrals usually can be generalized naturally to complex numbers simply by replacing a real variable x, for example, by complex z.

In Chapter 8, we shall see that the second-order differential equations of interest in physics may be solved by power series. The same power series may be used by replacing x by the complex variable z. The dependence of the solution $f(z)$ at a given z_0 on the behavior of $f(z)$ elsewhere gives us greater insight into the behavior of our solution and a powerful tool (analytic continuation) for extending the region in which the solution is valid.

Third, the change of a parameter k from real to imaginary transforms the Helmholtz equation into the diffusion equation. The same change transforms the Helmholtz equation solutions (Bessel and spherical Bessel functions) into the diffusion equation solutions (modified Bessel and modified spherical Bessel functions).

Fourth, integrals in the complex plane have a wide variety of useful applications:

- Evaluating definite integrals (in Section 7.2)
- Inverting power series
- Infinite product representations of analytic functions (in Section 7.2)
- Obtaining solutions of differential equations for large values of some variable (asymptotic solutions in Section 7.3)
- Investigating the stability of potentially oscillatory systems
- Inverting integral transforms (in Chapter 15)

Finally, many physical quantities that were originally real become complex as a simple physical theory is made more general. The real index of refraction of light becomes a complex quantity when absorption is included. The real energy associated with an energy level becomes complex, $E = m \pm i\Gamma$, when the finite lifetime of the level is considered. Electric circuits with resistance R, capacitance C, and inductance L typically lead to a complex impedance $Z = R + i(\omega L - \frac{1}{\omega C})$.

We start with complex arithmetic in Section 6.1 and then introduce complex functions and their derivatives in Section 6.2. This leads to the fundamental Cauchy integral formula in Sections 6.3 and 6.4; analytic continuation, singularities, and Taylor and Laurent expansions of functions in Section 6.5; and conformal mapping, branch point singularities, and multivalent functions in Sections 6.6 and 6.7.

6.1 Complex Algebra

As we know from practice with solving real quadratic equations for their real zeros, they often fail to yield a solution. A case in point is the following example.

EXAMPLE 6.1.1

Positive Quadratic Form For all real x

$$y(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$$

is positive definite; that is, in the real number field $y(x) = 0$ has no solutions. Of course, if we use the **symbol** $i = \sqrt{-1}$, we can formally write the solutions of $y(x) = 0$ as $\frac{1}{2}(-1 \pm i\sqrt{3})$ and check that

$$\left[\frac{1}{2}(-1 \pm i\sqrt{3})\right]^2 + \frac{1}{2}(-1 \pm i\sqrt{3}) + 1 = \frac{1}{4}(1 - 3 \mp 2i\sqrt{3} - 2 \pm 2i\sqrt{3}) + 1 = 0.$$

Although we can do arithmetic with i subject to the rule $i^2 = -1$, this symbol does not tell us what imaginary numbers really are. ■

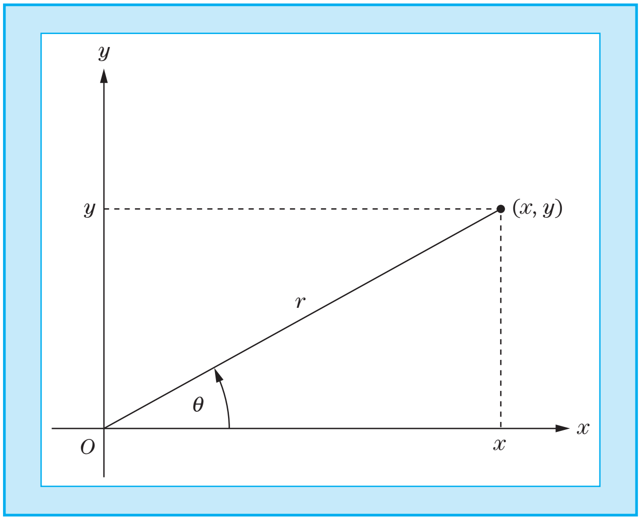
In order to make complex zeros visible we have to enlarge the real numbers on a line to complex numbers in a plane. **We define a complex number** such as a point with two coordinates in the Euclidean plane **as an ordered pair of two real numbers**, (a, b) as shown in Fig. 6.1. Similarly, **a complex variable is an ordered pair of two real variables**,

$$z \equiv (x, y). \tag{6.1}$$

The ordering is significant; x is called the **real part** of z and y the **imaginary part** of z. In general, (a, b) is not equal to (b, a) and (x, y) is not equal to (y, x). As usual, we continue writing a real number $(x, 0)$ simply as x, and call $i \equiv (0, 1)$ the imaginary unit. The x-axis is the real axis and the y-axis the imaginary axis of the complex number plane. Note that in electrical engineering the convention is $j = \sqrt{-1}$ and i is reserved for a current there. The complex numbers $\frac{1}{2}(-1 \pm i\sqrt{3})$ from Example 6.1.1 are the points $(-\frac{1}{2}, \pm\sqrt{3})$.

Figure 6.1

Complex Plane—Argand Diagram



A graphical representation is a powerful means to see a complex number or variable. By plotting x (the real part of z) as the abscissa and y (the imaginary part of z) as the ordinate, we have the complex plane or Argand plane shown in Fig. 6.1. If we assign specific values to x and y, then z corresponds to a point (x, y) in the plane. In terms of the ordering mentioned previously, it is obvious that the point (x, y) does not coincide with the point (y, x) except for the special case of $x = y$.

Complex numbers are points in the plane, and now we want to add, subtract, multiply, and divide them, just like real numbers. All our complex variable analyses can now be developed in terms of ordered pairs[1] of numbers (a, b), variables (x, y), and functions $(u(x, y), v(x, y))$.

We now define **addition** of complex numbers in terms of their Cartesian components as

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = z_2 + z_1, \tag{6.2}$$

that is, two-dimensional vector addition. In Chapter 1, the points in the xy-plane are identified with the two-dimensional displacement vector $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. As a result, two-dimensional vector analogs can be developed for much of our complex analysis. Exercise 6.1.2 is one simple example; Cauchy's theorem (Section 6.3) is another. Also, $-z + z = (-x, -y) + (x, y) = 0$ so that the negative of a complex number is uniquely specified. **Subtraction** of complex numbers then proceeds as addition: $z_1 - z_2 = (x_1 - x_2, y_1 - y_2)$.

Multiplication of complex numbers is defined as

$$z_1 z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1). \tag{6.3}$$

Using Eq. (6.3) we verify that $i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1$ so that we can also identify $i = \sqrt{-1}$ as usual, and further rewrite Eq. (6.1) as

$$z = (x, y) = (x, 0) + (0, y) = x + (0, 1) \cdot (y, 0) = x + iy. \tag{6.4}$$

Clearly, the i **is not necessary here but it is truly convenient and traditional**. It serves to keep pairs in order—somewhat like the unit vectors of vector analysis in Chapter 1.

With complex numbers at our disposal, we can determine the complex zeros of $z^2 + z + 1 = 0$ in Example 6.1.1 as $z = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}$ so that

$$z^2 + z + 1 = \left(z + \frac{1}{2} - \frac{i}{2}\sqrt{3}\right)\left(z + \frac{1}{2} + \frac{i}{2}\sqrt{3}\right)$$

factorizes completely.

Complex Conjugation

The operation of replacing i by $-i$ is called "taking the complex conjugate." The complex conjugate of z is denoted by z^*,[2] where

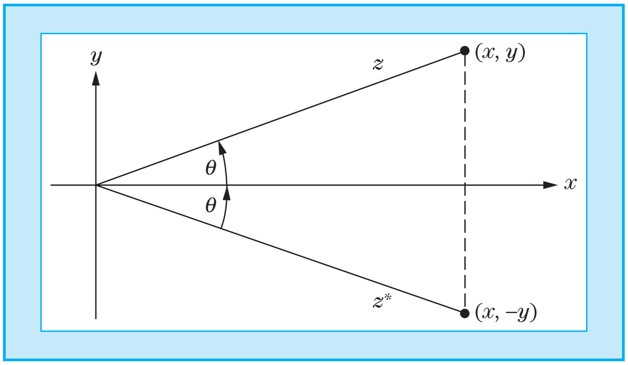
$$z^* = x - iy. \tag{6.5}$$

[1] This is precisely how a computer does complex arithmetic.

[2] The complex conjugate is often denoted by $\bar{z}$ in the mathematical literature.

Figure 6.2

Complex Conjugate Points



The complex variable z and its complex conjugate z^* are mirror images of each other reflected in the x-axis; that is, inversion of the y-axis (compare Fig. 6.2). The product zz^* leads to

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 = r^2. \tag{6.6}$$

Hence,

$$(zz^*)^{1/2} = |z|$$

is defined as the **magnitude or modulus** of z.

Division of complex numbers is most easily accomplished by replacing the denominator by a positive number as follows:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1 x_2 + y_1 y_2,\ x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}, \tag{6.7}$$

which displays its real and imaginary parts as ratios of real numbers with the same positive denominator. Here, $|z_2|^2 = x_2^2 + y_2^2$ is the **absolute value (squared)** of z_2, and $z_2^* = x_2 - iy_2$ is called the **complex conjugate** of z_2. We write $|z_2|^2 = z_2^* z_2$, which is the squared length of the associated Cartesian vector in the complex plane.

Furthermore, from Fig. 6.1 we may write in plane polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta \tag{6.8}$$

and

$$z = r(\cos\theta + i\sin\theta). \tag{6.9}$$

In this representation r is the **modulus or magnitude** of

$$z\,(r = |z| = (x^2 + y^2)^{1/2})$$

and the angle θ $(= \tan^{-1}(y/x))$ is labeled the **argument or phase** of z. Using a result that is suggested (but not rigorously proved)[3] by Section 5.6, we have the very useful polar representation

$$z = re^{i\theta}. \tag{6.10}$$

In order to prove this identity, we use $i^3 = -i$, $i^4 = 1$, etc. in the Taylor expansion of the exponential and trigonometric functions and separate even and odd powers in

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{\nu=0}^{\infty} \frac{(i\theta)^{2\nu}}{(2\nu)!} + \sum_{\nu=0}^{\infty} \frac{(i\theta)^{2\nu+1}}{(2\nu+1)!} \\ &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\theta^{2\nu}}{(2\nu)!} + i \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\theta^{2\nu+1}}{(2\nu+1)!} = \cos\theta + i \sin\theta. \end{aligned} \tag{6.11}$$

For the special values $\theta = \pi/2$, and $\theta = \pi$, we obtain

$$e^{i\pi/2} = \cos(\pi/2) + i \sin(\pi/2) = i, \quad e^{i\pi} = \cos(\pi) = -1,$$

intriguing connections between e, i, and π. Moreover, the exponential function $e^{i\theta}$ is periodic with period 2π, just like $\sin\theta$ and $\cos\theta$. As an immediate application we can derive the trigonometric addition rules from

$$\begin{aligned} \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) &= e^{i(\theta_1+\theta_2)} \\ &= e^{i\theta_1} e^{i\theta_2} = [\cos\theta_1 + i \sin\theta_1][\cos\theta_2 + i \sin\theta_2] \\ &= \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \\ &\quad + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1). \end{aligned}$$

Let us now convert a ratio of complex numbers to polar form explicitly.

EXAMPLE 6.1.2

Conversion to Polar Form We start by converting the denominator of a ratio to a real number:

$$\frac{2+i}{3-2i} = \frac{(2+i)(3+2i)}{3^2+2^2} = \frac{6-2+i(3+4)}{13} = \frac{4+7i}{13} = \sqrt{\frac{5}{13}}\, e^{i\theta_0},$$

where $\frac{4^2+7^2}{13^2} = \frac{65}{13^2} = \frac{5}{13}$ and $\tan\theta_0 = \frac{7}{4}$. Because $\arctan(\theta)$ has two branches in the range from zero to 2π, we pick the solution $\theta_0 = 60.255^\circ$, $0 < \theta_0 < \pi/2$, because the second solution $\theta_0 + \pi$ gives $e^{i(\theta_0+\pi)} = -e^{i\theta_0}$ (i.e., the wrong sign).

Alternately, we can convert $2 + i = \sqrt{5}e^{i\alpha}$ and $3 - 2i = \sqrt{13}e^{i\beta}$ to polar form with $\tan\alpha = \frac{1}{2}$, $\tan\beta = -\frac{2}{3}$ and then divide them to get

$$\frac{2+i}{3-2i} = \sqrt{\frac{5}{13}}\, e^{i(\alpha-\beta)}. \quad \blacksquare$$

[3] Strictly speaking, Chapter 5 was limited to real variables. However, we can define e^z as $\sum_{n=0}^{\infty} z^n/n!$ for complex z. The development of power-series expansions for complex functions is taken up in Section 6.5 (Laurent expansion).

The choice of polar representation [Eq. (6.10)] or Cartesian representation [Eqs. (6.1) and (6.4)] is a matter of convenience. Addition and subtraction of complex variables are easier in the Cartesian representation [Eq. (6.2)]. Multiplication, division, powers, and roots are easier to handle in polar form [Eqs. (6.8)–(6.10)].

Let us examine the geometric meaning of multiplying a function by a complex constant.

EXAMPLE 6.1.3

Multiplication by a Complex Number When we multiply the complex variable z by $i = e^{i\pi/2}$, for example, it is rotated counterclockwise by 90° to $iz = ix - y = (-y, x)$. When we multiply $z = re^{i\theta}$ by $e^{i\alpha}$, we get $re^{i(\theta+\alpha)}$, which is z rotated by the angle α.

Similarly, curves defined by $f(z) = \text{const.}$ are rotated when we multiply a function by a complex constant. When we set

$$f(z) = (x + iy)^2 = (x^2 - y^2) + 2ixy = c = c_1 + ic_2 = \text{const.},$$

we define two hyperbolas

$$x^2 - y^2 = c_1, \quad 2xy = c_2.$$

Upon multiplying $f(z) = c$ by a complex number $Ae^{i\alpha}$, we obtain

$$\begin{aligned} Ae^{i\alpha} f(z) &= A(\cos\alpha + i\sin\alpha)(x^2 - y^2 + 2ixy) \\ &= A[i(2xy\cos\alpha + (x^2 - y^2)\sin\alpha) - 2xy\sin\alpha + (x^2 - y^2)\cos\alpha]. \end{aligned}$$

The hyperbolas are scaled by the modulus A and rotated by the angle α. ■

Analytically or graphically, using the vector analogy, we may show that the modulus of the sum of two complex numbers is no greater than the sum of the moduli and no less than the difference (Exercise 6.1.2):

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|. \tag{6.12}$$

Because of the vector analogy, these are called the **triangle** inequalities.

Using the polar form [Eq. (6.8)] we find that the magnitude of a product is the product of the magnitudes,

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|. \tag{6.13}$$

Also,

$$\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2. \tag{6.14}$$

From our complex variable z complex functions $f(z)$ or $w(z)$ may be constructed. These complex functions may then be resolved into real and imaginary parts

$$w(z) = u(x, y) + iv(x, y), \tag{6.15}$$

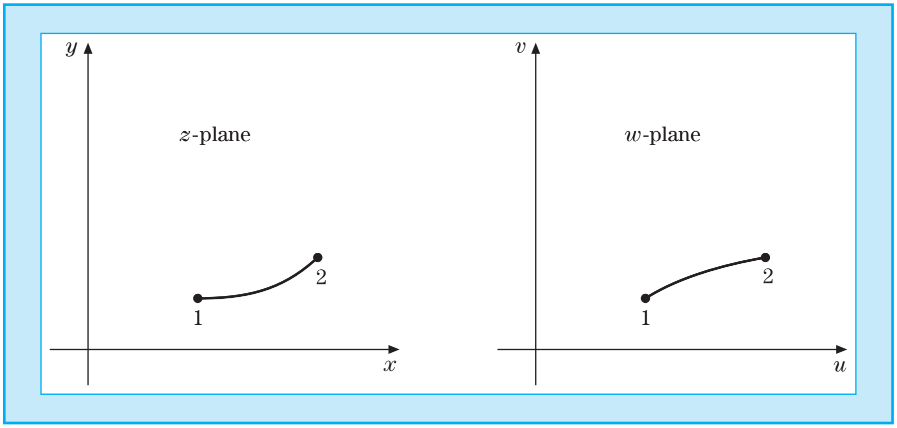


Figure 6.3

The Function $w(z) = u(x, y) + iv(x, y)$ Maps Points in the xy-Plane into Points in the uv-Plane

in which the separate functions $u(x, y)$ and $v(x, y)$ are pure real. For example, if $f(z) = z^2$, we have

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

The **real part** of a function $f(z)$ will be labeled $\Re f(z)$, whereas the **imaginary part** will be labeled $\Im f(z)$. In Eq. (6.15),

$$\Re w(z) = u(x, y), \qquad \Im w(z) = v(x, y). \tag{6.16}$$

The relationship between the independent variable z and the dependent variable w is perhaps best pictured as a mapping operation. A given $z = x + iy$ means a given point in the z-plane. The complex value of $w(z)$ is then a point in the w-plane. Points in the z-plane map into points in the w-plane, and curves in the z-plane map into curves in the w-plane, as indicated in Fig. 6.3.

Functions of a Complex Variable

All the elementary (real) functions of a real variable may be extended into the complex plane, replacing the real variable x by the complex variable z. This is an example of the analytic continuation mentioned in Section 6.5. The extremely important relations, Eqs. (6.4), (6.8), and (6.9), are illustrations. Moving into the complex plane opens up new opportunities for analysis.

EXAMPLE 6.1.4

De Moivre's Formula If Eq. (6.11) is raised to the nth power, we have

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n. \tag{6.17}$$

Using Eq. (6.11) now with argument $n\theta$, we obtain

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n. \tag{6.18}$$

This is De Moivre's formula.

Now if the right-hand side of Eq. (6.18) is expanded by the binomial theorem, we obtain $\cos n\theta$ as a series of powers of $\cos\theta$ and $\sin\theta$ (Exercise 6.1.5). Numerous other examples of relations among the exponential, hyperbolic, and trigonometric functions in the complex plane appear in the exercises. ■

Occasionally, there are complications. Taking the nth **root** of a complex number $z = re^{i\theta}$ gives $z^{1/n} = r^{1/n}e^{i\theta/n}$. This root is not the only solution, though, because $z = re^{i(\theta+2m\pi)}$ for any integer m yields $n-1$ additional roots $z^{1/n} = r^{1/n}e^{i\theta/n+2im\pi/n}$ for $m = 1, 2, \ldots, n-1$. Therefore, taking the nth **root is a multivalued function** or operation with n values, for a given complex number z. Let us look at a numerical example.

EXAMPLE 6.1.5

Square Root When we take the square root of a complex number of argument θ we get $\theta/2$. Starting from -1, which is $r = 1$ at $\theta = 180^\circ$, we end up with $r = 1$ at $\theta = 90^\circ$, which is i, or we get $\theta = -90^\circ$, which is $-i$ upon taking the root of $-1 = e^{-i\pi}$. Here is a more complicated ratio of complex numbers:

$$\sqrt{\frac{3-i}{4+2i}} = \sqrt{\frac{(3-i)(4-2i)}{4^2+2^2}} = \sqrt{\frac{12-2-i(4+6)}{20}} = \sqrt{\frac{1}{2}(1-i)}$$

$$= \sqrt{\frac{1}{\sqrt{2}}e^{-i(\pi/4-2n\pi)}} = \frac{1}{2^{1/4}}e^{-i\pi/8+in\pi} = \frac{\pm 1}{2^{1/4}}e^{-i\pi/8}$$

for $n = 0, 1$. ■

Another example is the logarithm of a complex variable z that may be expanded using the polar representation

$$\ln z = \ln re^{i\theta} = \ln r + i\theta. \tag{6.19}$$

Again, this is not complete due to the multiple branches of the inverse tangent function. To the phase angle, θ, we may add any integral multiple of 2π without changing z due to the period 2π of the tangent function. Hence, Eq. (6.19) should read

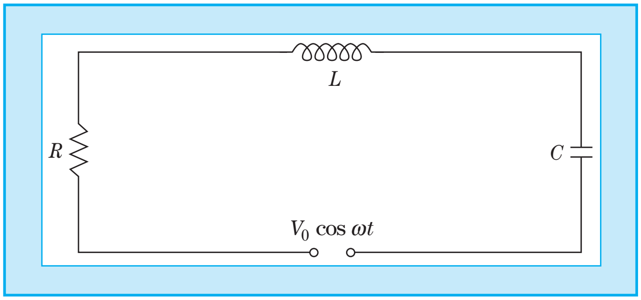
$$\ln z = \ln re^{i(\theta+2n\pi)} = \ln r + i(\theta + 2n\pi). \tag{6.20}$$

The parameter n may be any integer. This means that $\ln z$ is a **multivalued** function having an infinite number of values for a single pair of real values r and θ. To avoid ambiguity, we usually agree to set $n = 0$ and limit the phase to an interval of length 2π such as $(-\pi, \pi)$.[4] The line in the z-plane that is not crossed, the negative real axis in this case, is labeled a **cut line**. The value of $\ln z$ with $n = 0$ is called the **principal value** of $\ln z$. Further discussion of these functions, including the logarithm, appears in Section 6.6.

[4]There is no standard choice of phase: The appropriate phase depends on each problem.

Figure 6.4

Electric RLC Circuit with Alternating Current



EXAMPLE 6.1.6

Electric Circuits An electric circuit with a current I flowing through a resistor and driven by a voltage V is governed by Ohm's law, $V = IR$, where R is the resistance. If the resistance is replaced by an inductance L, then the voltage and current are related by $V = L\frac{dI}{dt}$. If the inductance is replaced by the capacitance C, then the voltage depends on the charge Q of the capacitor: $V = \frac{Q}{C}$. Taking the time derivative yields $C\frac{dV}{dt} = \frac{dQ}{dt} = I$. Therefore, a circuit with a resistor, a coil, and a capacitor in series (see Fig. 6.4) obeys the ordinary differential equation

$$L\frac{dI}{dt} + \frac{Q}{C} + IR = V = V_0 \cos \omega t \tag{6.21}$$

if it is driven by an alternating voltage with frequency ω. In electrical engineering it is a convention and tradition to use the complex voltage $V = V_0 e^{i\omega t}$ and a current $I = I_0 e^{i\omega t}$ of the same form, which is the steady-state solution of Eq. (6.21). This complex form will make the phase difference between current and voltage manifest. At the end, the physically observed values are taken to be the real parts (i.e., $V_0 \cos \omega t = V_0 \Re e^{i\omega t}$, etc.). If we substitute the exponential time dependence, use $dI/dt = i\omega I$, and integrate I once to get $Q = I/i\omega$ in Eq. (6.21), we find the following **complex form of Ohm's law**:

$$i\omega LI + \frac{I}{i\omega C} + RI = V \equiv ZI.$$

We define $Z = R + i(\omega L - \frac{1}{\omega C})$ as the impedance, a complex number, obtaining $V = ZI$, as shown.

More complicated electric circuits can now be constructed using the impedance alone—that is, without solving Eq. (6.21) anymore—according to the following combination rules:

- The resistance R of two resistors in series is $R = R_1 + R_2$.
- The inductance L of two inductors in series is $L = L_1 + L_2$.
- The resistance R of two parallel resistors obeys $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.
- The inductance L of two parallel inductors obeys $\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$.
- The capacitance of two capacitors in series obeys $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$.
- The capacitance of two parallel capacitors obeys $C = C_1 + C_2$.

In complex form these rules can be stated in a more compact form as follows:

- Two impedances in series combine as $Z = Z_1 + Z_2$;
- Two parallel impedances combine as $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$. ■

SUMMARY

Complex numbers extend the real number axis to the complex number plane so that any polynomial can be completely factored. Complex numbers add and subtract like two-dimensional vectors in Cartesian coordinates:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2).$$

They are best multiplied or divided in polar coordinates of the complex plane:

$$(x_1 + iy_1)(x_2 + iy_2) = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}, \ r_j^2 = x_j^2 + y_j^2, \ \tan\theta_j = y_j/x_j.$$

The complex exponential function is given by $e^z = e^x(\cos y + i \sin y)$. For $z = x + i0 = x$, $e^z = e^x$. The trigonometric functions become

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y,$$

$$\sin z = \frac{1}{2}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y.$$

The hyperbolic functions become

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) = \cos iz, \quad \sinh z = \frac{1}{2}(e^z - e^{-z}) = -i \sin iz.$$

The natural logarithm generalizes to $\ln z = \ln |z| + i(\theta + 2\pi n)$, $n = 0, \pm 1, \ldots$ and general powers are defined as $z^p = e^{p \ln z}$.

EXERCISES

6.1.1 (a) Find the reciprocal of $x + iy$, working entirely in the Cartesian representation.

(b) Repeat part (a), working in polar form but expressing the final result in Cartesian form.

6.1.2 Prove algebraically that

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Interpret this result in terms of vectors. Prove that

$$|z - 1| < |\sqrt{z^2 - 1}| < |z + 1|, \quad \text{for} \quad \Re(z) > 0.$$

6.1.3 We may define a complex conjugation operator K such that $Kz = z^*$. Show that K is not a linear operator.

6.1.4 Show that complex numbers have square roots and that the square roots are contained in the complex plane. What are the square roots of i?

6.1.5 Show that

(a) $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \cdots$.

(b) $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots$.

Note. The quantities $\binom{n}{m}$ are the binomial coefficients (Chapter 5): $\binom{n}{m} = n!/[(n-m)!m!]$.

6.1.6 Show that

(a) $$\sum_{n=0}^{N-1}\cos nx = \frac{\sin(Nx/2)}{\sin x/2}\cos(N-1)\frac{x}{2},$$

(b) $$\sum_{n=0}^{N-1}\sin nx = \frac{\sin(Nx/2)}{\sin x/2}\sin(N-1)\frac{x}{2}.$$

Hint. Parts (a) and (b) may be combined to form a geometric series (compare Section 5.1).

6.1.7 For $-1 < p < 1$, show that

(a) $$\sum_{n=0}^{\infty} p^n\cos nx = \frac{1 - p\cos x}{1 - 2p\cos x + p^2},$$

(b) $$\sum_{n=0}^{\infty} p^n\sin nx = \frac{p\sin x}{1 - 2p\cos x + p^2}.$$

These series occur in the theory of the Fabry–Perot interferometer.

6.1.8 Assume that the trigonometric functions and the hyperbolic functions are defined for complex argument by the appropriate power series

$$\begin{aligned}
\sin z &= \sum_{n=1,\text{odd}}^{\infty}(-1)^{(n-1)/2}\frac{z^n}{n!} = \sum_{s=0}^{\infty}(-1)^s\frac{z^{2s+1}}{(2s+1)!},\\
\cos z &= \sum_{n=0,\text{even}}^{\infty}(-1)^{n/2}\frac{z^n}{n!} = \sum_{s=0}^{\infty}(-1)^s\frac{z^{2s}}{(2s)!},\\
\sinh z &= \sum_{n=1,\text{odd}}^{\infty}\frac{z^n}{n!} = \sum_{s=0}^{\infty}\frac{z^{2s+1}}{(2s+1)!},\\
\cosh z &= \sum_{n=0,\text{even}}^{\infty}\frac{z^n}{n!} = \sum_{s=0}^{\infty}\frac{z^{2s}}{(2s)!}.
\end{aligned}$$

(a) Show that

$$i\sin z = \sinh iz,\ \sin iz = i\sinh z,$$
$$\cos z = \cosh iz,\ \cos iz = \cosh z.$$

(b) Verify that familiar functional relations such as

$$\cosh z = \frac{e^z + e^{-z}}{2},\ \sin(z_1 + z_2) = \sin z_1\cos z_2 + \sin z_2\cos z_1$$

still hold in the complex plane.

6.1.9 Using the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

established from comparison of power series, show that

(a) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$
$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y,$

(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y.$

This demonstrates that we may have $|\sin z|, |\cos z| > 1$ in the complex plane.

6.1.10 From the identities in Exercises 6.1.8 and 6.1.9, show that

(a) $\sinh(x + iy) = \sinh x \cos y + i\ \cosh x \sin y,$
$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$

(b) $|\sinh z|^2 = \sinh^2 x + \sin^2 y, \quad |\cosh z|^2 = \sinh^2 x + \cos^2 y.$

6.1.11 Prove that
(a) $|\sin z| \geq |\sin x|,$ (b) $|\cos z| \geq |\cos x|.$

6.1.12 Show that the exponential function e^z is periodic with a pure imaginary period of $2\pi i$.

6.1.13 Show that

(a) $\tanh(z/2) = \dfrac{\sinh x + i \sin y}{\cosh x + \cos y},$ (b) $\coth(z/2) = \dfrac{\sinh x - i \sin y}{\cosh x - \cos y}.$

6.1.14 Find all the zeros of
(a) $\sin z$, (b) $\cos z$, (c) $\sinh z$, (d) $\cosh z$.

6.1.15 Show that

(a) $\sin^{-1} z = -i \ln(iz \pm \sqrt{1 - z^2}),$ (d) $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1}),$

(b) $\cos^{-1} z = -i \ln(z \pm \sqrt{z^2 - 1}),$ (e) $\cosh^{-1} z = \ln(z + \sqrt{z^2 - 1}),$

(c) $\tan^{-1} z = \dfrac{i}{2} \ln\left(\dfrac{i + z}{i - z}\right),$ (f) $\tanh^{-1} z = \dfrac{1}{2} \ln\left(\dfrac{1 + z}{1 - z}\right).$

Hint. 1. Express the trigonometric and hyperbolic functions in terms of exponentials. 2. Solve for the exponential and then for the exponent. Note that $\sin^{-1} z = \arcsin z \neq (\sin z)^{-1}$, etc.

6.1.16 A plane wave of light of angular frequency ω is represented by

$$e^{i\omega(t - nx/c)}.$$

In a certain substance the simple real index of refraction n is replaced by the complex quantity $n - ik$. What is the effect of k on the wave? What does k correspond to physically? The generalization of a quantity from real to complex form occurs frequently in physics. Examples range from the complex Young's modulus of viscoelastic materials to the complex (optical) potential of the "cloudy crystal ball" model of the atomic nucleus. See the chapter on the optical model in M. A. Preston, *Structure of the Nucleus*. Addison-Wesley, Reading, MA (1993).

6.1.17 A damped simple harmonic oscillator is driven by the complex external force $Fe^{i\omega t}$. Show that the steady-state amplitude is given by

$$A = \frac{F}{m(\omega_0^2 - \omega^2) + i\omega b}.$$

Explain the resonance condition and relate m, ω_0, b to the oscillator parameters.
Hint. Find a complex solution $z(t) = Ae^{i\omega t}$ of the ordinary differential equation.

6.1.18 We see that for the angular momentum components defined in Exercise 2.5.10,

$$L_x - iL_y \neq (L_x + iL_y)^*.$$

Explain why this happens.

6.1.19 Show that the **phase** of $f(z) = u + iv$ is equal to the imaginary part of the logarithm of $f(z)$. Exercise 10.2.13 depends on this result.

6.1.20 (a) Show that $e^{\ln z}$ always equals z.
(b) Show that $\ln e^z$ does not always equal z.

6.1.21 Verify the consistency of the combination rules of impedances with those of resistances, inductances, and capacitances by considering circuits with resistors only, etc. Derive the combination rules from Kirchhoff's laws. Describe the origin of Kirchhoff's laws.

6.1.22 Show that negative numbers have logarithms in the complex plane. In particular, find ln(−1).

ANS. $\ln(-1) = i\pi$.

6.2 Cauchy–Riemann Conditions

Having established complex functions of a complex variable, we now proceed to differentiate them. The derivative of $f(z) = u(x, y) + iv(x, y)$, like that of a real function, is defined by

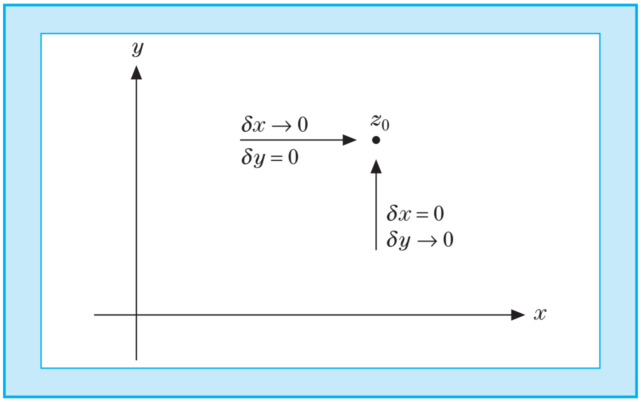
$$\lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{z + \delta z - z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z), \tag{6.22}$$

provided that the limit is **independent** of the particular approach to the point z. For real variables we require that the right-hand limit ($x \to x_0$ from above) and the left-hand limit ($x \to x_0$ from below) be equal for the derivative $df(x)/dx$ to exist at $x = x_0$. Now, with z (or z_0) some point in a plane, our requirement that the limit be independent of the direction of approach is very restrictive. Consider increments δx and δy of the variables x and y, respectively. Then

$$\delta z = \delta x + i\delta y. \tag{6.23}$$

Figure 6.5

Alternate Approaches to z_0



Also,

$$\delta f = \delta u + i\delta v \tag{6.24}$$

so that

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}. \tag{6.25}$$

Let us take the limit indicated by Eq. (6.23) by two different approaches as shown in Fig. 6.5. First, with $\delta y = 0$, we let $\delta x \to 0$. Equation (6.24) yields

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{6.26}$$

assuming the partial derivatives exist. For a second approach, we set $\delta x = 0$ and then let $\delta y \to 0$. This leads to

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta y\to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{6.27}$$

If we are to have a derivative df/dz, Eqs. (6.26) and (6.27) must be identical. Equating real parts to real parts and imaginary parts to imaginary parts (like components of vectors), we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6.28}$$

These are the famous **Cauchy–Riemann** conditions. They were discovered by Cauchy and used extensively by Riemann in his theory of analytic functions. These Cauchy–Riemann conditions are necessary for the existence of a derivative of $f(z)$; that is, if df/dz exists, the Cauchy–Riemann conditions must hold. They may be **interpreted geometrically** as follows. Let us write them as a product of ratios of partial derivatives

$$\frac{u_x}{u_y}\cdot\frac{v_x}{v_y} = -1, \tag{6.29}$$

with the abbreviations

$$\frac{\partial u}{\partial x} \equiv u_x, \qquad \frac{\partial u}{\partial y} \equiv u_y, \qquad \frac{\partial v}{\partial x} \equiv v_x, \qquad \frac{\partial v}{\partial y} \equiv v_y.$$

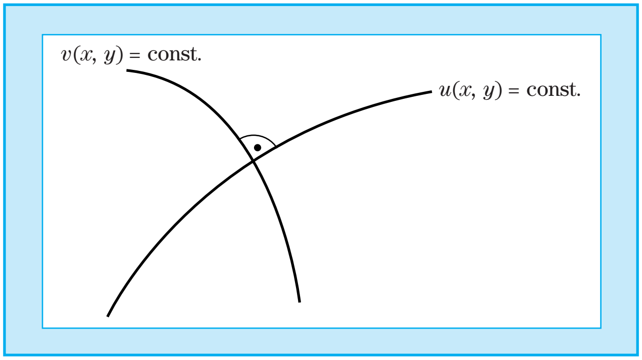


Figure 6.6

Orthogonal Tangents to $u(x, y) =$ const. $v(x, y) =$ const. Lines

Now recall the geometric meaning of $-u_x/u_y$ as the slope of the tangent [see Eq. (1.54)] of each curve $u(x, y) =$ const. and similarly for $v(x, y) =$ const. (Fig. 6.6). Thus, Eq. (6.29) means that the $u =$ const. and $v =$ const. curves are mutually orthogonal at each intersection because $\sin\beta = \sin(\alpha + 90^\circ) = \cos\alpha$ and $\cos\beta = -\sin\alpha$ imply $\tan\beta \cdot \tan\alpha = -1$ by taking the ratio. Alternatively,

$$u_x\, dx + u_y\, dy = 0 = v_y\, dx - v_x\, dy$$

states that if (dx, dy) is tangent to the u-curve, then the orthogonal $(-dy, dx)$ is tangent to the v-curve at the intersection point $z = (x, y)$. Equivalently, $u_x v_x + u_y v_y = 0$ implies that the **gradient vectors** (u_x, u_y) and (v_x, v_y) **are perpendicular**. Conversely, if the Cauchy–Riemann conditions are satisfied and the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, the derivative df/dz exists. This may be shown by writing

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y. \tag{6.30}$$

The justification for this expression depends on the continuity of the partial derivatives of u and v. Dividing by δz, we have

$$\begin{aligned}\frac{\delta f}{\delta z} &= \frac{(\partial u/\partial x + i(\partial v/\partial x))\delta x + (\partial u/\partial y + i(\partial u/\partial y))\delta y}{\delta x + i\delta y} \\ &= \frac{(\partial u/\partial x + i(\partial v/\partial x)) + (\partial u/\partial y + i(\partial v/\partial y))\delta y/\delta x}{1 + i(\delta y/\delta x)}.\end{aligned} \tag{6.31}$$

If $\delta f/\delta z$ is to have a unique value, the dependence on $\delta y/\delta x$ must be eliminated. Applying the Cauchy–Riemann conditions to the y derivatives, we obtain

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}. \tag{6.32}$$

Substituting Eq. (6.32) into Eq. (6.30), we may rewrite the δy and δx dependence as $\delta z = \delta x + i\delta y$ and obtain

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x},$$

which shows that $\lim \delta f/\delta z$ is independent of the direction of approach in the complex plane as long as the partial derivatives are continuous.

It is worthwhile to note that the Cauchy–Riemann conditions guarantee that the curves $u = c_1 =$ constant will be orthogonal to the curves $v = c_2 =$ constant (compare Section 2.1). This property is fundamental in application to potential problems in a variety of areas of physics. If $u = c_1$ is a line of electric force, then $v = c_2$ is an equipotential line (surface) and vice versa. Also, it is easy to show from Eq. (6.28) that both u and v satisfy Laplace's equation. A further implication for potential theory is developed in Exercise 6.2.1.

We have already generalized the elementary functions to the complex plane by replacing the real variable x by complex z. Let us now check that their derivatives are the familiar ones.

EXAMPLE 6.2.1

Derivatives of Elementary Functions We define the elementary functions by their Taylor expansions (see Section 5.6, with $x \to z$, and Section 6.5)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}.$$

We differentiate termwise [which is justified by absolute convergence for e^z, $\cos z$, $\sin z$ for all z and for $\ln(1+z)$ for $|z| < 1$] and see that

$$\frac{d}{dz} z^n = \lim_{\delta z \to 0} \frac{(z+\delta z)^n - z^n}{\delta z}$$

$$= \lim_{\delta z \to 0} [z^n + n z^{n-1}\delta z + \cdots + (\delta z)^n - z^n]/\delta z = n z^{n-1},$$

$$\frac{de^z}{dz} = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = e^z,$$

$$\frac{d \sin z}{dz} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1) z^{2n}}{(2n+1)!} = \cos z,$$

$$\frac{d \cos z}{dz} = \sum_{n=1}^{\infty} (-1)^n \frac{2n\, z^{2n-1}}{(2n)!} = -\sin z,$$

$$\frac{d \ln(1+z)}{dz} = \frac{d}{dz} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} z^{n-1} = \frac{1}{1+z},$$

that is, the real derivative results all generalize to the complex field, simply replacing $x \to z$. ■

Biographical Data

Riemann, Bernhard Georg Friedrich. Riemann, a German mathematician, was born in 1826 in Hannover and died of tuberculosis in 1866 in Selasca, Italy. Son of a Lutheran pastor, he changed from studying theology to mathematics at the University of Göttingen where, in 1851, his Ph.D. thesis was approved by Gauss. He contributed to many branches of mathematics despite dying before the age of 40, the most famous being the development of metric (curved) spaces from their intrinsic geometric properties such as curvature. This topic was the subject of his Habilitation thesis, or venia legendi, which Gauss attended and deeply impressed him. Half a century later Riemannian geometry would become the basis for Einstein's General Relativity. Riemann's profound analysis of the complex zeta function laid the foundations for the first proof of the prime number theorem in 1898 by French mathematicians J. Hadamard and C. de la Vallée-Poussin and other significant advances in the theory of analytic functions of one complex variable. His hypothesis about the distribution of the nontrivial zeros of the zeta function, with many consequences in analytic prime number theory, remains the most famous unsolved problem in mathematics today.

Analytic Functions

Finally, if $f(z)$ is differentiable at $z = z_0$ and in some small region around z_0, we say that $f(z)$ is **analytic**[5] at $z = z_0$. If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an **entire** function. Our theory of complex variables is one of analytic functions of a complex variable, which indicates the crucial importance of the Cauchy–Riemann conditions. The concept of analyticity used in advanced theories of modern physics plays a crucial role in dispersion theory (of elementary particles or light). If $f'(z)$ does not exist at $z = z_0$, then z_0 is labeled a singular point and consideration of it is postponed until Section 7.1.

To illustrate the Cauchy–Riemann conditions, consider two very simple examples.

EXAMPLE 6.2.2

Let $f(z) = z^2$. Then the real part $u(x, y) = x^2 - y^2$ and the imaginary part $v(x, y) = 2xy$. Following Eq. (6.28),

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

We see that $f(z) = z^2$ satisfies the Cauchy–Riemann conditions throughout the complex plane. Since the partial derivatives are clearly continuous, we conclude that $f(z) = z^2$ is analytic. ■

[5]Some writers use the terms **holomorphic** or **regular**.

EXAMPLE 6.2.3

Let $f(z) = z^*$. Now $u = x$ and $v = -y$. Applying the Cauchy–Riemann conditions, we obtain

$$\frac{\partial u}{\partial x} = 1, \quad \text{whereas} \quad \frac{\partial v}{\partial y} = -1.$$

The Cauchy–Riemann conditions are not satisfied and $f(z) = z^*$ is not an analytic function of z. It is interesting to note that $f(z) = z^*$ is continuous, thus providing an example of a function that is everywhere continuous but nowhere differentiable. ■

SUMMARY

The derivative of a real function of a real variable is essentially a local characteristic in that it provides information about the function only in a local neighborhood—for instance, as a truncated Taylor expansion. The existence of a derivative of a function of a complex variable has much more far-reaching implications. The real and imaginary parts of analytic functions must separately satisfy Laplace's equation. This is Exercise 6.2.1. Furthermore, an analytic function is guaranteed derivatives of all orders (Section 6.4). In this sense the derivative not only governs the local behavior of the complex function but also controls the distant behavior.

EXERCISES

6.2.1 The functions $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of an analytic function $w(z)$.

(a) Assuming that the required derivatives exist, show that

$$\nabla^2 u = \nabla^2 v = 0.$$

Solutions of Laplace's equation, such as $u(x, y)$ and $v(x, y)$, are called **harmonic** functions.

(b) Show that

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0$$

and give a geometric interpretation.

Hint. The technique of Section 1.5 allows you to construct vectors normal to the curve $u(x, y) = c_i$ and $v(x, y) = c_j$.

6.2.2 Show whether or not the function $f(z) = \Re(z) = x$ is analytic.

6.2.3 Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that $u(x, y)$ and $v(x, y)$ **cannot both have either a maximum or a minimum** in the interior of any region in which $w(z)$ is analytic. (They can have **saddle points**; see Section 7.3.)

6.2.4 Let $A = \partial^2 w/\partial x^2$, $B = \partial^2 w/\partial x \partial y$, $C = \partial^2 w/\partial y^2$. From the calculus of functions of two variables, $w(x, y)$, we have a **saddle point** if

$$B^2 - AC > 0.$$

With $f(z) = u(x, y) + iv(x, y)$, apply the Cauchy–Riemann conditions and show that both $u(x, y)$ and $v(x, y)$ do not have **a maximum or a minimum** in a finite region of the complex plane. (See also Section 7.3.)

6.2.5 Find the analytic function

$$w(z) = u(x, y) + iv(x, y)$$

if (a) $u(x, y) = x^3 - 3xy^2$, (b) $v(x, y) = e^{-y} \sin x$.

6.2.6 If there is some common region in which $w_1 = u(x, y) + iv(x, y)$ and $w_2 = w_1^* = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

6.2.7 The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $f^*(z^*)$ is also analytic.

6.2.8 A proof of the Schwarz inequality (Section 9.4) involves minimizing an expression

$$f = \psi_{aa} + \lambda\psi_{ab} + \lambda^*\psi_{ab}^* + \lambda\lambda^*\psi_{bb} \geq 0.$$

The ψ are integrals of products of functions; ψ_{aa} and ψ_{bb} are real, ψ_{ab} is complex, and λ is a complex parameter.

(a) Differentiate the preceding expression with respect to λ^*, treating λ as an independent parameter, independent of λ^*. Show that setting the derivative $\partial f/\partial\lambda^*$ equal to zero yields

$$\lambda = -\psi_{ab}^*/\psi_{bb}.$$

(b) Show that $\partial f/\partial\lambda = 0$ leads to the same result.

(c) Let $\lambda = x + iy$, $\lambda^* = x - iy$. Set the x and y derivatives equal to zero and show that again

$$\lambda = \psi_{ab}^*/\psi_{bb}.$$

6.2.9 The function $f(z)$ is analytic. Show that the derivative of $f(z)$ with respect to z^* does not exist unless $f(z)$ is a constant.
Hint. Use the chain rule and take $x = (z + z^*)/2$, $y = (z - z^*)/2i$.
Note. This result emphasizes that our analytic function $f(z)$ is not just a complex function of two real variables x and y. It is a function of the complex variable $x + iy$.

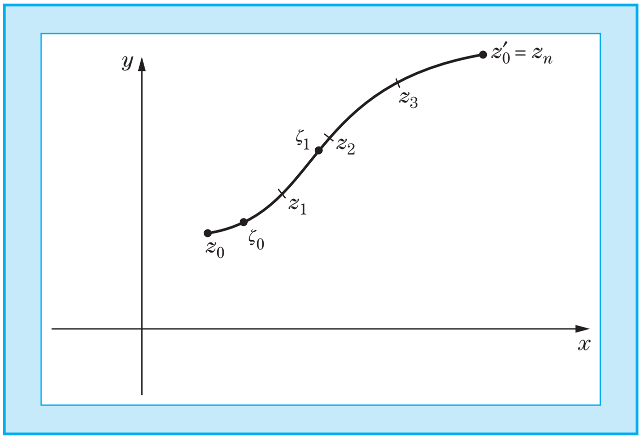
6.3 Cauchy's Integral Theorem

Contour Integrals

With differentiation under control, we turn to integration. The integral of a complex variable over a contour in the complex plane may be defined in close analogy to the (Riemann) integral of a real function integrated along the real x-axis and line integrals of vectors in Chapter 1. The contour integral may be

Figure 6.7

Integration Path



defined by

$$\int_{z_1}^{z_2} f(z)dz = \int_{x_1,y_1}^{x_2,y_2} [u(x, y) + iv(x, y)][dx + i\,dy]$$
$$= \int_{x_1,y_1}^{x_2,y_2} [u(x, y)dx - v(x, y)dy] + i \int_{x_1,y_1}^{x_2,y_2} [v(x, y)dx + u(x, y)dy] \tag{6.33}$$

with the path joining (x_1, y_1) and (x_2, y_2) specified. If the path C is parameterized as $x(s)$, $y(s)$, then $dx \to \frac{dx}{ds}ds$, and $dy \to \frac{dy}{ds}ds$. This reduces the complex integral to the complex sum of real integrals. It is somewhat analogous to the replacement of a vector integral by the vector sum of scalar integrals (Section 1.9).

We can also proceed by dividing the contour from z_0 to z_0' into n intervals by picking $n-1$ intermediate points $z_1, z_2, \ldots,$ on the contour (Fig. 6.7). Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}), \tag{6.34a}$$

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with

$$|z_j - z_{j-1}| \to 0$$

for all j. If the $\lim_{n\to\infty} S_n$ exists and is independent of the details of choosing the points z_j and ζ_j as long as they lie on the contour, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)dz. \tag{6.34b}$$

The right-hand side of Eq. (6.34b) is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$). When we **integrate** along the contour **in the opposite direction**, dz changes sign and the **integral changes sign**.

An important example is the following contour integral.

EXAMPLE 6.3.1

Cauchy Integral for Powers Let us evaluate the contour integral $\int_C z^n\,dz$, where C is a **circle** of radius $r > 0$ around the origin $z = 0$ in the positive mathematical sense (counterclockwise). In polar coordinates of Eq. (6.10) we parameterize the circle as $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. For **integer** $n \neq -1$, we then obtain

$$\int_C z^n dz = r^{n+1}\int_0^{2\pi} i\exp[i(n+1)\theta]d\theta$$
$$= (n+1)^{-1}r^{n+1}[e^{i(n+1)\theta}]|_{\theta=0}^{2\pi} = 0 \tag{6.35}$$

because 2π is a period of $e^{i(n+1)\theta}$, whereas for $n = -1$

$$\int_C \frac{dz}{z} = \int_0^{2\pi} i\,d\theta = 2\pi i, \tag{6.36}$$

again independent of r.

Alternatively, we can **integrate around a rectangle** with the corners z_1, z_2, z_3, z_4 to obtain for $n \neq -1$

$$\int z^n dz = \frac{z^{n+1}}{n+1}\bigg|_{z_1}^{z_2} + \frac{z^{n+1}}{n+1}\bigg|_{z_2}^{z_3} + \frac{z^{n+1}}{n+1}\bigg|_{z_3}^{z_4} + \frac{z^{n+1}}{n+1}\bigg|_{z_4}^{z_1} = 0$$

because each corner point appears once as an upper and a lower limit that cancel. For $n = -1$ the corresponding real parts of the logarithms cancel similarly, but their imaginary parts involve the increasing arguments of the points from z_1 to z_4 and, when we come back to the first corner z_1 its argument has increased by 2π due to the multivaluedness of the logarithm so that $2\pi i$ is left over as the value of the integral. Thus, **the value of the integral involving a multivalued function must be that which is reached in a continuous fashion on the path being taken**. These integrals are examples of Cauchy's integral theorem, which we prove for general functions in the next section. ■

Stokes's Theorem Proof of Cauchy's Integral Theorem

Cauchy's integral theorem is the first of two basic theorems in the theory of the behavior of functions of a complex variable. We present a proof under relatively restrictive conditions of physics applications—conditions that are intolerable to the mathematician developing a beautiful abstract theory but that are usually satisfied in physical problems. Cauchy's theorem states the following:

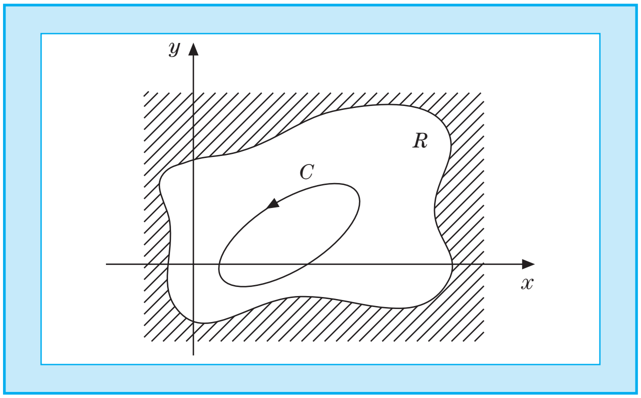


Figure 6.8

A Closed Contour C within a Simply Connected Region R

If a function $f(z)$ is analytic (therefore single-valued) and its partial derivatives are continuous throughout some simply connected region R,[6] for every closed path C (Fig. 6.8) in R the line integral of $f(z)$ around C is zero or

$$\oint_C f(z)dz = 0. \tag{6.37}$$

The symbol $\oint$ is used to emphasize that the path is closed.[7]

In this form the Cauchy integral theorem may be proved by direct application of Stokes's theorem (Section 1.11). With $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + i\,dy$,

$$\begin{aligned}\oint_C f(z)dz &= \oint_C (u + iv)(dx + i\,dy)\\ &= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy).\end{aligned} \tag{6.38}$$

These two line integrals may be converted to surface integrals by Stokes's theorem, a procedure that is justified if the partial derivatives are continuous within C. In applying Stokes's theorem, note that the final two integrals of Eq. (6.38) are real. Using

$$\mathbf{V} = \hat{\mathbf{x}}V_x + \hat{\mathbf{y}}V_y,$$

Stokes's (or Green's) theorem states that (A is area enclosed by C)

$$\oint_C (V_x\,dx + V_y\,dy) = \int_A \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy. \tag{6.39}$$

[6]A simply connected region or domain is one in which every closed contour in that region encloses only the points contained in it. If a region is not simply connected, it is called multiply connected. As an example of a multiply connected region, consider the z-plane with the interior of the unit circle **excluded**.

[7]Recall that in Section 1.12 such a function $f(z)$, identified as a force, was labeled conservative.

For the first integral in the last part of Eq. (6.38), let $u = V_x$ and $v = -V_y$.[8] Then

$$\oint_C (u\,dx - v\,dy) = \oint_C (V_x\,dx + V_y\,dy)$$
$$= \int_A \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy = -\int_A \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy. \tag{6.40}$$

For the second integral on the right side of Eq. (6.38), we let $u = V_y$ and $v = V_x$. Using Stokes's theorem again, we obtain

$$\oint_C (v\,dx + u\,dy) = \int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy. \tag{6.41}$$

On application of the Cauchy–Riemann conditions that must hold, since $f(z)$ is assumed analytic, each integrand vanishes and

$$\oint f(z)dz = -\int_A \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i\int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0. \tag{6.42}$$

A consequence of the Cauchy integral theorem is that for analytic functions the line integral is a function only of its end points, independent of the path of integration,

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)dz, \tag{6.43}$$

again exactly like the case of a conservative force (Section 1.12).

In summary, a Cauchy integral around a closed contour $\oint f(z)dz = 0$ when the function $f(z)$ is analytic in the simply connected region whose boundary is the closed path of the integral. The Cauchy integral is a two-dimensional analog of line integrals of conservative forces.

Multiply Connected Regions

The original statement of our theorem demanded a simply connected region. This restriction may easily be relaxed by the creation of a barrier, a contour line. Consider the multiply connected region of Fig. 6.9, in which $f(z)$ is not defined for the interior R'. Cauchy's integral theorem is not valid for the contour C, as shown, but we can construct a contour C' for which the theorem holds. We draw a line from the interior forbidden region R' to the forbidden region exterior to R and then run a new contour C', as shown in Fig. 6.10. The new contour C' through $ABDEFGA$ never crosses the contour line that converts R into a simply connected region. The three-dimensional analog of this technique

[8] In the proof of Stokes's theorem (Section 1.12), V_x and V_y are any two functions (with continuous partial derivatives).

Figure 6.9

A Closed Contour C in a Multiply Connected Region

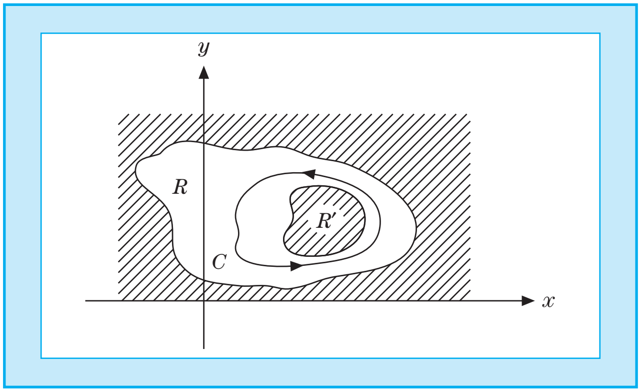
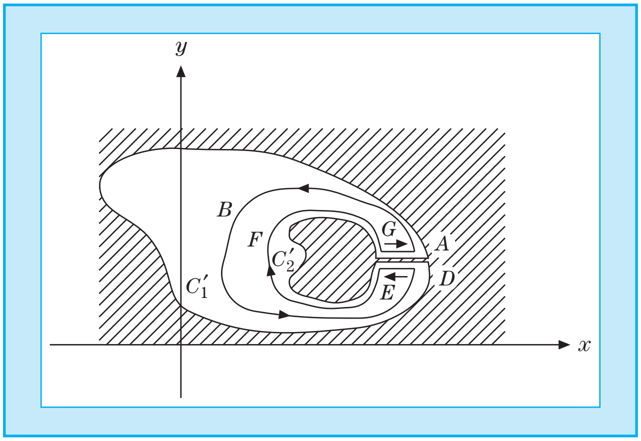


Figure 6.10

Conversion of a Multiply Connected Region into a Simply Connected Region



was used in Section 1.13 to prove Gauss's law. By Eq. (6.43),

$$\int_G^A f(z)dz = -\int_E^D f(z)dz, \tag{6.44}$$

with $f(z)$ having been continuous across the contour line and line segments DE and GA arbitrarily close together. Then

$$\oint_{C'} f(z)dz = \int_{ABD} f(z)dz + \int_{EFG} f(z)dz = 0 \tag{6.45}$$

by Cauchy's integral theorem, with region R now simply connected. Applying Eq. (6.43) once again with $ABD \to C'_1$ and $EFG \to -C'_2$, we obtain

$$\oint_{C'_1} f(z)dz = \oint_{C'_2} f(z)dz, \tag{6.46}$$

in which C'_1 and C'_2 are both traversed in the same (counterclockwise) direction.

Let us emphasize that the contour line here is a matter of mathematical convenience to permit the application of Cauchy's integral theorem. Since $f(z)$ is analytic in the annular region, it is necessarily single-valued and continuous across any such contour line.

Biographical Data

Cauchy, Augustin Louis, Baron. Cauchy, a French mathematician, was born in 1789 in Paris and died in 1857 in Sceaux, France. In 1805, he entered the Ecole Polytechnique, where Ampère was one of his teachers. In 1816, he replaced Monge in the Académie des Sciences, when Monge was expelled for political reasons. The father of modern complex analysis, his most famous contribution is his integral formula for analytic functions and their derivatives, but he also contributed to partial differential equations and the ether theory in electrodynamics.

EXERCISES

6.3.1 Show that $\int_{z_1}^{z_2} f(z)dz = -\int_{z_2}^{z_1} f(z)dz$.

6.3.2 Prove that

$$\left|\int_C f(z)dz\right| \leq |f|_{\max} \cdot L,$$

where $|f|_{\max}$ is the maximum value of $|f(z)|$ along the contour C and L is the length of the contour.

6.3.3 Verify that

$$\int_{0,0}^{1,1} z^* dz$$

depends on the path by evaluating the integral for the two paths shown in Fig. 6.11. Recall that $f(z) = z^*$ is not an analytic function of z and that Cauchy's integral theorem therefore does not apply.

6.3.4 Show that

$$\oint \frac{dz}{z^2 + z} = 0,$$

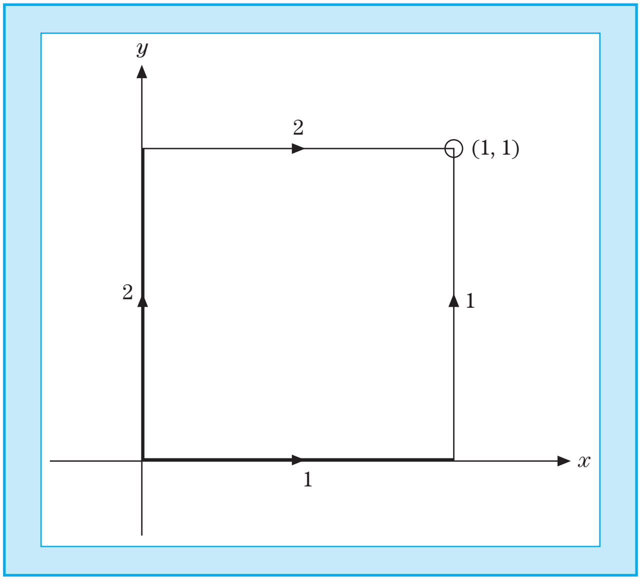
in which the contour C is (i) a circle defined by $|z| = R > 1$ and (ii) a square with the corners $\pm 2 \pm 2i$.

Hint. Direct use of the Cauchy integral theorem is illegal. Why? The integral may be evaluated by transforming to polar coordinates and using tables. The preferred technique is the calculus of residues (Section 7.2). This yields 0 for $R > 1$ and $2\pi i$ for $R < 1$.

6.3.5 Evaluate $\int_0^{2+i} |z|^2\, dz$ along a straight line from the origin to $2 + i$ and on a second path along the real axis from the origin to 2 continuing from 2

Figure 6.11

Contour



to $2+i$ parallel to the imaginary axis. Compare with the same integrals where $|z|^2$ is replaced by z^2. Discuss why there is path dependence in one case but not in the other.

6.4 Cauchy's Integral Formula

As in the preceding section, we consider a function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C. We seek to prove the Cauchy integral formula,

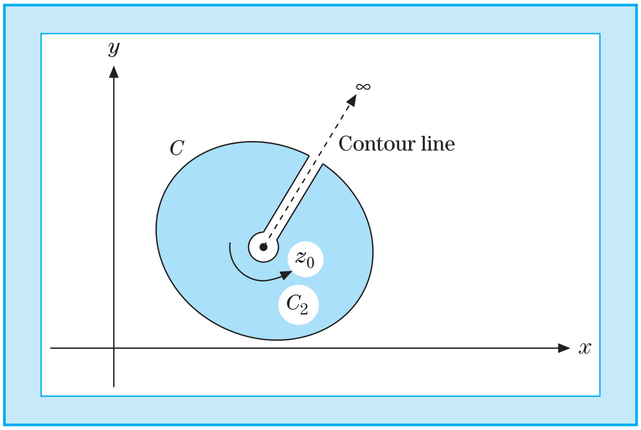
$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = f(z_0), \tag{6.47}$$

in which z_0 is some point in the interior region bounded by C. This is the second of the two basic theorems. Note that since z is on the contour C while z_0 is in the interior, $z-z_0 \neq 0$ and the integral Eq. (6.47) is well defined. Looking at the integrand $\frac{f(z)}{z-z_0}$, we realize that although $f(z)$ is analytic within C, the denominator vanishes at $z=z_0$. If $f(z_0)\neq 0$ and z_0 lies inside C, the **integrand is singular, and this singularity is defined as a first-order or simple pole**. The presence of the pole is essential for Cauchy's formula to hold and in the $n=-1$ case of Example 6.3.1 as well. If the contour is deformed as shown in Fig. 6.12 (or Fig. 6.10, Section 6.3), Cauchy's integral theorem applies. By Eq. (6.46),

$$\oint_C \frac{f(z)}{z-z_0}dz - \oint_{C_2}\frac{f(z)}{z-z_0}dz = 0, \tag{6.48}$$

Figure 6.12

Exclusion of a Singular Point



where C is the original outer contour and C_2 is the circle surrounding the point z_0 traversed in a **counterclockwise** direction. Let $z = z_0 + re^{i\theta}$, using the polar representation because of the circular shape of the path around z_0. Here, r is small and will eventually be made to approach zero. We have

$$\oint_{C_2} \frac{f(z)}{z - z_0} dz = \oint_{C_2} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta.$$

Taking the limit as $r \to 0$, we obtain

$$\oint_{C_2} \frac{f(z)}{z - z_0} dz = if(z_0) \int_{C_2} d\theta = 2\pi i f(z_0) \tag{6.49}$$

since $f(z)$ is analytic and therefore continuous at $z = z_0$. This proves the Cauchy integral formula [Eq. (6.47)].

Here is a remarkable result. The value of an analytic function $f(z)$ is given at an interior point $z = z_0$ once the values on the boundary C are specified. This is closely analogous to a two-dimensional form of Gauss's law (Section 1.13) in which the magnitude of an interior line charge would be given in terms of the cylindrical surface integral of the electric field $\mathbf{E}$. A further analogy is the determination of a function in real space by an integral of the function and the corresponding Green's function (and their derivatives) over the bounding surface. Kirchhoff diffraction theory is an example of this.

It has been emphasized that z_0 is an interior point. What happens if z_0 is exterior to C? In this case, the entire integrand is analytic on and within C. Cauchy's integral theorem (Section 6.3) applies and the integral vanishes. We have

$$\frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0} = \begin{cases} f(z_0), & z_0 \text{ interior} \\ 0, & z_0 \text{ exterior.} \end{cases}$$

Derivatives

Cauchy's integral formula may be used to obtain an expression for the derivative of $f(z)$. From Eq. (6.47), with $f(z)$ analytic,

$$\frac{f(z_0+\delta z_0)-f(z_0)}{\delta z_0} = \frac{1}{2\pi i\delta z_0}\left(\oint \frac{f(z)}{z-z_0-\delta z_0}dz - \oint \frac{f(z)}{z-z_0}dz\right).$$

Then, by definition of derivative [Eq. (6.22)],

$$f'(z_0) = \lim_{\delta z_0\to 0}\frac{1}{2\pi i\delta z_0}\oint \frac{\delta z_0 f(z)}{(z-z_0-\delta z_0)(z-z_0)}dz$$
$$= \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}dz. \qquad (6.50)$$

This result could have been obtained by differentiating Eq. (6.47) under the integral sign with respect to z_0. This formal or turning-the-crank approach is valid, but the justification for it is contained in the preceding analysis. Again, the integrand $f(z)/(z-z_0)^2$ is singular at $z=z_0$ if $f(z_0)\neq 0$, and **this singularity is defined to be a second-order pole**.

This technique for constructing derivatives may be repeated. We write $f'(z_0+\delta z_0)$ and $f'(z_0)$ using Eq. (6.50). Subtracting, dividing by δz_0, and finally taking the limit as $\delta z_0 \to 0$, we have

$$f^{(2)}(z_0) = \frac{2}{2\pi i}\oint \frac{f(z)dz}{(z-z_0)^3}.$$

Note that $f^{(2)}(z_0)$ is independent of the direction of δz_0 as it must be. If $f(z_0)\neq 0$, then $f(z)/(z-z_0)^3$ has **a singularity, which is defined to be a third-order pole**. Continuing, we get[9]

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)dz}{(z-z_0)^{n+1}}; \qquad (6.51)$$

that is, the requirement that $f(z)$ be analytic guarantees not only a first derivative but also derivatives of **all** orders. Note that **the integrand has a pole of order** $n+1$ at $z=z_0$ if $f(z_0)\neq 0$. The derivatives of $f(z)$ are automatically analytic. Notice that this statement assumes the Goursat version of the Cauchy integral theorem [assuming $f'(z)$ exists but need not be assumed to be continuous; for a proof, see 5th ed. of Arfken and Weber's *Math. Methods*]. This is also why Goursat's contribution is so significant in the development of the theory of complex variables.

Morera's Theorem

A further application of Cauchy's integral formula is in the proof of Morera's theorem, which is the converse of Cauchy's integral theorem. The theorem states the following:

[9] This expression is the starting point for defining derivatives of **fractional order**. See Erdelyi, A. (Ed.) (1954). *Tables of Integral Transforms*, Vol. 2. McGraw-Hill, New York. For recent applications to mathematical analysis, see Osler, T. J. (1972). An integral analogue of Taylor's series and its use in computing Fourier transforms. *Math. Comput.* **26**, 449.

> If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)\,dz = 0$ for every closed contour C within R, then $f(z)$ is analytic throughout R.

Let us integrate $f(z)$ from z_1 to z_2. Since every closed path integral of $f(z)$ vanishes, the integral is independent of path and depends only on its end points. We label the result of the integration $F(z)$, with

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(z)dz. \tag{6.52}$$

As an identity,

$$\frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) = \frac{\int_{z_1}^{z_2} [f(t) - f(z_1)]dt}{z_2 - z_1}, \tag{6.53}$$

using t as another complex variable. Now we take the limit as $z_2 \to z_1$:

$$\lim_{z_2 \to z_1} \frac{\int_{z_1}^{z_2} [f(t) - f(z_1)]dt}{z_2 - z_1} = 0 \tag{6.54}$$

since $f(t)$ is continuous.[10] Therefore,

$$\lim_{z_2 \to z_1} \frac{F(z_2) - F(z_1)}{z_2 - z_1} = F'(z)|_{z=z_1} = f(z_1) \tag{6.55}$$

by definition of derivative [Eq. (6.22)]. We have proved that $F'(z)$ at $z = z_1$ exists and equals $f(z_1)$. Since z_1 is any point in R, we see that $F(z)$ is analytic. Then by Cauchy's integral formula [compare Eq. (6.51)] $F'(z) = f(z)$ is also analytic, proving Morera's theorem.

Drawing once more on our electrostatic analog, we might use $f(z)$ to represent the electrostatic field $\mathbf{E}$. If the net charge within every closed region in R is zero (Gauss's law), the charge density is everywhere zero in R. Alternatively, in terms of the analysis of Section 1.12, $f(z)$ represents a conservative force (by definition of conservative), and then we find that it is always possible to express it as the derivative of a potential function $F(z)$.

An important application of Cauchy's integral formula is the following Cauchy inequality. If $f(z) = \sum a_n z^n$ is analytic and bounded, $|f(z)| \le M$ on a circle of radius r about the origin, then

$$|a_n| r^n \le M \qquad (\textbf{Cauchy's inequality}) \tag{6.56}$$

gives upper bounds for the coefficients of its Taylor expansion. To prove Eq. (6.56), let us define $M(r) = \max_{|z|=r} |f(z)|$ and use the Cauchy integral for a_n:

$$|a_n| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \le M(r) \frac{2\pi r}{2\pi r^{n+1}}.$$

[10] We can quote the mean value theorem of calculus here.

An immediate consequence of the inequality (6.56) is **Liouville's theorem**:

If $f(z)$ is analytic and bounded in the complex plane, it is a constant.

In fact, if $|f(z)| \leq M$ for all z, then Cauchy's inequality [Eq. (6.56)] gives $|a_n| \leq Mr^{-n} \to 0$ as $r \to \infty$ for $n > 0$. Hence, $f(z) = a_0$.

Conversely, the slightest deviation of an analytic function from a constant value implies that there must be at least one singularity somewhere in the infinite complex plane. Apart from the trivial constant functions, then, singularities are a fact of life, and we must learn to live with them. However, we shall do more than that. We shall next expand a function in a Laurent series at a singularity, and we shall use singularities to develop the powerful and useful calculus of residues in Chapter 7.

A famous application of Liouville's theorem yields the **fundamental theorem of algebra** (due to C. F. Gauss), which states that any polynomial $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ with $n > 0$ and $a_n \neq 0$ has n roots. To prove this, suppose $P(z)$ has no zero. Then $f(z) = 1/P(z)$ is analytic and bounded as $|z| \to \infty$. Hence, $f(z) = 1/P$ is a constant by Liouville's theorem—a contradiction. Thus, $P(z)$ has at least one root that we can divide out. Then we repeat the process for the resulting polynomial of degree $n - 1$. This leads to the conclusion that $P(z)$ has exactly n roots.

SUMMARY

In summary, if an analytic function $f(z)$ is given on the boundary C of a simply connected region R, then the values of the function and all its derivatives are known at any point inside that region R in terms of Cauchy integrals

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z - z_0}, \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

These Cauchy integrals are extremely important in numerous physics applications.

EXERCISES

6.4.1 Show that

$$\frac{1}{2\pi i} \oint z^{m-n-1} dz, \quad m \text{ and } n \text{ integers}$$

(with the contour encircling the origin once counterclockwise) is a representation of the Kronecker δ_{mn}.

6.4.2 Solve Exercise 6.3.4 by separating the integrand into partial fractions and then applying Cauchy's integral theorem for multiply connected regions. *Note.* Partial fractions are explained in Section 15.7 in connection with Laplace transforms.

6.4.3 Evaluate

$$\oint_C \frac{dz}{z^2 - 1},$$

where C is the circle $|z| = 2$. Alternatively, integrate around a square with corners $\pm 2 \pm 2i$.

6.4.4 Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\oint_C \frac{f'(z)}{z - z_0} dz = \oint_C \frac{f(z)}{(z - z_0)^2} dz.$$

6.4.5 You know that $f(z)$ is analytic on and within a closed contour C. You suspect that the nth derivative $f^{(n)}(z_0)$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Using mathematical induction, prove that this expression is correct.

6.4.6 (a) A function $f(z)$ is analytic within a closed contour C (and continuous on C). If $f(z) \neq 0$ within C and $|f(z)| \leq M$ on C, show that

$$|f(z)| \leq M$$

for all points within C.
Hint. Consider $w(z) = 1/f(z)$.

(b) If $f(z) = 0$ within the contour C, show that the foregoing result does not hold—that it is possible to have $|f(z)| = 0$ at one or more points in the interior with $|f(z)| > 0$ over the entire bounding contour. Cite a specific example of an analytic function that behaves this way.

6.4.7 Using the Cauchy integral formula for the nth derivative, convert the following Rodrigues's formulas into Cauchy integrals with appropriate contours:

(a) Legendre

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

ANS. $\frac{(-1)^n}{2^n} \cdot \frac{1}{2\pi i} \oint \frac{(1 - z^2)^n}{(z - x)^{n+1}} dz.$

(b) Hermite

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

(c) Laguerre

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}).$$

Note. From these integral representations one can develop generating functions for these special functions. Compare Sections 11.4, 13.1, and 13.2.

6.4.8 Obtain $\oint_C z^* dz$, where C is the unit circle in the first quadrant. Compare with the integral from $z = 1$ parallel to the imaginary axis to $1 + i$ and from there to i parallel to the real axis.

6.4.9 Evaluate $\int_{2\pi}^{2\pi+i\infty} e^{iz} dz$ along two different paths of your choice.

6.5 Laurent Expansion

Taylor Expansion

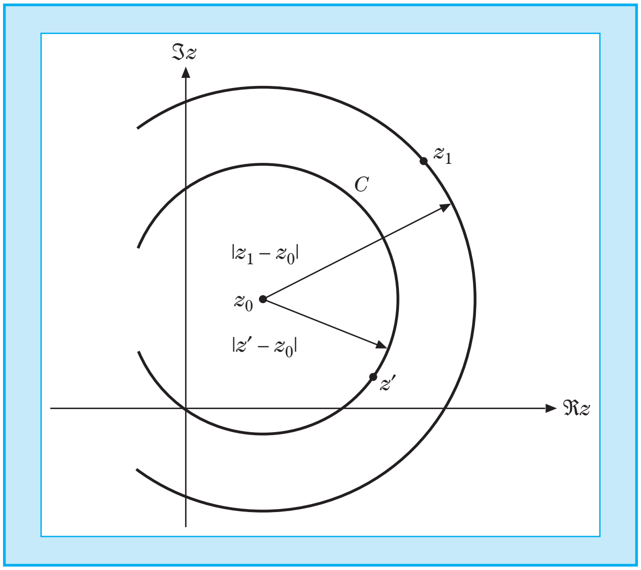
The Cauchy integral formula of the preceding section opens up the way for another derivation of Taylor's series (Section 5.6), but this time for functions of a complex variable. Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point on the Argand diagram for which $f(z)$ is not analytic. We construct a circle C centered at $z = z_0$ with radius $|z' - z_0| < |z_1 - z_0|$ (Fig. 6.13). Since z_1 was assumed to be the nearest point at which $f(z)$ was not analytic, $f(z)$ is necessarily analytic on and within C.

From Eq. (6.47), the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{z' - z} \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z' - z_0) - (z - z_0)} \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z' - z_0)[1 - (z - z_0)/(z' - z_0)]}, \end{aligned} \tag{6.57}$$

Figure 6.13

Circular Domain for Taylor Expansion



where z' is a point on the contour C and z is any point interior to C. We expand the denominator of the integrand in Eq. (6.57) by the binomial theorem, which generalizes to complex variables as in Example 6.2.1 for other elementary functions. Or, we note the identity (for complex t)

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n, \tag{6.58}$$

which may easily be verified by multiplying both sides by $1 - t$. The infinite series, following the methods of Section 5.2, is convergent for $|t| < 1$. Upon replacing the positive terms a_n in a real series by absolute values $|a_n|$ of complex numbers, the convergence criteria of Chapter 5 translate into valid convergence theorems for complex series.

Now for a point z interior to C, $|z - z_0| < |z' - z_0|$, and using Eq. (6.58), Eq. (6.57) becomes

$$f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n f(z') dz'}{(z' - z_0)^{n+1}}. \tag{6.59}$$

Interchanging the order of integration and summation [valid since Eq. (6.58) is uniformly convergent for $|t| < 1$], we obtain

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z') dz'}{(z' - z_0)^{n+1}}. \tag{6.60}$$

Referring to Eq. (6.51), we get

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!}, \tag{6.61}$$

which is our desired Taylor expansion. Note that it is based only on the assumption that $f(z)$ is analytic for $|z - z_0| < |z_1 - z_0|$. Just as for real variable power series (Section 5.7), this expansion is unique for a given z_0.

From the Taylor expansion for $f(z)$ a binomial theorem may be derived (Exercise 6.5.2).

Schwarz Reflection Principle

From the binomial expansion of $g(z) = (z - x_0)^n$ for integral n we see that the complex conjugate of the function is the function of the complex conjugate, for real x_0

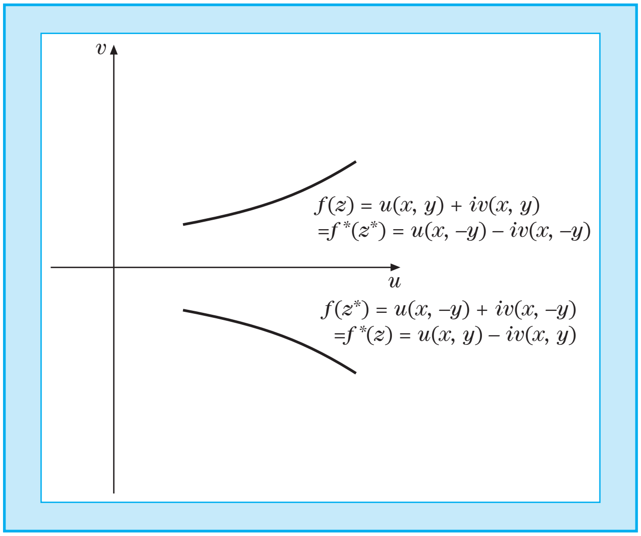
$$g^*(z) = (z - x_0)^{n*} = (z^* - x_0)^n = g(z^*). \tag{6.62}$$

This leads us to the Schwarz reflection principle:

> If a function $f(z)$ is (1) analytic over some region including the real axis and (2) real when z is real, then
>
> $$f^*(z) = f(z^*). \tag{6.63}$$

Figure 6.14

Schwarz Reflection



(Fig. 6.14). It may be proved as follows. Expanding $f(z)$ about some (nonsingular) point x_0 on the real axis,

$$f(z) = \sum_{n=0}^{\infty} (z - x_0)^n \frac{f^{(n)}(x_0)}{n!} \tag{6.64}$$

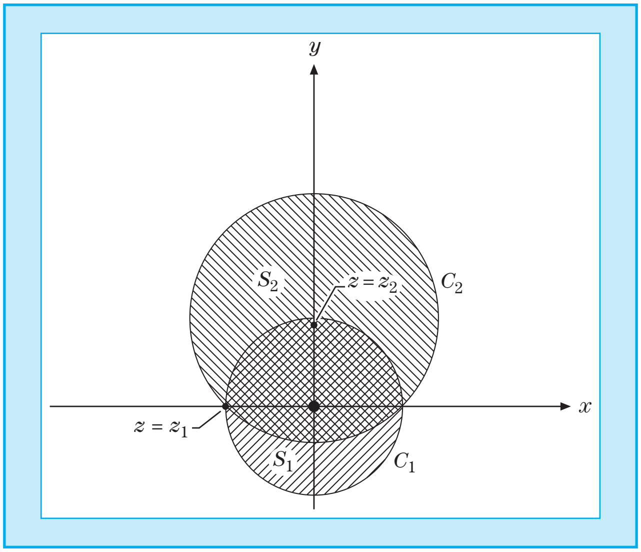
by Eq. (6.60). Since $f(z)$ is analytic at $z = x_0$, this Taylor expansion exists. Since $f(z)$ is real when z is real, $f^{(n)}(x_0)$ must be real for all n. Then when we use Eq. (6.62), Eq. (6.63) (the Schwarz reflection principle) follows immediately. Exercise 6.5.6 is another form of this principle. The Schwarz reflection principle applies to all elementary functions and those in Example 6.2.1 in particular.

Analytic Continuation

It is natural to think of the values $f(z)$ of an analytic function f as a single entity that is usually defined in some restricted region S_1 of the complex plane, for example, by a Taylor series (Fig. 6.15). Then f is analytic inside the **circle of convergence** C_1, whose radius is given by the distance r_1 from the center of C_1 to the **nearest singularity** of f at z_1 (in Fig. 6.15). If we choose a point inside C_1 that is farther than r_1 from the singularity z_1 and make a Taylor expansion of f about it (z_2 in Fig. 6.15), then the circle of convergence C_2 will usually extend beyond the first circle C_1. In the overlap region of both circles C_1, C_2 the function f is uniquely defined. In the region of the circle C_2 that extends beyond C_1, $f(z)$ is uniquely defined by the Taylor series about the center of C_2 and analytic there, although the Taylor series about the center of C_1 is no longer convergent there. After Weierstrass, this process is called **analytic continuation**. It defines the analytic functions in terms of its original definition (e.g., in C_1) and all its continuations.

Figure 6.15

Analytic Continuation



A specific example is the **meromorphic** function

$$f(z) = \frac{1}{1+z}, \tag{6.65}$$

which has a simple pole at $z = -1$ and is analytic elsewhere. The geometric series expansion

$$\frac{1}{1+z} = 1 - z + z^2 + \cdots = \sum_{n=0}^{\infty} (-z)^n \tag{6.66}$$

converges for $|z| < 1$ (i.e., inside the circle C_1 in Fig. 6.15).

Suppose we expand $f(z)$ about $z = i$ so that

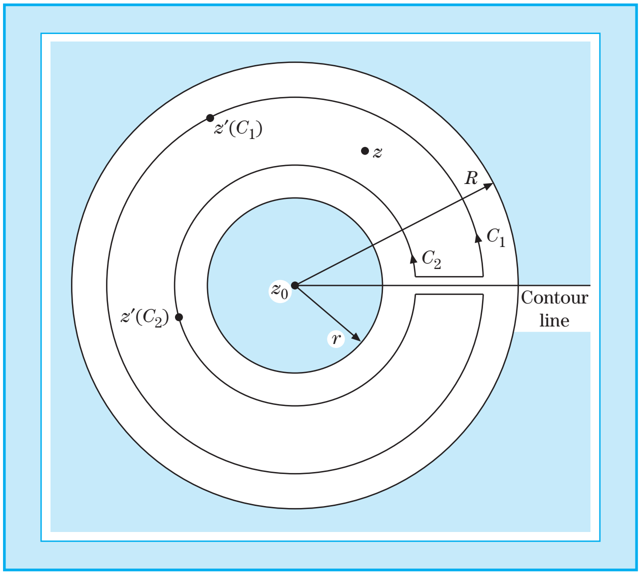
$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+(z-i)} = \frac{1}{(1+i)(1+(z-i)/(1+i))}$$
$$= \left[1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} - \cdots\right] \frac{1}{1+i} \tag{6.67}$$

converges for $|z - i| < |1 + i| = \sqrt{2}$. Our circle of convergence is C_2 in Fig. 6.15. Now $f(z)$ is defined by the expansion [Eq. (6.67)] in S_2 that overlaps S_1 and extends further out in the complex plane.[11] This extension is an analytic

[11]One of the most powerful and beautiful results of the more abstract theory of functions of a complex variable is that if two analytic functions coincide in any region, such as the overlap of S_1 and S_2, or coincide on any line segment, they are the same function in the sense that they will coincide everywhere as long as they are both well defined. In this case, the agreement of the expansions [Eqs. (6.66) and (6.67)] over the region common to S_1 and S_2 would establish the identity of the functions these expansions represent. Then Eq. (6.67) would represent an analytic continuation or extension of $f(z)$ into regions not covered by Eq. (6.66). We could equally well say that $f(z) = 1/(1+z)$ is an analytic continuation of either of the series given by Eqs. (6.66) and (6.67).

Figure 6.16

$|z' - z_0|_{C_1} > |z - z_0|$; $|z' - z_0|_{C_2} < |z - z_0|$



continuation, and when we have only isolated singular points to contend with, the function can be extended indefinitely. Equations (6.65)–(6.67) are three different representations of the same function. Each representation has its own domain of convergence. Equation (6.66) is a Maclaurin series. Equation (6.67) is a Taylor expansion about $z = i$.

Analytic continuation may take many forms and the series expansion just considered is not necessarily the most convenient technique. As an alternate technique we shall use a recurrence relation in Section 10.1 to extend the factorial function around the isolated singular points, $z = -n$, $n = 1, 2, 3, \ldots$.

Laurent Series

We frequently encounter functions that are analytic in an annular region, for example, of inner radius r and outer radius R, as shown in Fig. 6.16. Drawing an imaginary contour line to convert our region into a simply connected region, we apply Cauchy's integral formula, and for two circles, C_2 and C_1, centered at $z = z_0$ and with radii r_2 and r_1, respectively, where $r < r_2 < r_1 < R$, we have[12]

$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{z'-z} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{z'-z}. \qquad (6.68)$$

Note that in Eq. (6.68) an explicit minus sign has been introduced so that contour C_2 (like C_1) is to be traversed in the positive (counterclockwise) sense. The treatment of Eq. (6.68) now proceeds exactly like that of Eq. (6.57) in the development of the Taylor series. Each denominator is written as $(z' - z_0) - (z - z_0)$

[12]We may take r_2 arbitrarily close to r and r_1 arbitrarily close to R, maximizing the area enclosed between C_1 and C_2.

and expanded by the binomial theorem, which now follows from the Taylor series [Eq. (6.61)].

Noting that for C_1, $|z' - z_0| > |z - z_0|$, whereas for C_2, $|z' - z_0| < |z - z_0|$, we find

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_{C_1}\frac{f(z')dz'}{(z' - z_0)^{n+1}}$$
$$+\frac{1}{2\pi i}\sum_{n=1}^{\infty}(z - z_0)^{-n}\oint_{C_2}(z' - z_0)^{n-1}f(z')dz'. \quad (6.69)$$

The minus sign of Eq. (6.68) has been absorbed by the binomial expansion. Labeling the first series S_1 and the second S_2,

$$S_1 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_{C_1}\frac{f(z')dz'}{(z' - z_0)^{n+1}}, \quad (6.70)$$

which is the regular Taylor expansion, convergent for $|z - z_0| < |z' - z_0| = r_1$, that is, for all z **interior** to the larger circle, C_1. For the second series in Eq. (6.68), we have

$$S_2 = \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z - z_0)^{-n}\oint_{C_2}(z' - z_0)^{n-1}f(z')dz' \quad (6.71)$$

convergent for $|z - z_0| > |z' - z_0| = r_2$, that is, for all z **exterior** to the smaller circle C_2. Remember, C_2 goes counterclockwise.

These two series are combined into one series[13] (a Laurent series) by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \quad (6.72)$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}. \quad (6.73)$$

Since, in Eq. (6.72), convergence of a binomial expansion is no problem, C may be any contour within the annular region $r < |z - z_0| < R$ encircling z_0 once in a counterclockwise sense. The integrals are independent of the contour, and Eq. (6.72) is the **Laurent series or Laurent expansion of** $f(z)$.

The use of the contour line (Fig. 6.16) is convenient in converting the annular region into a simply connected region. Since our function is analytic in this annular region (and therefore single-valued), the contour line is not essential and, indeed, does not appear in the final result [Eq. (6.72)]. For $n \geq 0$, the integrand $f(z)/(z - z_0)^{n+1}$ is singular at $z = z_0$ if $f(z_0) \neq 0$. The integrand has a pole of order $n + 1$ at $z = z_0$. If f has a first-order zero at $z = z_0$, then $f(z)/(z - z_0)^{n+1}$ has a pole of order n, etc. The presence of poles is essential for the validity of the Laurent formula.

[13] Replace n by $-n$ in S_2 and add.

Laurent series coefficients need not come from evaluation of contour integrals (which may be very intractable). Other techniques such as ordinary series expansions often provide the coefficients.

Numerous examples of Laurent series appear in Chapter 7. We start here with a simple example to illustrate the application of Eq. (6.72).

EXAMPLE 6.5.1

Laurent Expansion by Integrals Let $f(z) = [z(z-1)]^{-1}$. If we choose $z_0 = 0$, then $r = 0$ and $R = 1$, $f(z)$ diverging at $z = 1$. From Eqs. (6.73) and (6.72),

$$a_n = \frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)}$$
$$= \frac{-1}{2\pi i}\oint \sum_{m=0}^{\infty}(z')^m \frac{dz'}{(z')^{n+2}}. \qquad (6.74)$$

Again, interchanging the order of summation and integration (uniformly convergent series), we have

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint \frac{dz'}{(z')^{n+2-m}}. \qquad (6.75)$$

If we employ the polar form, as before Eq. (6.35) (of Example 6.3.1),

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint \frac{rie^{i\theta}d\theta}{r^{n+2-m}e^{i(n+2-m)\theta}}$$
$$= -\frac{1}{2\pi i}\cdot 2\pi i\sum_{m=0}^{\infty}\delta_{n+2-m,1}. \qquad (6.76)$$

In other words,

$$a_n = \begin{cases} -1 & \text{for } n \ge -1, \\ 0 & \text{for } n < -1. \end{cases} \qquad (6.77)$$

The Laurent expansion about $z = 0$ [Eq. (6.72)] becomes

$$\frac{1}{z(z-1)} = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots = -\sum_{n=-1}^{\infty} z^n. \qquad (6.78)$$

For this simple function the Laurent series can, of course, be obtained by a direct binomial expansion or partial fraction and geometric series expansion as follows. We expand in partial fractions

$$f(z) = \frac{1}{z(z-1)} = \frac{b_0}{z} + \frac{b_1}{z-1},$$

where we determine b_0 at $z \to 0$,

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{1}{z-1} = -1 = b_0 + \lim_{z\to 0}\frac{b_1 z}{z-1} = b_0,$$

and b_1 at $z \to 1$ similarly,

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{1}{z} = 1 = b_1 + \lim_{z\to 1}\frac{b_0(z-1)}{z} = b_1.$$

Expanding $1/(z-1)$ in a geometric series yields the Laurent series [Eq. (6.78)]. ■

The Laurent series differs from the Taylor series by the obvious feature of negative powers of $(z - z_0)$. For this reason, the Laurent series will always diverge at least at $z = z_0$ and perhaps as far out as some distance r (Fig. 6.16).

EXAMPLE 6.5.2

Laurent Expansion by Series Expand $f(z) = \exp(z)\exp(1/z)$ in a Laurent series $f(z) = \sum_n a_n z^n$ about the origin.

This function is analytic in the complex plane except at $z = 0$ and $z \to \infty$. Moreover, $f(1/z) = f(z)$, so that $a_{-n} = a_n$. Multiplying the power series

$$e^z e^{\frac{1}{z}} = \sum_{m=0}^{\infty}\frac{z^m}{m!}\sum_{n=0}^{\infty}\frac{1}{z^n n!},$$

we get the constant term a_0 from the products of the $m = n$ terms as

$$a_0 = \sum_{m=0}^{\infty}\frac{1}{(m!)^2}.$$

The coefficient of z^k comes from the products of the terms $\frac{z^{m+k}}{(m+k)!}$ and $\frac{1}{z^m m!}$; that is,

$$a_k = a_{-k} = \sum_{m=0}^{\infty}\frac{1}{m!(m+k)!}.$$

From the ratio test or the absence of singularities in the finite complex plane for $z \neq 0$, this Laurent series converges for $|z| > 0$. ■

Biographical Data

Laurent, Pierre-Alphonse. Laurent, a French mathematician, was born in 1813 and died in 1854. He contributed to complex analysis, his famous theorem being published in 1843.

SUMMARY

The Taylor expansion of an analytic function about a regular point follows from Cauchy's integral formulas. The radius of convergence of a Taylor series around a regular point is given by its distance to the nearest singularity. An analytic function can be expanded in a power series with positive and negative (integer) exponents about an arbitrary point, which is called its Laurent series; it converges in an annular region around a singular point and becomes its Taylor series around a regular point. If there are infinitely many negative exponents in its Laurent series the function has an essential singularity; if the Laurent series breaks off with a finite negative exponent it has a pole of that order at the expansion point. Analytic continuation of an analytic function from some neighborhood of a regular point to its natural domain by means of successive Taylor or Laurent series, an integral representation, or functional equation is a concept unique to the theory of analytic functions that highlights its power.

EXERCISES

6.5.1 Develop the Taylor expansion of $\ln(1+z)$.

$$ANS.\quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}.$$

6.5.2 Derive the binomial expansion

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2}z^2 + \cdots = \sum_{n=0}^{\infty}\binom{m}{n}z^n$$

for m any real number. The expansion is convergent for $|z| < 1$.

6.5.3 A function $f(z)$ is analytic on and within the unit circle. Also, $|f(z)| < 1$ for $|z| \le 1$ and $f(0) = 0$. Show that $|f(z)| < |z|$ for $|z| \le 1$.
Hint. One approach is to show that $f(z)/z$ is analytic and then express $[f(z_0)/z_0]^n$ by the Cauchy integral formula. Finally, consider absolute magnitudes and take the nth root. This exercise is sometimes called **Schwarz's theorem**.

6.5.4 If $f(z)$ is a real function of the complex variable $z = x + iy$ [i.e., $f(x) = f^*(x)$], and the Laurent expansion about the origin, $f(z) = \sum a_n z^n$, has $a_n = 0$ for $n < -N$, show that all of the coefficients, a_n, are real.
Hint. Show that $z^N f(z)$ is analytic (via Morera's theorem; Section 6.4).

6.5.5 A function $f(z) = u(x, y) + iv(x, y)$ satisfies the conditions for the Schwarz reflection principle. Show that
(a) u is an even function of y. (b) v is an odd function of y.

6.5.6 A function $f(z)$ can be expanded in a Laurent series about the origin with the coefficients a_n real. Show that the complex conjugate of this function of z is the same function of the complex conjugate of z; that is,

$$f^*(z) = f(z^*).$$

Verify this explicitly for
(a) $f(z) = z^n$, n an integer, (b) $f(z) = \sin z$.
If $f(z) = iz$, $(a_1 = i)$, show that the foregoing statement does not hold.

6.5.7 The function $f(z)$ is analytic in a domain that includes the real axis. When z is real $(z = x)$, $f(x)$ is pure imaginary.
(a) Show that

$$f(z^*) = -[f(z)]^*.$$

(b) For the specific case $f(z) = iz$, develop the Cartesian forms of $f(z)$, $f(z^*)$, and $f^*(z)$. Do not quote the general result of part (a).

6.5.8 Develop the first three nonzero terms of the Laurent expansion of

$$f(z) = (e^z - 1)^{-1}$$

about the origin.

6.5.9 Prove that the Laurent expansion of a given function about a given point is unique; that is, if

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n = \sum_{n=-N}^{\infty} b_n (z - z_0)^n,$$

show that $a_n = b_n$ for all n.
Hint. Use the Cauchy integral formula.

6.5.10 (a) Develop a Laurent expansion of $f(z) = [z(z-1)]^{-1}$ about the point $z = 1$ valid for small values of $|z - 1|$. Specify the exact range over which your expansion holds. This is an analytic continuation of Eq. (6.78).
(b) Determine the Laurent expansion of $f(z)$ about $z = 1$ but for $|z-1|$ large.
Hint. Use partial fraction of this function and the geometric series.

6.5.11 (a) Given $f_1(z) = \int_0^\infty e^{-zt} dt$ (with t real), show that the domain in which $f_1(z)$ exists (and is analytic) is $\Re(z) > 0$.
(b) Show that $f_2(z) = 1/z$ equals $f_1(z)$ over $\Re(z) > 0$ and is therefore an analytic continuation of $f_1(z)$ over the entire z-plane except for $z = 0$.
(c) Expand $1/z$ about the point $z = i$. You will have $f_3(z) = \sum_{n=0}^{\infty} a_n (z - i)^n$. What is the domain of $f_3(z)$?

ANS. $\dfrac{1}{z} = -i \displaystyle\sum_{n=0}^{\infty} i^n (z - i)^n, \; |z - i| < 1.$

6.5.12 Expand $f(z) = \sin(\frac{z}{1-z})$ in a Laurent series about $z = 1$.

6.5.13 Expand

$$f(z) = \frac{z^3 - 2z^2 + 1}{(z - 3)(z^2 + 3)}$$

in a Laurent series about (i) $z = 3$, (ii) $z = \pm i\sqrt{3}$, (iii) $z = 1$, and (iv) $z = \frac{1}{2}(1 \pm \sqrt{5})$.

6.5.14 Find the Laurent series of $\ln((1 + z^2)/(1 - z^2))$ at ∞.

6.5.15 Write z in polar form and set up the relations that have to be satisfied for $\ln z = \ln |z| + i \arg z$ and $\ln(1 + z)$ defined by its Maclaurin series to be consistent.

6.6 Mapping

In the preceding sections, we defined analytic functions and developed some of their main features. Now we introduce some of the more geometric aspects of functions of complex variables—aspects that will be useful in visualizing the integral operations in Chapter 7 and that are valuable in their own right in solving Laplace's equation in two-dimensional systems.

In ordinary analytic geometry we may take $y = f(x)$ and then plot y versus x. Our problem here is more complicated because z is a function of two real variables x and y. We use the notation

$$w = f(z) = u(x, y) + iv(x, y). \tag{6.79}$$

Then for a point in the z-plane (specific values for x and y) there may correspond specific values for $u(x, y)$ and $v(x, y)$ that then yield a point in the w-plane. As points in the z-plane transform or are mapped into points in the w-plane, lines or areas in the z-plane will be mapped into lines or areas in the w-plane. Our immediate purpose is to see how lines and areas map from the z-plane to the w-plane for a number of simple functions.

Translation

$$w = z + z_0. \tag{6.80}$$

The function w is equal to the variable z plus a constant, $z_0 = x_0 + iy_0$. By Eqs. (6.2) and (6.80),

$$u = x + x_0, \qquad v = y + y_0, \tag{6.81}$$

representing a pure translation of the coordinate axes as shown in Fig. 6.17.

Figure 6.17

Translation

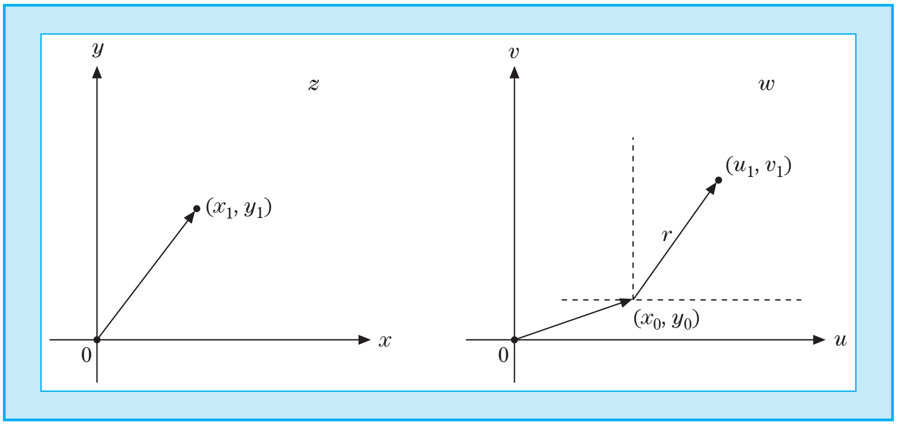
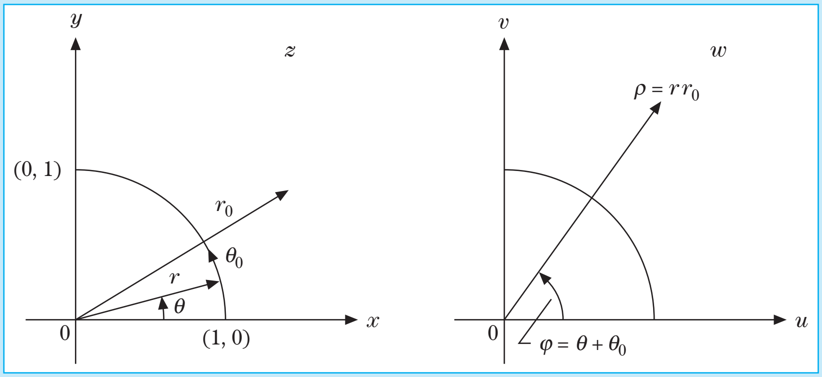


Figure 6.18

Rotation



Rotation

$$w = zz_0. \tag{6.82}$$

Here, it is convenient to return to the polar representation, using

$$w = \rho e^{i\varphi}, \quad z = re^{i\theta}, \quad \text{and} \quad z_0 = r_0 e^{i\theta_0}, \tag{6.83}$$

then

$$\rho e^{i\varphi} = rr_0 e^{i(\theta+\theta_0)} \tag{6.84}$$

or

$$\rho = rr_0, \qquad \varphi = \theta + \theta_0. \tag{6.85}$$

Two things have occurred. First, the modulus r has been modified, either expanded or contracted, by the factor r_0. Second, the argument θ has been increased by the additive constant θ_0 (Fig. 6.18). This represents a rotation of the complex variable through an angle θ_0. For the special case of $z_0 = i$, we have a pure rotation through $\pi/2$ radians.

Inversion

$$w = \frac{1}{z}. \tag{6.86}$$

Again, using the polar form, we have

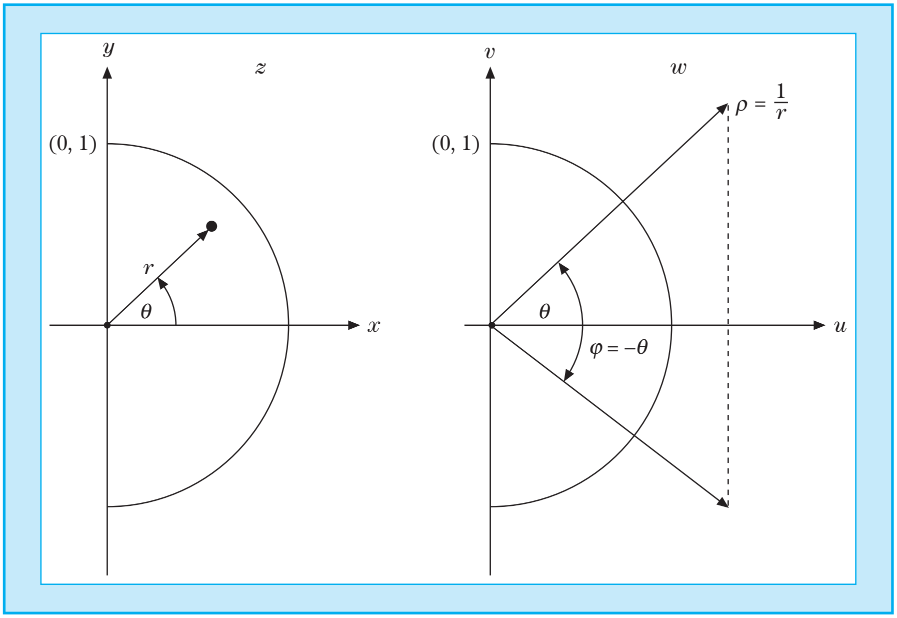
$$\rho e^{i\varphi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}, \tag{6.87}$$

which shows that

$$\rho = \frac{1}{r}, \qquad \varphi = -\theta. \tag{6.88}$$

Figure 6.19

Inversion



The radial part of Eq. (6.87) shows that inversion clearly. The interior of the unit circle is mapped onto the exterior and vice versa (Fig. 6.19). In addition, the angular part of Eq. (6.87) shows that the polar angle is reversed in sign. Equation (6.86) therefore also involves a reflection of the y-axis (like the complex conjugate equation).

To see how lines in the z-plane transform into the w-plane, we simply return to the Cartesian form:

$$u + iv = \frac{1}{x + iy}. \tag{6.89}$$

Rationalizing the right-hand side by multiplying numerator and denominator by z^* and then equating the real parts and the imaginary parts, we have

$$\begin{aligned} u &= \frac{x}{x^2 + y^2}, \qquad x = \frac{u}{u^2 + v^2}, \\ v &= -\frac{y}{x^2 + y^2}, \quad y = -\frac{v}{u^2 + v^2}. \end{aligned} \tag{6.90}$$

A circle centered at the origin in the z-plane has the form

$$x^2 + y^2 = r^2 \tag{6.91}$$

and by Eq. (6.90) transforms into

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = r^2. \tag{6.92}$$

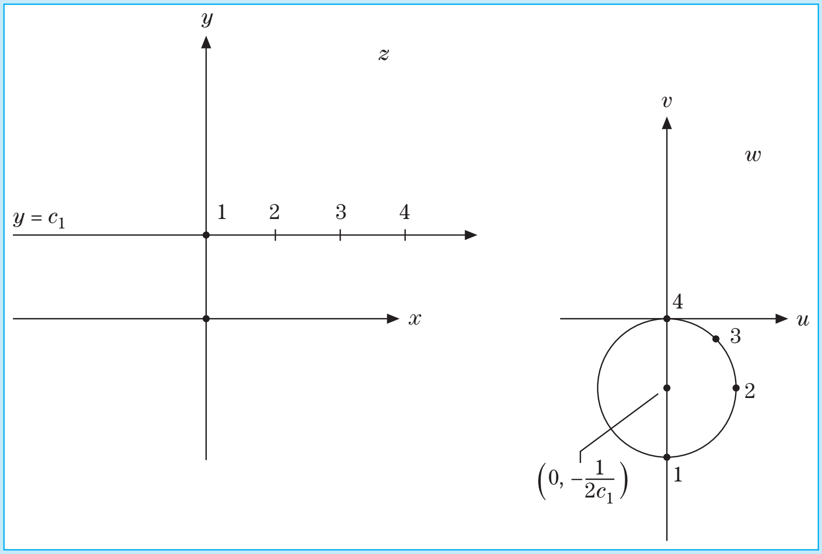


Figure 6.20

Inversion, Line ↔ Circle

Simplifying Eq. (6.92), we obtain

$$u^2 + v^2 = \frac{1}{r^2} = \rho^2, \tag{6.93}$$

which describes a circle in the w-plane also centered at the origin.

The horizontal line $y = c_1$ transforms into

$$\frac{-v}{u^2 + v^2} = c_1 \tag{6.94}$$

or

$$u^2 + \left(v + \frac{1}{2c_1}\right)^2 = \frac{1}{(2c_1)^2}, \tag{6.95}$$

which describes a circle in the w-plane of radius $(\frac{1}{2})c_1$ and centered at $u = 0$, $v = -\frac{1}{2}c_1$ (Fig. 6.20). We pick up the other three possibilities, $x = \pm c_1$, $y = -c_1$, by rotating the xy-axes. In general, any straight line or circle in the z-plane will transform into a straight line or a circle in the w-plane (compare Exercise 6.6.1).

Branch Points and Multivalent Functions

The three transformations just discussed all involved one-to-one correspondence of points in the z-plane to points in the w-plane. Now to illustrate the variety of transformations that are possible and the problems that can arise, we introduce first a two-to-one correspondence and then a many-to-one correspondence. Finally, we take up the inverses of these two transformations.

Consider first the transformation

$$w = z^2, \tag{6.96}$$

which leads to

$$\rho = r^2, \qquad \varphi = 2\theta. \tag{6.97}$$

Clearly, our transformation is nonlinear because the modulus is squared, but the significant feature of Eq. (6.96) is that the phase angle or argument is doubled. This means that the

- first quadrant of z, $0 \le \theta < \frac{\pi}{2} \to$ upper half-plane of w, $0 \le \varphi < \pi$,
- upper half-plane of z, $0 \le \theta < \pi \to$ whole plane of w, $0 \le \varphi < 2\pi$.

The lower half-plane of z maps into the already covered entire plane of w, thus covering the w-plane a second time. This is our two-to-one correspondence: two distinct points in the z-plane, z_0 and $z_0 e^{i\pi} = -z_0$, corresponding to the single point $w = z_0^2$.

In Cartesian representation,

$$u + iv = (x + iy)^2 = x^2 - y^2 + i2xy, \tag{6.98}$$

leading to

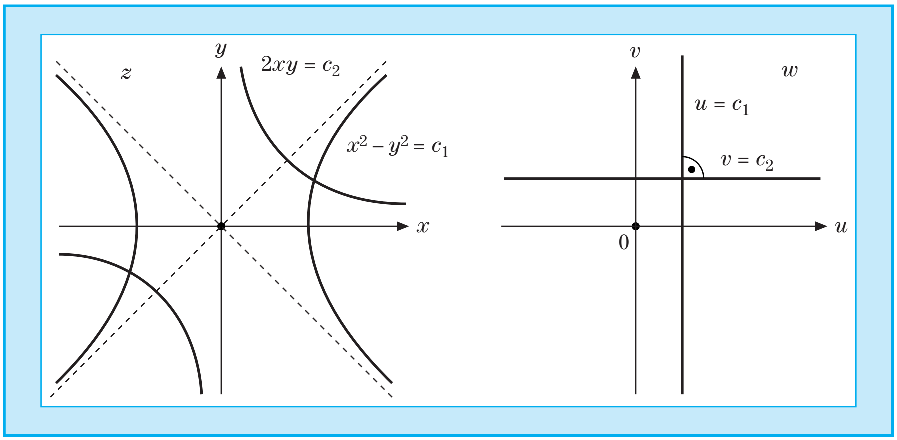
$$u = x^2 - y^2, \qquad v = 2xy. \tag{6.99}$$

Hence, the lines $u = c_1$, $v = c_2$ in the w-plane correspond to $x^2 - y^2 = c_1$, $2xy = c_2$, rectangular (and orthogonal) hyperbolas in the z-plane (Fig. 6.21). To every point on the hyperbola $x^2 - y^2 = c_1$ in the right half-plane, $x > 0$, one point on the line $u = c_1$ corresponds and vice versa. However, every point on the line $u = c_1$ also corresponds to a point on the hyperbola $x^2 - y^2 = c_1$ in the left half-plane, $x < 0$, as already explained.

It will be shown in Section 6.7 that if lines in the w-plane are orthogonal the corresponding lines in the z-plane are also orthogonal, as long as the transformation is analytic. Since $u = c_1$ and $v = c_2$ are constructed perpendicular

Figure 6.21

Mapping—Hyperbolic Coordinates



to each other, the corresponding hyperbolas in the z-plane are orthogonal. We have constructed a new orthogonal system of hyperbolic lines. Exercise 2.3.3 was an analysis of this system. Note that if the hyperbolic lines are electric or magnetic lines of force, then we have a quadrupole lens useful in focusing beams of high-energy particles.

The inverse of the fourth transformation [Eq. (6.96)] is

$$w = z^{1/2}. \qquad (6.100)$$

From the relation

$$\rho e^{i\varphi} = r^{1/2} e^{i\theta/2}, \qquad (6.101)$$

and

$$2\varphi = \theta, \qquad (6.102)$$

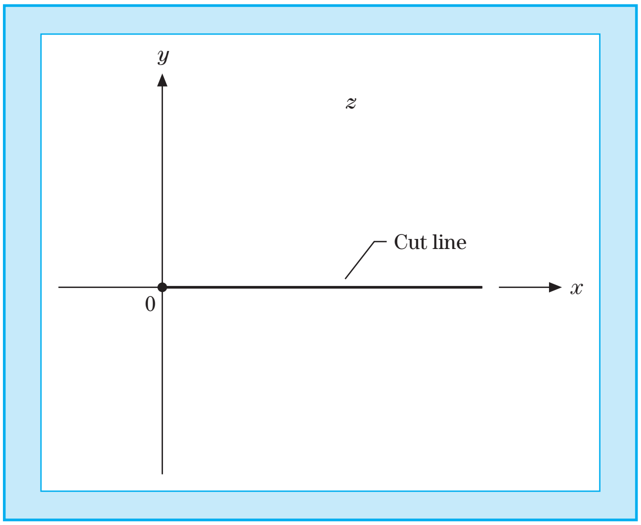
we now have two points in the w-plane (arguments φ and $\varphi + \pi$) corresponding to one point in the z-plane (except for the point $z = 0$). In other words, θ and $\theta + 2\pi$ correspond to φ and $\varphi + \pi$, two distinct points in the w-plane. This is the complex variable analog of the simple real variable equation $y^2 = x$, in which two values of y, plus and minus, correspond to each value of x. Replacing $z \rightarrow 1/z$ for $z \rightarrow 0$ in Eq. (6.100) shows that our function $w(z)$ behaves similarly around the point at infinity.

The important point here is that we can make the function w of Eq. (6.100) a single-valued function instead of a double-valued function if we agree to restrict θ to a range such as $0 \leq \theta < 2\pi$. This may be done by agreeing never to cross the line $\theta = 0$ in the z-plane (Fig. 6.22). Such a line of demarcation is called a **cut line**.

The **cut line joins the two branch point singularities** at 0 and ∞, where the function is clearly not analytic. Any line from $z = 0$ to infinity would serve

Figure 6.22

A Cut Line



equally well. The purpose of the cut line is to restrict the argument of z. The points z and $z\exp(2\pi i)$ coincide in the z-plane but yield different points w and $-w = w\exp(\pi i)$ in the w-plane. Hence, in the absence of a cut line the function $w = z^{1/2}$ is ambiguous. Alternatively, since the function $w = z^{1/2}$ is double valued, we can also glue two sheets of the complex z-plane together along the cut line so that $\arg(z)$ increases beyond 2π along the cut line and steps down from 4π on the second sheet to the start on the first sheet. This construction is called the **Riemann surface** of $w = z^{1/2}$. We shall encounter branch points and cut lines frequently in Chapter 7.

The transformation

$$w = e^z \tag{6.103}$$

leads to

$$\rho e^{i\varphi} = e^{x+iy} \tag{6.104}$$

or

$$\rho = e^x, \qquad \varphi = y. \tag{6.105}$$

If y ranges from $0 \le y < 2\pi$ (or $-\pi < y \le \pi$), then φ covers the same range. However, this is the whole w-plane. In other words, a horizontal strip in the z-plane of width 2π maps into the entire w-plane. Furthermore, any point $x + i(y+2n\pi)$, in which n is any integer, maps into the same point [by Eq. (6.104)], in the w-plane. We have a many-(infinitely many)-to-one correspondence.

Finally, as the inverse of the fifth transformation [Eq. (6.103)], we have

$$w = \ln z. \tag{6.106}$$

By expanding it, we obtain

$$u + iv = \ln re^{i\theta} = \ln r + i\theta. \tag{6.107}$$

For a given point z_0 in the z-plane, the argument θ is unspecified within an integral multiple of 2π. This means that

$$v = \theta + 2n\pi, \tag{6.108}$$

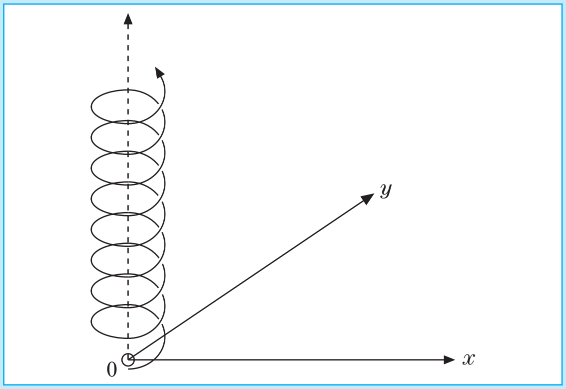
and, as in the exponential transformation, we have an infinitely many-to-one correspondence.

Equation (6.106) has a nice physical representation. If we go around the unit circle in the z-plane, $r = 1$, and by Eq. (6.107), $u = \ln r = 0$; however, $v = \theta$, and θ is steadily increasing and continues to increase as θ continues, past 2π.

The cut line joins the branch point at the origin with infinity. As θ increases past 2π, we glue a new sheet of the complex z-plane along the cut line, etc. Going around the unit circle in the z-plane is like the advance of a screw as it is rotated or the ascent of a person walking up a spiral staircase (Fig. 6.23), which is the **Riemann surface** of $w = \ln z$.

Figure 6.23

The Riemann Surface for ln z, a Multivalued Function



As in the preceding example, we can also make the correspondence unique [and Eq. (6.106) unambiguous] by restricting θ to a range such as $0 \leq \theta < 2\pi$ by taking the line $\theta = 0$ (positive real axis) as a cut line. This is equivalent to taking one and only one complete turn of the spiral staircase.

It is because of the multivalued nature of $\ln z$ that the contour integral

$$\oint \frac{dz}{z} = 2\pi i \neq 0,$$

integrating about the origin. This property appears in Exercise 6.4.1 and is the basis for the entire calculus of residues (Chapter 7).

SUMMARY

The concept of mapping is a very broad and useful one in mathematics. Our mapping from a complex z-plane to a complex w-plane is a simple generalization of one definition of function: a mapping of x (from one set) into y in a second set.

A more sophisticated form of mapping appears in Section 1.14, in which we use the Dirac delta function $\delta(x - a)$ to map a function $f(x)$ into its value at the point a. In Chapter 15, integral transforms are used to map one function $f(x)$ in x-space into a second (related) function $F(t)$ in t-space.

EXERCISES

6.6.1 How do circles centered on the origin in the z-plane transform for

(a) $w_1(z) = z + \frac{1}{z}$, (b) $w_2(z) = z - \frac{1}{z}$, for $z \neq 0$?

What happens when $|z| \to 1$?

6.6.2 What part of the z-plane corresponds to the interior of the unit circle in the w-plane if

(a) $w = \frac{z-1}{z+1}$, (b) $w = \frac{z-i}{z+i}$?

6.6.3 Discuss the transformations

(a) $w(z) = \sin z$, (c) $w(z) = \sinh z$,
(b) $w(z) = \cos z$, (d) $w(z) = \cosh z$.

Show how the lines $x = c_1$, $y = c_2$ map into the w-plane. Note that the last three transformations can be obtained from the first one by appropriate translation and/or rotation.

6.6.4 Show that the function

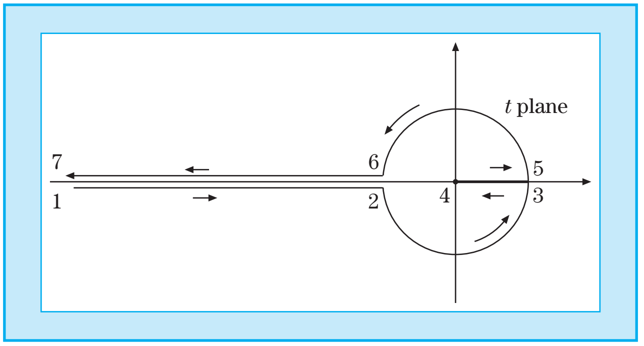
$$w(z) = (z^2 - 1)^{1/2}$$

is single-valued if we take $-1 \leq x \leq 1$, $y = 0$ as a cut line.

6.6.5 An integral representation of the Bessel function follows the contour in the t-plane shown in Fig. 6.24. Map this contour into the θ-plane with $t = e^\theta$. Many additional examples of mapping are given in Chapters 11–13.

Figure 6.24

Bessel Function Integration Contour



6.6.6 For noninteger m, show that the binomial expansion of Exercise 6.5.2 holds only for a suitably defined branch of the function $(1 + z)^m$. Show how the z-plane is cut. Explain why $|z| < 1$ may be taken as the circle of convergence for the expansion of this branch, in light of the cut you have chosen.

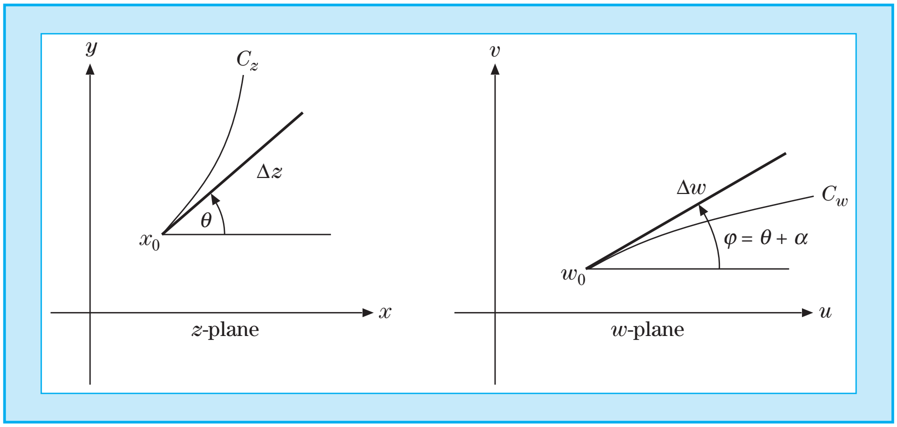
6.6.7 The Taylor expansion of Exercises 6.5.2 and 6.6.6 is **not** suitable for branches other than the one suitably defined branch of the function $(1+z)^m$ for noninteger m. [Note that other branches cannot have the same Taylor expansion since they must be distinguishable.] Using the same branch cut of the earlier exercises for all other branches, find the corresponding Taylor expansions detailing the phase assignments and Taylor coefficients.

6.7 Conformal Mapping

In Section 6.6, hyperbolas were mapped into straight lines and straight lines were mapped into circles. However, in all these transformations one feature, angles, stayed constant, which we now address. This constancy was a result of the fact that all the transformations of Section 6.6 were analytic.

Figure 6.25

Conformal Mapping—Preservation of Angles



As long as $w = f(z)$ is an analytic function, we have

$$\frac{df}{dz} = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{6.109}$$

Assuming that this equation is in polar form, we may equate modulus to modulus and argument to argument. For the latter (assuming that $df/dz \neq 0$),

$$\begin{aligned} \arg \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta z \to 0} \arg \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z = \arg \frac{df}{dz} = \alpha, \end{aligned} \tag{6.110}$$

where α, the argument of the derivative, may depend on z but is a constant for fixed z, independent of the direction of approach. To see the significance of this, consider two curves, C_z in the z-plane and the corresponding curve C_w in the w-plane (Fig. 6.25). The increment Δz is shown at an angle of θ relative to the real (x) axis, whereas the corresponding increment Δw forms an angle of φ with the real (u) axis. From Eq. (6.110),

$$\varphi = \theta + \alpha, \tag{6.111}$$

or any line in the z-plane is rotated through an angle α in the w-plane as long as w is an analytic transformation and the derivative is not zero.[14]

Since this result holds for any line through z_0, it will hold for a pair of lines. Then for the angle between these two lines

$$\varphi_2 - \varphi_2 = (\theta_2 + \alpha) - (\theta_1 + \alpha) = \theta_2 - \theta_1, \tag{6.112}$$

which shows that the included angle is preserved under an analytic transformation. Such **angle-preserving transformations are called conformal**. The rotation angle α will, in general, depend on z. In addition, $|f'(z)|$ will usually be a function of z.

[14]If $df/dz = 0$, its argument or phase is undefined and the (analytic) transformation will not necessarily preserve angles.

SUMMARY

Historically, conformal transformations have been of great importance to scientists and engineers in solving Laplace's equation for problems of electrostatics, hydrodynamics, heat flow, and so on. Unfortunately, the conformal transformation approach, however elegant, is limited to problems that can be reduced to two dimensions. The method is often beautiful if there is a high degree of symmetry present but often impossible if the symmetry is broken or absent.

Because of these limitations and primarily because high-speed electronic computers offer a useful alternative (iterative solution of the partial differential equation), the details and applications of conformal mappings are omitted.

EXERCISES

6.7.1 Expand $w(x)$ in a Taylor series about the point $z = z_0$, where $f'(z_0) = 0$. (Angles are not preserved.) Show that if the first $n-1$ derivatives vanish but $f^{(n)}(z_0) \neq 0$, then angles in the z-plane with vertices at $z = z_0$ appear in the w-plane multiplied by n.

6.7.2 In the transformation

$$e^z = \frac{a - w}{a + w},$$

how do the coordinate lines in the z-plane transform? What coordinate system have you constructed?

6.7.3 Develop a conformal transformation that maps a circle in the z-plane into a circle in the w-plane. Consider first circles with centers at the origin and then those with arbitrary centers. Plot several cases using graphical software.

6.7.4 Develop a conformal transformation that maps straight lines parallel to the coordinate axes in the z-plane into parabolas in the w-plane. Plot several parabolas using graphical software.

Additional Reading

Ahlfors, L. V. (1979). *Complex Analysis*, 3rd ed. McGraw-Hill, New York. This text is detailed, thorough, rigorous, and extensive.

Churchill, R. V., Brown, J. W., and Verkey, R. F. (1989). *Complex Variables and Applications*, 5th ed. McGraw-Hill, New York. This is an excellent text for both the beginning and advanced student. It is readable and quite complete. A detailed proof of the Cauchy–Goursat theorem is given in Chapter 5.

Greenleaf, F. P. (1972). *Introduction to Complex Variables*. Saunders, Philadelphia. This very readable book has detailed, careful explanations.

Kurala, A. (1972). *Applied Functions of a Complex Variable*. Wiley–Interscience, New York. An intermediate-level text designed for scientists and engineers. Includes many physical applications.

Levinson, N., and Redheffer, R. M. (1970). *Complex Variables*. Holden-Day, San Francisco. This text is written for scientists and engineers who are interested in applications.

Morse, P. M., and Feshbach, H. (1953). *Methods of Theoretical Physics*. McGraw-Hill, New York. Chapter 4 is a presentation of portions of the theory of functions of a complex variable of interest to theoretical physicists.

Remmert, R. (1991). *Theory of Complex Functions*. Springer, New York.

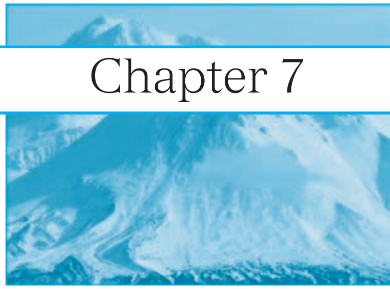
Sokolnikoff, I. S., and Redheffer, R. M. (1966). *Mathematics of Physics and Modern Engineering*, 2nd ed. McGraw-Hill, New York. Chapter 7 covers complex variables.

Spiegel, M. R. (1985). *Complex Variables*. McGraw-Hill, New York. An excellent summary of the theory of complex variables for scientists.

Titchmarsh, E. C. (1958). *The Theory of Functions*, 2nd ed. Oxford Univ. Press, New York. A classic.

Watson, G. N. (1917/1960). *Complex Integration and Cauchy's Theorem*. Hafner, New York. A short work containing a rigorous development of the Cauchy integral theorem and integral formula. Applications to the calculus of residues are included. *Cambridge Tracts in Mathematics, and Mathematical Physics*, No. 15.

Other references are given at the end of Chapter 15.



Chapter 7

Functions of a Complex Variable II

Calculus of Residues

7.1 Singularities

In this chapter we return to the line of analysis that started with the Cauchy–Riemann conditions in Chapter 6 and led to the Laurent expansion (Section 6.5). The Laurent expansion represents a generalization of the Taylor series in the presence of singularities. We define the point z_0 as an **isolated singular point** of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic and single valued in a punctured disk $0 < |z - z_0| < R$ for some positive R. For rational functions, which are quotients of polynomials, $f(z) = P(z)/Q(z)$, the only singularities arise from zeros of the denominator if the numerator is nonzero there. For example, $f(z) = \frac{z^3-2z^2+1}{(z-3)(z^2+3)}$ from Exercise 6.5.13 has simple poles at $z = \pm i\sqrt{3}$ and $z = 3$ and is regular everywhere else. A function that is analytic throughout the finite complex plane **except** for isolated poles is called **meromorphic**. Examples are **entire** functions that have no singularities in the finite complex plane, such as e^z, $\sin z$, $\cos z$, rational functions with a finite number of poles, or $\tan z$, $\cot z$ with infinitely many isolated simple poles at $z = n\pi$ and $z = (2n+1)\pi/2$ for $n = 0, \pm 1, \pm 2, \ldots$, respectively.

From Cauchy's integral we learned that a loop integral of a function around a simple pole gives a nonzero result, whereas higher order poles do not contribute to the integral (Example 6.3.1). We consider in this chapter the generalization of this case to meromorphic functions leading to the residue theorem, which has important applications to many integrals that physicists and engineers encounter, some of which we will discuss. Here, singularities, and simple poles in particular, play a dominant role.

Poles

In the Laurent expansion of $f(z)$ about z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{7.1}$$

if $a_n = 0$ for $n < -m < 0$ and $a_{-m} \neq 0$, we say that z_0 is a pole of order m. For instance, if $m = 1$—that is, if $a_{-1}/(z - z_0)$ is the first nonvanishing term in the Laurent series—we have a pole of order 1, often called a simple pole. Example 6.5.1 is a relevant case: The function

$$f(z) = [z(z-1)]^{-1} = -\frac{1}{z} + \frac{1}{z-1}$$

has a simple pole at the origin and at $z = 1$. Its square, $f^2(z)$, has poles of order 2 at the same places and $[f(z)]^m$ has poles of order $m = 1, 2, 3 \ldots$. In contrast, the function

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{z^n n!}$$

from Example 6.5.2 has poles of any order at $z = 0$.

If there are poles of any order (i.e., the summation in the Laurent series at z_0 continues to $n = -\infty$), then z_0 is a pole of infinite order and is called an **essential singularity**. These essential singularities have many pathological features. For instance, we can show that in any small neighborhood of an essential singularity of $f(z)$ the function $f(z)$ comes arbitrarily close to any (and therefore every) preselected complex quantity w_0.[1] Literally, the entire w-plane is mapped into the neighborhood of the point z_0, the essential singularity. One point of fundamental difference between a pole of finite order and an essential singularity is that a pole of order m can be removed by multiplying $f(z)$ by $(z - z_0)^m$. This obviously cannot be done for an essential singularity.

The behavior of $f(z)$ as $z \to \infty$ is defined in terms of the behavior of $f(1/t)$ as $t \to 0$. Consider the function

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}. \tag{7.2}$$

As $z \to \infty$, we replace the z by $1/t$ to obtain

$$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,t^{2n+1}}. \tag{7.3}$$

Clearly, from the definition, $\sin z$ has an essential singularity at infinity. This result could be anticipated from Exercise 6.1.9 since

$$\sin z = \sin iy = i \sinh y, \quad \text{when} \quad x = 0,$$

[1]This theorem is due to Picard. A proof is given by E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. Oxford Univ. Press, New York (1939).

which approaches infinity exponentially as $y \to \infty$. Thus, although the absolute value of $\sin x$ for real x is equal to or less than unity, the absolute value of $\sin z$ is not bounded. The same applies to $\cos z$.

Branch Points

There is another sort of singularity that will be important in the later sections of this chapter and that we encountered in Chapter 6 in the context of inverting powers and the exponential function, namely roots and logarithms. Consider

$$f(z) = z^a,$$

in which a **is not an integer**.[2] As z moves around the unit circle from e^0 to $e^{2\pi i}$,

$$f(z) \to e^{2\pi ai} \neq e^{0i}$$

for nonintegral a. As in Section 6.6, we have a branch point at the origin and another at infinity. If we set $z = 1/t$, a similar analysis for $t \to 0$ shows that $t = 0$; that is, ∞ is also a branch point. The points e^{0i} and $e^{2\pi i}$ in the z-plane coincide but these **coincident points lead to different values** of $f(z)$; that is, $f(z)$ is a **multivalued function**. The problem is resolved by constructing a **cut line joining both branch points** so that $f(z)$ will be uniquely specified for a given point in the z-plane. For z^a the cut line can go out at any angle. Note that the point at infinity must be included here; that is, the cut line may join finite branch points via the point at infinity. The next example is a case in point. If $a = p/q$ is a rational number, then q is called the order of the branch point because one needs to go around the branch point q times before coming back to the starting point or, equivalently, the Riemann surface of $z^{1/q}$ and $z^{p/q}$ is made up of q sheets, as discussed in Chapter 6. If a is irrational, then the order of the branch point is infinite, just as for the logarithm.

Note that a function with a branch point and a required cut line will not be continuous across the cut line. In general, there will be a phase difference on opposite sides of this cut line. Exercise 7.2.23 is an example of this situation. Hence, line integrals on opposite sides of this branch point cut line will not generally cancel each other. Numerous examples of this case appear in the exercises.

The contour line used to convert a multiply connected region into a simply connected region (Section 6.3) is completely different. Our function is continuous across this contour line, and no phase difference exists.

EXAMPLE 7.1.1

Function with Two Branch Points Consider the function

$$f(z) = (z^2 - 1)^{1/2} = (z + 1)^{1/2}(z - 1)^{1/2}. \tag{7.4}$$

[2] $z = 0$ is technically a singular point because z^a has only a finite number of derivatives, whereas an analytic function is guaranteed an infinite number of derivatives (Section 6.4). The problem is that $f(z)$ is not single-valued as we encircle the origin. The Cauchy integral formula may not be applied.

The first factor on the right-hand side, $(z+1)^{1/2}$, has a branch point at $z = -1$. The second factor has a branch point at $z = +1$. Each branch point has order 2 because the Riemann surface is made up of two sheets. At infinity $f(z)$ has a simple pole. This is best seen by substituting $z = 1/t$ and making a binomial expansion at $t = 0$

$$(z^2 - 1)^{1/2} = \frac{1}{t}(1 - t^2)^{1/2} = \frac{1}{t}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n t^{2n} = \frac{1}{t} - \frac{1}{2}t - \frac{1}{8}t^3 + \cdots.$$

The **cut line has to connect both branch points** so that it is not possible to encircle either branch point completely. To check the possibility of taking the line segment joining $z = +1$ and $z = -1$ as a cut line, let us follow the phases of these two factors as we move along the contour shown in Fig. 7.1.

For convenience in following the changes of phase, let $z + 1 = re^{i\theta}$ and $z - 1 = \rho e^{i\varphi}$. Then the phase of $f(z)$ is $(\theta + \varphi)/2$. We start at point 1, where both $z + 1$ and $z - 1$ have a phase of zero. Moving from point 1 to point 2, φ, the phase of $z - 1 = \rho e^{i\varphi}$ increases by π. ($z - 1$ becomes negative.) φ then stays constant until the circle is completed, moving from 6 to 7. θ, the phase of $z + 1 = re^{i\theta}$, shows a similar behavior increasing by 2π as we move from 3 to 5. The phase of the function $f(z) = (z+1)^{1/2}(z-1)^{1/2} = r^{1/2}\rho^{1/2}e^{i(\theta+\varphi)/2}$ is $(\theta + \varphi)/2$. This is tabulated in the final column of Table 7.1.

Table 7.1

Phase Angle

Point	θ	φ	$(\theta+\varphi)/2$
1	0	0	0
2	0	π	$\pi/2$
3	0	π	$\pi/2$
4	π	π	π
5	2π	π	$3\pi/2$
6	2π	π	$3\pi/2$
7	2π	2π	2π

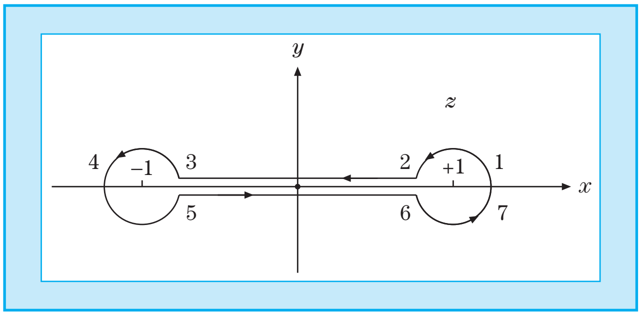


Figure 7.1

Cut Line Joining Two Branch Points at ±1

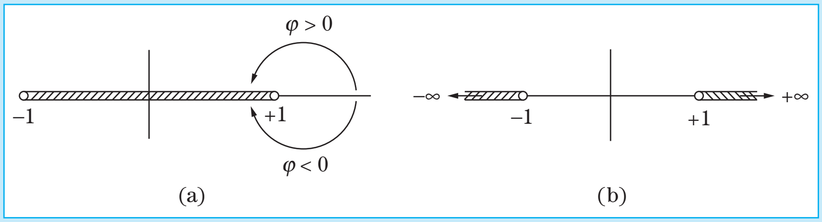


Figure 7.2

Branch Points Joined by (a) a Finite Cut Line and (b) Two Cut Lines from 1 to ∞ and -1 to $-\infty$ that Form a Single Cut Line Through the Point at Infinity. Phase Angles are Measured as Shown in (a)

Two features emerge:

1. The phase at points 5 and 6 is not the same as the phase at points 2 and 3. This behavior can be expected at a branch point cut line.
2. The phase at point 7 exceeds that at point 1 by 2π and the function $f(z) = (z^2 - 1)^{1/2}$ is therefore **single-valued** for the contour shown, encircling **both** branch points.

If we take the x-axis $-1 \le x \le 1$ as a cut line, $f(z)$ is uniquely specified. Alternatively, the positive x-axis for $x > 1$ and the negative x-axis for $x < -1$ may be taken as cut lines. In this case, the branch points at ± 1 are joined by the cut line via the point at infinity. Again, the branch points cannot be encircled and the function remains single-valued. ■

Generalizing from this example, the phase of a function

$$f(z) = f_1(z) \cdot f_2(z) \cdot f_3(z) \cdots$$

is the algebraic sum of the phase of its individual factors:

$$\arg f(z) = \arg f_1(z) + \arg f_2(z) + \arg f_3(z) + \cdots.$$

The phase of an individual factor may be taken as the arctangent of the ratio of its imaginary part to its real part,

$$\arg f_i(z) = \tan^{-1}(v_i/u_i),$$

but one should be aware of the different branches of arctangent. For the case of a factor of the form

$$f_i(z) = (z - z_0),$$

the phase corresponds to the phase angle of a two-dimensional vector from $+z_0$ to z, with the phase increasing by 2π as the point $+z_0$ is encircled provided, it is measured without crossing a cut line ($z_0 = 1$ in Fig. 7.2a). Conversely, the traversal of any closed loop not encircling z_0 does not change the phase of $z - z_0$.

SUMMARY

Poles are the simplest singularities, and functions that have only poles besides regular points are called meromorphic. Examples are $\tan z$ and ratios of polynomials. Branch points are singularities characteristic of multivalent functions. Examples are fractional powers of the complex variable z, whereas the logarithm has branch points of infinite order at the origin of the complex plane and at infinity. Essential singularities are the most complicated ones, and many functions have one, such as $\cos z$, $\sin z$ at infinity.

EXERCISES

7.1.1 The function $f(z)$ expanded in a Laurent series exhibits a pole of order m at $z = z_0$. Show that the coefficient of $(z - z_0)^{-1}$, a_{-1}, is given by

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)]_{z=z_0},$$

with

$$a_{-1} = [(z - z_0)f(z)]_{z=z_0},$$

when the pole is a simple pole ($m = 1$). These equations for a_{-1} are extremely useful in determining the residue to be used in the residue theorem of the next section.

Hint. The technique that was so successful in proving the uniqueness of power series (Section 5.7) will work here also.

7.1.2 A function $f(z)$ can be represented by

$$f(z) = \frac{f_1(z)}{f_2(z)},$$

where $f_1(z)$ and $f_2(z)$ are analytic. The denominator $f_2(z)$ vanishes at $z = z_0$, showing that $f(z)$ has a pole at $z = z_0$. However, $f_1(z_0) \neq 0$, $f_2'(z_0) \neq 0$. Show that a_{-1}, the coefficient of $(z - z_0)^{-1}$ in a Laurent expansion of $f(z)$ at $z = z_0$, is given by

$$a_{-1} = \frac{f_1(z_0)}{f_2'(z_0)}.$$

This result leads to the Heaviside expansion theorem (Section 15.12).

7.1.3 In analogy with Example 7.1.1, consider in detail the phase of each factor and the resultant overall phase of $f(z) = (z^2 + 1)^{1/2}$ following a contour similar to that of Fig. 7.1 but encircling the new branch points.

7.1.4 As an example of an essential singularity, consider $e^{1/z}$ as z approaches zero. For any complex number z_0, $z_0 \neq 0$, show that

$$e^{1/z} = z_0$$

has an infinite number of solutions.

7.1.5 If the analytic function $f(z)$ goes to zero for $|z| \to \infty$, show that its residue (a_{-1} as defined in Exercise 7.1.1) at infinity is $-\lim_{z\to\infty} zf(z)$; if $f(z)$ has a finite (nonzero) limit at infinity, show that its residue at infinity is $-\lim_{z\to\infty} z^2 f'(z)$.

7.2 Calculus of Residues

Residue Theorem

If the Laurent expansion of a function $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is integrated term by term by using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain (Example 6.3.1)

$$a_n \oint (z-z_0)^n dz = a_n \frac{(z-z_0)^{n+1}}{n+1}\bigg|_{z_1}^{z_1} = 0, \quad n \neq -1. \tag{7.5}$$

However, if $n = -1$, using the polar form $z = z_0 + re^{i\theta}$ we find that

$$a_{-1} \oint (z-z_0)^{-1} dz = a_{-1} \oint \frac{ire^{i\theta} d\theta}{re^{i\theta}} = 2\pi i a_{-1}. \tag{7.6}$$

The first and simplest case of a residue occurred in Example 6.3.1 involving $\oint z^n dz = 2\pi n i \delta_{n,-1}$, where the integration is anticlockwise around a circle of radius r. Of all powers z^n, only $1/z$ contributes.

Summarizing Eqs. (7.5) and (7.6), we have

$$\frac{1}{2\pi i} \oint f(z)\, dz = a_{-1}. \tag{7.7}$$

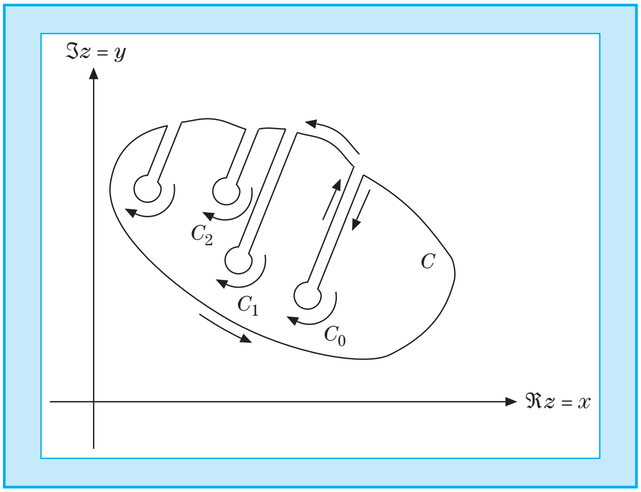
The constant a_{-1}, the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion, is called the residue of $f(z)$ at $z = z_0$.

A set of isolated singularities can be handled by deforming our contour as shown in Fig. 7.3. Cauchy's integral theorem (Section 6.3) leads to

$$\oint_C f(z)\, dz + \oint_{C_0} f(z)\, dz + \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \cdots = 0. \tag{7.8}$$

Figure 7.3

Excluding Isolated Singularities



The circular integral around any given singular point is given by Eq. (7.7),

$$\oint_{C_i} f(z)\,dz = -2\pi i a_{-1z_i}, \tag{7.9}$$

assuming a Laurent expansion about the singular point $z = z_i$. The negative sign comes from the clockwise integration as shown in Fig. 7.3. Combining Eqs. (7.8) and (7.9), we have

$$\begin{aligned}\oint_C f(z)\,dz &= 2\pi i(a_{-1z_0} + a_{-1z_1} + a_{-1z_2} + \cdots)\\ &= 2\pi i \;\text{(sum of enclosed residues)}.\end{aligned} \tag{7.10}$$

This is the **residue theorem**. The problem of evaluating one or more contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points (poles of order 1). In the remainder of this section, we apply the residue theorem to a wide variety of definite integrals of mathematical and physical interest. The residue theorem will also be needed in Chapter 15 for a variety of integral transforms, particularly the inverse Laplace transform. We also use the residue theorem to develop the concept of the Cauchy principal value.

Using the transformation $z = 1/w$ for $w \to 0$, we can find the nature of a singularity at $z \to \infty$ and the residue of a function $f(z)$ with just isolated singularities and no branch points. In such cases, we know that

$$\sum \{\text{residues in the finite } z\text{-plane}\} + \{\text{residue at } z \to \infty\} = 0.$$

Evaluation of Definite Integrals

The calculus of residues is useful in evaluating a wide variety of definite integrals in both physical and purely mathematical problems. We consider, first, **integrals of the form**

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta, \tag{7.11}$$

where f is finite for all values of θ. We also require f to be a rational function of $\sin\theta$ and $\cos\theta$ so that it will be single-valued. Let

$$z = e^{i\theta}, \quad dz = ie^{i\theta}\,d\theta.$$

From this,

$$d\theta = -i\frac{dz}{z}, \quad \sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}. \tag{7.12}$$

Our integral becomes

$$I = -i\oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z}, \tag{7.13}$$

with the path of integration the unit circle. By the residue theorem [Eq. (7.10)],

$$I = (-i)2\pi i \sum \text{residues within the unit circle.} \tag{7.14}$$

Note that we want to determine the residues of $f(z)/z$. Illustrations of integrals of this type are provided by Exercises 7.2.6–7.2.9.

EXAMPLE 7.2.1

Reciprocal Cosine Our problem is to evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon \cos\theta}, \qquad |\varepsilon| < 1.$$

By Eq. (7.13), this becomes

$$I = -i \oint_{\text{unit circle}} \frac{dz}{z[1 + (\varepsilon/2)(z + z^{-1})]}$$
$$= -i\frac{2}{\varepsilon} \oint \frac{dz}{z^2 + (2/\varepsilon)z + 1}.$$

The denominator has roots

$$z_- = -\frac{1}{\varepsilon} - \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2} \qquad \text{and} \qquad z_+ = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1-\varepsilon^2}$$

and can be written as $(z - z_+)(z - z_-)$. Here, z_+ is within the unit circle; z_- is outside. Near z_+ the denominator can be expanded as

$$z^2 + \frac{2}{\epsilon}z + 1 = 0 + (z - z_+)\frac{d}{dz}\left(z^2 + \frac{2}{\epsilon}z + 1\right)\bigg|_{z_+} = (z - z_+)\left(2z_+ + \frac{2}{\epsilon}\right)$$

so that the residue at z_+ is $\frac{1}{2z_+ + 2/\epsilon}$. (See Exercise 7.1.1.) Then by Eq. (7.14),

$$I = -i\frac{2}{\varepsilon} \cdot 2\pi i \frac{1}{2z + 2/\varepsilon}\bigg|_{z=-1/\varepsilon+(1/\varepsilon)\sqrt{1-\varepsilon^2}}.$$

Hence,

$$\int_0^{2\pi} \frac{d\theta}{1 + \varepsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}, \qquad |\varepsilon| < 1. \quad ■$$

Now consider a **class of definite integrals** that have the form

$$I = \int_{-\infty}^{\infty} f(x)\, dx \tag{7.15}$$

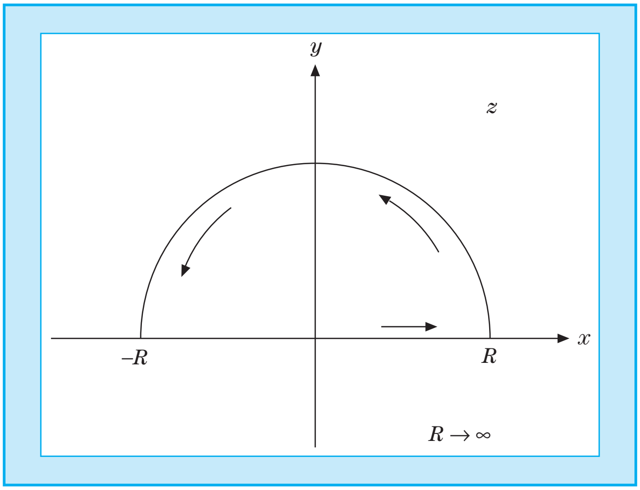
and satisfy the two conditions:

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles. (It will be assumed that there are no poles on the real axis. If poles are present on the real axis, they may be included or excluded as discussed elsewhere.)
- $f(z)$ vanishes as strongly[3] as $1/z^2$ for $|z| \to \infty$, $\quad 0 \le \arg\ z \le \pi$.

[3] We could use $f(z)$ vanishes faster than $1/z$; that is, the second condition is overly sufficient, and we wish to have $f(z)$ single-valued.

Figure 7.4

Path of Integration is a Half Circle in the Upper Half Plane



With these conditions, we may take as a contour of integration the real axis and a semicircle in the upper half-plane as shown in Fig. 7.4. We let the radius R of the semicircle become infinitely large. Then

$$\oint f(z)\,dz = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{0}^{\pi} f(Re^{i\theta})iRe^{i\theta}\,d\theta$$
$$= 2\pi i \sum \text{residues (upper half-plane)} \qquad (7.16)$$

From the second condition, the second integral (over the semicircle) vanishes and

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{residues (upper half-plane).} \qquad (7.17)$$

Note that a corresponding result is obtained when f is analytic in the lower half-plane and we use a contour in the lower half-plane. In that case, the contour will be tracked clockwise and the residues will enter with a minus sign.

EXAMPLE 7.2.2

Inverse Polynomial Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}. \qquad (7.18)$$

From Eq. (7.16),

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \sum \text{residues (upper half-plane).}$$

Here and in every other similar problem, we have the question: Where are the poles? Rewriting the integrand as

$$\frac{1}{z^2+1} = \frac{1}{z+i}\cdot\frac{1}{z-i}, \qquad (7.19)$$

we see that there are simple poles (order 1) at $z = i$ and $z = -i$.

A simple pole at $z = z_0$ indicates (and is indicated by) a Laurent expansion of the form

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n. \tag{7.20}$$

The residue a_{-1} is easily isolated as (Exercise 7.1.1)

$$a_{-1} = (z - z_0) f(z) \,|_{z=z_0}. \tag{7.21}$$

Using Eq. (7.21), we find that the residue at $z = i$ is $1/2i$, whereas that at $z = -i$ is $-1/2i$. Another way to see this is to write the partial fraction decomposition:

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Then

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \cdot \frac{1}{2i} = \pi. \tag{7.22}$$

Here, we used $a_{-1} = 1/2i$ for the residue of the one included pole at $z = i$. Readers should satisfy themselves that it is possible to use the lower semicircle and that this choice will lead to the same result: $I = \pi$. ■

A more delicate problem is provided by the next example.

EXAMPLE 7.2.3

Evaluation of Definite Integrals Consider **definite integrals of the form**

$$I = \int_{-\infty}^{\infty} f(x) e^{iax} dx, \tag{7.23}$$

with a real and positive. This is a Fourier transform (Chapter 15). We assume the two conditions:

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles.
- $\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg\ z \le \pi.$ (7.24)

Note that this is a less restrictive condition than the second condition imposed on $f(z)$ for integrating $\int_{-\infty}^{\infty} f(x)\,dx$.

We employ the contour shown in Fig. 7.4 because the exponential factor goes rapidly to zero in the upper half-plane. The application of the calculus of residues is the same as the one just considered, but here we have to work harder to show that the integral over the (infinite) semicircle goes to zero. This integral becomes

$$I_R = \int_0^{\pi} f(Re^{i\theta})\, e^{iaR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta. \tag{7.25}$$

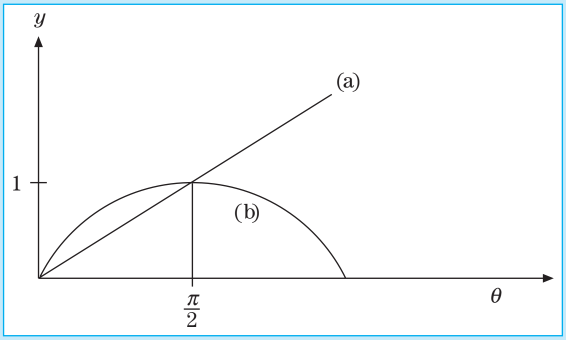


Figure 7.5

(a) $y = (2/\pi)\theta$;
(b) $y = \sin\theta$

Let R be so large that $|f(z)| = |f(Re^{i\theta})| < \varepsilon$. Then

$$|I_R| \le \varepsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta. \tag{7.26}$$

In the range $[0, \pi/2]$,

$$\frac{2}{\pi}\theta \le \sin\theta.$$

Therefore (Fig. 7.5),

$$|I_R| \le 2\varepsilon R \int_0^{\pi/2} e^{-aR2\theta/\pi}\, d\theta. \tag{7.27}$$

Now, integrating by inspection, we obtain

$$|I_R| \le 2\varepsilon R \frac{1 - e^{-aR}}{aR2/\pi}.$$

Finally,

$$\lim_{R\to\infty} |I_R| \le \frac{\pi}{a}\varepsilon. \tag{7.28}$$

From condition (7.24), $\varepsilon \to 0$ as $R \to \infty$ and

$$\lim_{R\to\infty} |I_R| = 0. \tag{7.29}$$

This useful result is sometimes called **Jordan's lemma**. With it, we are prepared to deal with Fourier integrals of the form shown in Eq. (7.23).

Using the contour shown in Fig. 7.4, we have

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx + \lim_{R\to\infty} I_R = 2\pi i \sum \text{residues (upper half-plane).}$$

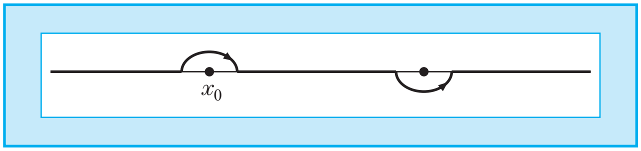


Figure 7.6

Bypassing Singular Points

Since the integral over the upper semicircle I_R vanishes as $R \to \infty$ (Jordan's lemma),

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \sum \text{residues (upper half-plane)} \quad (a > 0). \tag{7.30}$$

This result actually holds more generally for complex a with $\Re(a) > 0$. ■

Cauchy Principal Value

Occasionally, an isolated first-order pole will be directly on the contour of integration. In this case, we may deform the contour to include or exclude the residue as desired by including a semicircular detour of **infinitesimal radius**. This is shown in Fig. 7.6. The integration over the semicircle then gives, with $z - x_0 = \delta e^{i\varphi}$, $dz = i\delta e^{i\varphi}d\varphi$,

$$\int \frac{dz}{z - x_0} = i\int_{\pi}^{2\pi} d\varphi = i\pi, \text{ i.e., } \pi i a_{-1} \quad \text{if counterclockwise,}$$

$$\int \frac{dz}{z - x_0} = i\int_{\pi}^{0} d\varphi = -i\pi, \text{ i.e., } -\pi i a_{-1} \text{ if clockwise.}$$

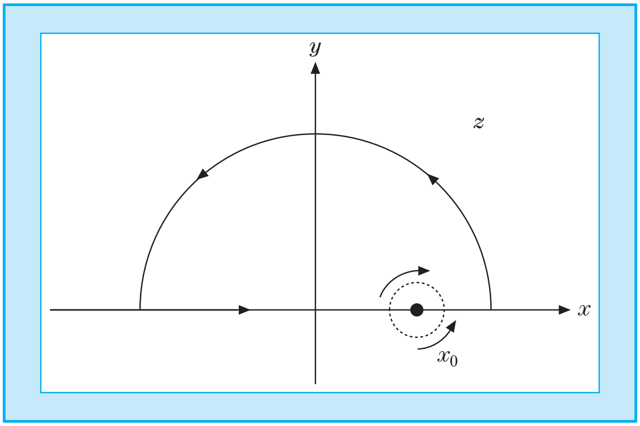
This contribution, $+$ or $-$, appears on the left-hand side of Eq. (7.10). If our detour were clockwise, the residue would not be enclosed and there would be no corresponding term on the right-hand side of Eq. (7.10). However, if our detour were counterclockwise, this residue would be enclosed by the contour C and a term $2\pi i a_{-1}$ would appear on the right-hand side of Eq. (7.10). The net result for either a clockwise or counterclockwise detour is that a simple pole on the contour is counted as one-half what it would be if it were within the contour.

For instance, let us suppose that $f(z)$ with a simple pole at $z = x_0$ is integrated over the entire real axis assuming $|f(z)| \to 0$ for $|z| \to \infty$ fast enough (faster than $1/|z|$) that the integrals in question are finite. The contour is closed with an infinite semicircle in the upper half-plane (Fig. 7.7). Then

$$\oint f(z)\,dz = \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{C_{x_0}} f(z)\,dz$$
$$+ \int_{x_0+\delta}^{\infty} f(x)\,dx + \int_C \text{infinite semicircle}$$
$$= 2\pi i \sum \textbf{enclosed} \text{ residues.} \tag{7.31}$$

Figure 7.7

Closing the Contour with an Infinite Radius Semicircle



If the small semicircle C_{x_0} includes x_0 (by going below the x-axis; counterclockwise), x_0 is enclosed, and its contribution appears **twice**—as $\pi i a_{-1}$ in $\int_{C_{x_0}}$ and as $2\pi i a_{-1}$ in the term $2\pi i \sum$ enclosed residues—for a net contribution of $\pi i a_{-1}$ on the right-hand side of Eq. (7.31). If the upper small semicircle is selected, x_0 is excluded. The only contribution is from the **clockwise** integration over C_{x_0}, which yields $-\pi i a_{-1}$. Moving this to the extreme right of Eq. (7.11), we have $+\pi i a_{-1}$, as before.

The integrals along the x-axis may be combined and the semicircle radius permitted to approach zero. We therefore define

$$\lim_{\delta \to 0} \left\{ \int_{-\infty}^{x_0 - \delta} f(x)\, dx + \int_{x_0 + \delta}^{\infty} f(x)\, dx \right\} = P \int_{-\infty}^{\infty} f(x)\, dx. \tag{7.32}$$

P indicates the Cauchy **principal value** and represents the preceding limiting process. Note that the Cauchy principal value is a balancing or canceling process; for even-order poles, $P \int_{-\infty}^{\infty} f(x)\, dx$ is not finite because there is no cancellation. In the vicinity of our singularity at $z = x_0$,

$$f(x) \approx \frac{a_{-1}}{x - x_0}. \tag{7.33}$$

This is odd, relative to x_0. The symmetric or even interval (relative to x_0) provides cancellation of the shaded areas (Fig. 7.8). The contribution of the singularity is in the integration about the semicircle.

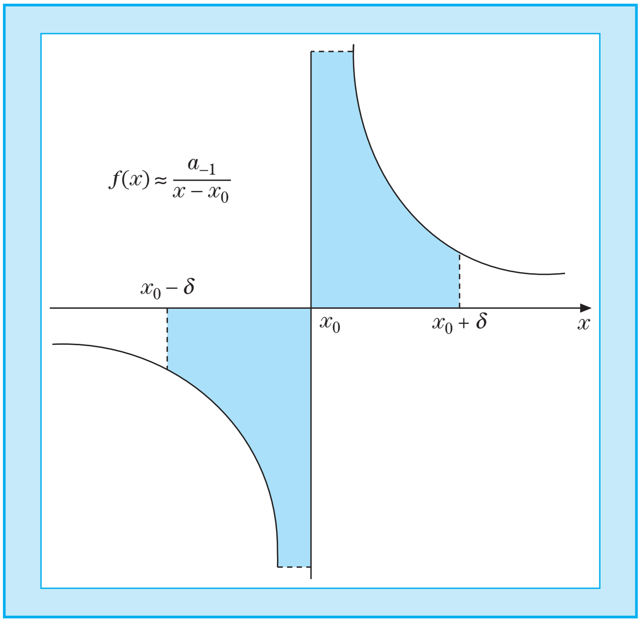
Sometimes, this same limiting technique is applied to the integration limits $\pm\infty$. If there is no singularity, we may define

$$P \int_{-\infty}^{\infty} f(x)\, dx = \lim_{a \to \infty} \int_{-a}^{a} f(x)\, dx. \tag{7.34}$$

An alternate treatment moves the pole off the contour and then considers the limiting behavior as it is brought back, in which the singular points are moved off the contour in such a way that the integral is forced into the form desired to satisfy the boundary conditions of the physical problem (for Green's functions this is often the case; see Examples 7.2.5 and 16.3.2). The principal value limit

Figure 7.8

Contour



is not necessary when a pole is removed by a zero of a numerator function. The integral

$$\int_{-\infty}^{\infty} \frac{\sin z}{z} dz = 2 \int_0^{\infty} \frac{\sin z}{z} dz = \pi,$$

evaluated next, is a case in point.

EXAMPLE 7.2.4

Singularity on Contour of Integration The problem is to evaluate

$$I = \int_0^{\infty} \frac{\sin x}{x} dx. \qquad (7.35)$$

This may be taken as half the imaginary part[4] of

$$I_z = P \int_{-\infty}^{\infty} \frac{e^{iz} dz}{z}. \qquad (7.36)$$

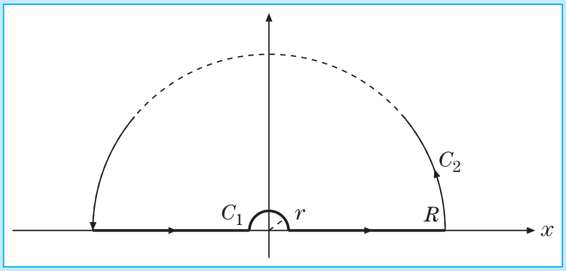
Now the only pole is a simple pole at $z = 0$ and the residue there by Eq. (7.21) is $a_{-1} = 1$. We choose the contour shown in Fig. 7.9 (i) to avoid the pole, (ii) to include the real axis, and (iii) to yield a vanishingly small integrand for $z = iy$, $y \to \infty$. Note that in this case a large (infinite) semicircle in the lower half-plane would be disastrous. We have

$$\oint \frac{e^{iz} dz}{z} = \int_{-R}^{-r} e^{ix} \frac{dx}{x} + \int_{C_1} \frac{e^{iz} dz}{z} + \int_r^R \frac{e^{ix} dx}{x} + \int_{C_2} \frac{e^{iz} dz}{z} = 0, \qquad (7.37)$$

[4]One can use $\int [(e^{iz} - e^{-iz})/2iz] dz$, but then two different contours will be needed for the two exponentials.

Figure 7.9

Singularity on Contour



the final zero coming from the residue theorem [Eq. (7.10)]. By Jordan's lemma,

$$\int_{C_2} \frac{e^{iz}dz}{z} = 0, \tag{7.38}$$

and

$$\oint \frac{e^{iz}dz}{z} = \int_{C_1} \frac{e^{iz}dz}{z} + P\int_{-\infty}^{\infty} \frac{e^{ix}dx}{x} = 0. \tag{7.39}$$

The integral over the small semicircle yields $(-)\pi i$ times the residue of 1, the minus as a result of going clockwise. Taking the imaginary part,[5] we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \tag{7.40}$$

or

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}. \tag{7.41}$$

The contour of Fig. 7.9, although convenient, is not at all unique. Another choice of contour for evaluating Eq. (7.35) is presented as Exercise 7.2.14. ■

EXAMPLE 7.2.5

Quantum Mechanical Scattering The quantum mechanical analysis of scattering leads to the function

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x dx}{x^2 - \sigma^2}, \tag{7.42}$$

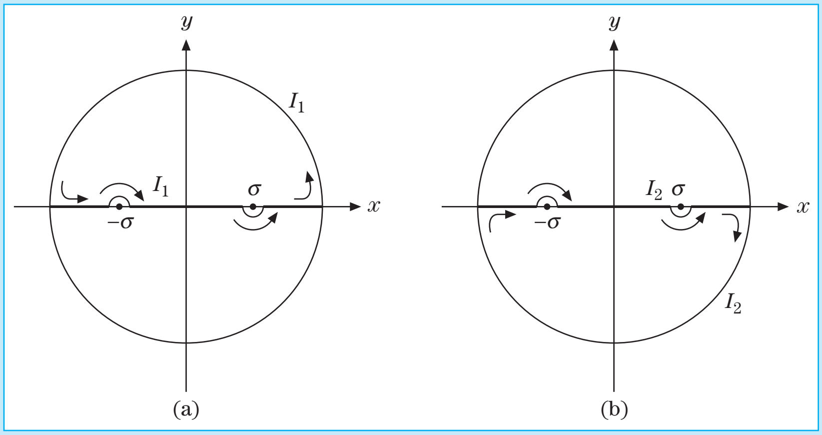
where σ is real and positive. From the physical conditions of the problem there is a further requirement: $I(\sigma)$ must have the form $e^{i\sigma}$ so that it will represent an outgoing scattered wave.

[5]Alternatively, we may combine the integrals of Eq. (7.37) as

$$\int_{-R}^{-r} e^{ix}\frac{dx}{x} + \int_{r}^{R} e^{ix}\frac{dx}{x} = \int_{r}^{R} (e^{ix} - e^{-ix})\frac{dx}{x} = 2i\int_{r}^{R} \frac{\sin x}{x} dx.$$

Figure 7.10

Contours



Using

$$\sin z = \frac{1}{2i}e^{iz} - \frac{1}{2i}e^{-iz}, \tag{7.43}$$

we write Eq. (7.42) in the complex plane as

$$I(\sigma) = I_1 + I_2, \tag{7.44}$$

with

$$I_1 = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 - \sigma^2} dz,$$

$$I_2 = -\frac{1}{2i}\int_{-\infty}^{\infty} \frac{ze^{-iz}}{z^2 - \sigma^2} dz. \tag{7.45}$$

Integral I_1 is similar to Example 7.2.2 and, as in that case, we may complete the contour by an infinite semicircle in the upper half-plane as shown in Fig. 7.10a. For I_2, the exponential is negative and we complete the contour by an infinite semicircle in the lower half-plane, as shown in Fig. 7.10b. As in Example 7.2.2, neither semicircle contributes anything to the integral–Jordan's lemma.

There is still the problem of locating the poles and evaluating the residues. We find poles at $z = +\sigma$ and $z = -\sigma$ **on the contour of integration**. The residues are (Exercises 7.1.1 and 7.2.1)

	$z = \sigma$	$z = -\sigma$
I_1	$\frac{e^{i\sigma}}{2}$	$\frac{e^{-i\sigma}}{2}$
I_2	$\frac{e^{-i\sigma}}{2}$	$\frac{e^{i\sigma}}{2}$

.

Detouring around the poles, as shown in Fig. 7.10 (it matters little whether we go above or below), we find that the residue theorem leads to

$$PI_1 - \pi i \left(\frac{1}{2i}\right)\frac{e^{-i\sigma}}{2} + \pi i \left(\frac{1}{2i}\right)\frac{e^{i\sigma}}{2} = 2\pi i \left(\frac{1}{2i}\right)\frac{e^{i\sigma}}{2} \tag{7.46}$$

because we have enclosed the singularity at $z = \sigma$ but excluded the one at $z = -\sigma$. In similar fashion, but noting that the contour for I_2 is clockwise,

$$PI_2 - \pi i \left(\frac{-1}{2i}\right)\frac{e^{i\sigma}}{2} + \pi i \left(\frac{-1}{2i}\right)\frac{e^{-i\sigma}}{2} = -2\pi i \left(\frac{-1}{2i}\right)\frac{e^{i\sigma}}{2}. \tag{7.47}$$

Adding Eqs. (7.46) and (7.47), we have

$$PI(\sigma) = PI_1 + PI_2 = \frac{\pi}{2}(e^{i\sigma} + e^{-i\sigma}) = \pi \cos\sigma. \tag{7.48}$$

This is a perfectly good evaluation of Eq. (7.42), but unfortunately the cosine dependence is appropriate for a standing wave and not for the outgoing scattered wave as specified.

To obtain the desired form, we try a different technique. We note that the integral, Eq. (7.42), is not absolutely convergent and its value will depend on the method of evaluation. Instead of dodging around the singular points, let us move them off the real axis. Specifically, let $\sigma \to \sigma + i\gamma$, $-\sigma \to -\sigma - i\gamma$, where γ is positive but small and will eventually be made to approach zero; that is, for I_1 we include one pole and for I_2 the other one:

$$I_+(\sigma) = \lim_{\gamma \to 0} I(\sigma + i\gamma). \tag{7.49}$$

With this simple substitution, the first integral I_1 becomes

$$I_1(\sigma + i\gamma) = 2\pi i \left(\frac{1}{2i}\right)\frac{e^{i(\sigma+i\gamma)}}{2} \tag{7.50}$$

by direct application of the residue theorem. Also,

$$I_2(\sigma + i\gamma) = -2\pi i \left(\frac{-1}{2i}\right)\frac{e^{i(\sigma+i\gamma)}}{2}. \tag{7.51}$$

Adding Eqs. (7.50) and (7.51) and then letting $\gamma \to 0$, we obtain

$$\begin{aligned} I_+(\sigma) &= \lim_{\gamma \to o} [I_1(\sigma + i\gamma) + I_2(\sigma + i\gamma)] \\ &= \lim_{\gamma \to 0} \pi e^{i(\sigma+i\gamma)} = \pi e^{i\sigma}, \end{aligned} \tag{7.52}$$

a result that does fit the boundary conditions of our scattering problem.

It is interesting to note that the substitution $\sigma \to \sigma - i\gamma$ would have led to

$$I_-(\sigma) = \pi e^{-i\sigma}, \tag{7.53}$$

which could represent an incoming wave. Our earlier result [Eq. (7.48)] is seen to be the arithmetic average of Eqs. (7.52) and (7.53). This average is the Cauchy principal value of the integral. Note that we have these possibilities [Eqs. (7.48), (7.52), and (7.53)] because our integral is not uniquely defined until we specify the particular limiting process (or average) to be used. ■

Pole Expansion of Meromorphic Functions

Analytic functions $f(z)$ that have only separated poles as singularities are called **meromorphic**. Examples are ratios of polynomials, $\cot z$ (see Example 7.2.7) and $\frac{f'(z)}{f(z)}$ of entire functions. For simplicity, we assume that these poles at finite $z = z_n$ with $0 < |z_1| < |z_2| < \cdots$ are all simple, with residues b_n. Then an expansion of $f(z)$ in terms of $b_n(z - z_n)^{-1}$ depends only on **intrinsic properties** of $f(z)$, in contrast to the Taylor expansion about an arbitrary analytic point of $f(z)$ or the Laurent expansion about some singular point of $f(z)$.

EXAMPLE 7.2.6

Rational Functions Rational functions are ratios of polynomials that can be completely factored according to the fundamental theorem of algebra. A partial fraction expansion then generates the pole expansion. Let us consider a few simple examples. We check that the meromorphic function

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$$

has simple poles at $z = 0$, $z = -1$ with residues $\frac{z}{z(z+1)}|_{z=0} = \frac{1}{z+1}|_{z=0} = 1$ and $\frac{z+1}{z(z+1)}|_{z=-1} = \frac{1}{z} = -1$, respectively. At ∞, $f(z)$ has a second order zero. Similarly, the meromorphic function

$$f(z) = \frac{1}{z^2 - 1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

has simple poles at $z = \pm 1$ with residues $\frac{z-1}{z^2-1}|_{z=1} = \frac{1}{z+1}|_{z=1} = \frac{1}{2}$ and $\frac{z+1}{z^2-1}|_{z=-1} = \frac{1}{z-1}|_{z=-1} = -\frac{1}{2}$. At infinity, $f(z)$ has a second-order zero also. ■

Let us consider a series of concentric circles C_n about the origin so that C_n includes $z_1, z_2, \ldots, z_n$ but no other poles, its radius $R_n \to \infty$ as $n \to \infty$. To guarantee convergence we assume that $|f(z)| < \varepsilon R_n$ for an arbitrarily small positive constant ε and all z on C_n, and f is regular at the origin. Then the series

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n\{(z - z_n)^{-1} + z_n^{-1}\} \tag{7.54}$$

converges to $f(z)$.

To prove this theorem (due to Mittag–Leffler) we use the residue theorem to evaluate the following contour integral for z inside C_n and not equal to a singular point of $f(w)/w$:

$$I_n = \frac{1}{2\pi i}\int_{C_n} \frac{f(w)}{w(w-z)}dw$$
$$= \sum_{m=1}^{n} \frac{b_m}{z_m(z_m - z)} + \frac{f(z) - f(0)}{z}, \tag{7.55}$$

where w in the denominator is needed for convergence and $w - z$ to produce $f(z)$ via the residue theorem. On C_n we have for $n \to \infty$,

$$|I_n| \le 2\pi R_n \frac{\max_{w \text{ on } C_n} |f(w)|}{2\pi R_n(R_n - |z|)} < \frac{\varepsilon R_n}{R_n - |z|} \le \varepsilon$$

for $R_n \gg |z|$ so that $|I_n| \to 0$ as $n \to \infty$. Using $I_n \to 0$ in Eq. (7.55) proves Eq. (7.54).

If $|f(z)| < \varepsilon R_n^{p+1}$ for some positive integer p, then we evaluate similarly the integral

$$I_n = \frac{1}{2\pi i}\int \frac{f(w)}{w^{p+1}(w-z)}dw \to 0 \quad \text{as} \quad n \to \infty$$

and obtain the analogous pole expansion

$$f(z) = f(0) + zf'(0) + \cdots + \frac{z^p f^{(p)}(0)}{p!} + \sum_{n=1}^{\infty} \frac{b_n z^{p+1}/z_n^{p+1}}{z - z_n}. \tag{7.56}$$

Note that **the convergence of the series** in Eqs. (7.54) and (7.56) **is implied by the bound** of $|f(z)|$ for $|z| \to \infty$.

EXAMPLE 7.2.7

Pole Expansion of Cotangent The meromorphic function $f(z) = \pi \cot \pi z$ has simple poles at $z = \pm n$, for $n = 0, 1, 2, \ldots$ with residues $\frac{\pi \cos \pi z}{d \sin \pi z/dz}|_{z=n} = \frac{\pi \cos \pi n}{\pi \cos \pi n} = 1$. Before we apply the Mittag–Leffler theorem, we have to subtract the pole at $z = 0$. Then the pole expansion becomes

$$\pi \cot \pi z - \frac{1}{z} = \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$
$$= \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

We check this result by taking the logarithm of the product for the sine [Eq. (7.60)] and differentiating

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{-1}{n(1 - \frac{z}{n})} + \frac{1}{n(1 + \frac{z}{n})}\right) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Finally, let us also compare the pole expansion of the rational functions of Example 7.2.6 with the earlier partial fraction forms. From the Mittag–Leffler theorem, we get

$$\frac{1}{z^2 - 1} = -1 + \frac{1}{2}\left(\frac{1}{z-1} + 1\right) - \frac{1}{2}\left(\frac{1}{z+1} - 1\right),$$

which is in agreement with our earlier result. In order to apply the Mittag–Leffler theorem to $\frac{1}{z(z+1)}$, we first must subtract the pole at $z = 0$. We obtain

$$\frac{1}{z(z+1)} - \frac{1}{z} = \frac{-1}{z+1} = -1 - \left(\frac{1}{z+1} - 1\right),$$

which again is consistent with Example 7.2.6. ■

Product Expansion of Entire Functions

A function $f(z)$ that is analytic for all finite z is called an **entire** function. The logarithmic derivative f'/f is a meromorphic function with a pole expansion, which can be used to get a product expansion of $f(z)$.

If $f(z)$ has a simple zero at $z = z_n$, then $f(z) = (z - z_n)g(z)$ with analytic $g(z)$ and $g(z_n) \neq 0$. Hence, the logarithmic derivative

$$\frac{f'(z)}{f(z)} = (z - z_n)^{-1} + \frac{g'(z)}{g(z)} \tag{7.57}$$

has a simple pole at $z = z_n$ with residue 1, and g'/g is analytic there. If f'/f satisfies the conditions that led to the pole expansion in Eq. (7.54), then

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\frac{1}{z_n} + \frac{1}{z - z_n}\right] \tag{7.58}$$

holds. Integrating Eq. (7.58) yields

$$\begin{aligned}\int_0^z \frac{f'(z)}{f(z)}dz &= \ln f(z) - \ln f(0)\\ &= \frac{zf'(0)}{f(0)} + \sum_{n=1}^{\infty}\left\{\ln(z - z_n) - \ln(-z_n) + \frac{z}{z_n}\right\},\end{aligned}$$

and exponentiating we obtain the product expansion

$$f(z) = f(0)\exp\left(\frac{zf'(0)}{f(0)}\right)\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}. \tag{7.59}$$

Examples are the product expansions for

$$\sin z = z\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(1 - \frac{z}{n\pi}\right)e^{z/n\pi} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right),$$

$$\cos z = \prod_{n=1}^{\infty}\left\{1 - \frac{z^2}{(n-1/2)^2\pi^2}\right\}. \tag{7.60}$$

Note that the sine-product expansion is derived by applying Eq. (7.59) to $f(z) = \sin z/z$ rather than to $\sin z$, so that $f(0) = 1$ and

$$f'(0) = \left(\frac{\cos z}{z} - \frac{\sin z}{z^2}\right)\bigg|_{z=0} = \left(\frac{1}{z} - \frac{z}{2} - \frac{1}{z} + \frac{z}{6} + \cdots\right)\bigg|_{z=0} = 0,$$

inserting the power expansions for sin z and cos z. Another example is the product expansion of the gamma function, which will be discussed in Chapter 10.

As a consequence of Eq. (7.57), the contour integral of the logarithmic derivative may be used to count the number N_f of zeros (including their multiplicities) of the function $f(z)$ inside the contour C:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_f. \tag{7.61}$$

This follows from the leading term of the Taylor expansion of f at a zero z_0, $f(z) = (z - z_0)f'(z_0)$, with $f'(z_0) \neq 0$, so that

$$\frac{f'(z_0)}{f(z)} = \frac{1}{z - z_0} \quad \text{with} \quad \frac{1}{2\pi i}\oint \frac{dz}{z - z_0} = 1,$$

where we integrate about a small circle around z_0. For a zero of order m (a positive integer), where $f^{(m)}(z_0) \neq 0$ is the lowest nonvanishing derivative, the leading term of the Taylor expansion becomes

$$f(z) = \frac{(z - z_0)^m}{m!}f^m(z_0), \qquad \frac{f'(z)}{f(z)} = \frac{m}{z - z_0}.$$

Thus, the logarithmic derivative counts the zero with its multiplicity m. Moreover, using

$$\int \frac{f'(z)}{f(z)}dz = \ln f(z) = \ln|f(z)| + i \arg\ f(z), \tag{7.62}$$

we see that the real part in Eq. (7.62) does not change as z moves once around the contour, whereas the corresponding change in arg f, called $\Delta_C \arg(f)$, must be

$$\Delta_C \arg(f) = 2\pi N_f. \tag{7.63}$$

This leads to **Rouché's theorem**:

> If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

To show this we use

$$2\pi N_{f+g} = \Delta_C \arg(f + g) = \Delta_C \arg(f) + \Delta_C \arg\left(1 + \frac{g}{f}\right).$$

Since $|g| < |f|$ on C, the point $w = 1 + g(z)/f(z)$ is always an interior point of the circle in the w-plane with center at 1 and radius 1. Hence, $\arg(1 + g/f)$ must return to its original value when z moves around C (it passes to the right of the origin); it cannot decrease or increase by a multiple of 2π so that $\Delta_C \arg(1 + g/f) = 0$.

Rouché's theorem may be used for an alternative proof of the fundamental theorem of algebra: A polynomial $\sum_{m=0}^{n} a_m z^m$ with $a_n \neq 0$ has n zeros. We define $f(z) = a_n z^n$. Then f has an n-fold zero at the origin and no other zeros. Let $g(z) = \sum_0^{n-1} a_m z^m$. We apply Rouché's theorem to a circle C with center at the origin and radius $R > 1$. On C, $|f(z)| = |a_n|R^n$ and

$$|g(z)| \leq |a_0| + |a_1|R + \cdots + |a_{n+1}|R^{n-1} \leq \left(\sum_{m=0}^{n-1} |a_m|\right) R^{n-1}.$$

Hence, $|g(z)| < |f(z)|$ for z on C provided $R > (\sum_{m=0}^{n-1} |a_m|)/|a_n|$. For all sufficiently large circles C, therefore, $f + g = \sum_{m=0}^{n} a_m z^m$ has n zeros inside C according to Rouché's theorem.

SUMMARY

The residue theorem

$$\oint_C f(z)dz = 2\pi i \sum_{z_j \in C} [a_{-1z_j}] = 2\pi i \sum (\text{residues enclosed by } C)$$

and its applications to definite integrals are of central importance for mathematicians and physicists. When it is applied to merophorphic functions it yields an intrinsic pole expansion that depends on the first-order pole locations and their residues provided the functions behave reasonably at $|z| \to \infty$. When it is applied to the logarithmic derivative of an entire function, it leads to its product expansion.

The residue theorem is the workhorse for solving definite integrals, at least for physicists and engineers. However, the mathematician also employs it to derive pole expansions for meromorphic functions and product expansions for entire functions. It forms part of the tool kit of every scientist.

EXERCISES

7.2.1 Determine the nature of the singularities of each of the following functions and evaluate the residues ($a > 0$):

(a) $\dfrac{1}{z^2 + a^2}$. (b) $\dfrac{1}{(z^2 + a^2)^2}$.

(c) $\dfrac{z^2}{(z^2 + a^2)^2}$. (d) $\dfrac{\sin 1/z}{z^2 + a^2}$.

(e) $\dfrac{ze^{+iz}}{z^2 + a^2}$. (f) $\dfrac{ze^{+iz}}{z^2 + a^2}$.

(g) $\dfrac{e^{+iz}}{z^2 - a^2}$. (h) $\dfrac{z^{-k}}{z + 1}$, $0 < k < 1$.

Hint. For the point at infinity, use the transformation $w = 1/z$ for $|z| \to 0$. For the residue, transform $f(z)\,dz$ into $g(w)\,dw$ and look at the behavior of $g(w)$.

7.2.2 The statement that the integral halfway around a singular point is equal to one-half the integral all the way around was limited to simple poles.

Show, by a specific example, that

$$\int_{\text{semicircle}} f(z)\,dz = \frac{1}{2}\oint_{\text{circle}} f(z)\,dz$$

does not necessarily hold if the integral encircles a pole of higher order. *Hint*. Try $f(z) = z^{-2}$.

7.2.3 A function $f(z)$ is analytic along the real axis except for a third-order pole at $z = x_0$. The Laurent expansion about $z = x_0$ has the form

$$f(z) = \frac{a_{-3}}{(z - x_0)^3} + \frac{a_{-1}}{(z - x_0)} + g(z),$$

with $g(z)$ analytic at $z = x_0$. Show that the Cauchy principal value technique is applicable in the sense that

(a) $\lim_{\delta\to 0}\left\{\int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx\right\}$ is well behaved.

(b) $\int_{C_{x_0}} f(z)\,dz = \pm i\pi a_{-1}$,

where C_{x_0} denotes a **small semicircle** about $z = x_0$.

7.2.4 The unit step function is defined as

$$u(s-a) = \begin{cases} 0, & s < a \\ 1, & s > a. \end{cases}$$

Show that $u(s)$ has the integral representations

(a) $u(s) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ixs}}{x - i\varepsilon}dx$, (b) $u(s) = \frac{1}{2} + \frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{e^{ixs}}{x}dx$.

Note. The parameter s is real.

7.2.5 Most of the special functions of mathematical physics may be generated (defined) by a generating function of the form

$$g(t, x) = \sum_n f_n(x)t^n.$$

Given the following integral representations, derive the corresponding generating function:

(a) Bessel

$$J_n(x) = \frac{1}{2\pi i}\oint e^{(x/2)(t-1/t)}t^{-n-1}dt.$$

(b) Legendre

$$P_n(x) = \frac{1}{2\pi i}\oint (1 - 2tx + t^2)^{-1/2}t^{-n-1}dt.$$

(c) Hermite

$$H_n(x) = \frac{n!}{2\pi i} \oint e^{-t^2+2tx} t^{-n-1} dt.$$

(d) Laguerre

$$L_n(x) = \frac{1}{2\pi i} \oint \frac{e^{-xt/(1-t)}}{(1-t)t^{n+1}} dt.$$

Each of the contours encircles the origin and no other singular points.

7.2.6 Generalizing Example 7.2.1, show that

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \frac{2\pi}{(a^2-b^2)^{1/2}}, \quad \text{for } a > |b|.$$

What happens if $|b| > |a|$?

7.2.7 Show that

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}, \quad a > 1.$$

7.2.8 Show that

$$\int_0^{2\pi} \frac{d\theta}{1-2t\cos\theta+t^2} = \frac{2\pi}{1-t^2}, \quad \text{for } |t| < 1.$$

What happens if $|t| > 1$? What happens if $|t| = 1$?

7.2.9 With the calculus of residues show that

$$\int_0^{\pi} \cos^{2n}\theta\, d\theta = \pi \frac{(2n)!}{2^{2n}(n!)^2} = \pi \frac{(2n-1)!!}{(2n)}, \quad n = 0, 1, 2, \ldots.$$

(The double factorial notation is defined in Chapter 5 and Section 10.1.)
Hint. $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1}), \quad |z| = 1.$

7.2.10 Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2} dx, \quad a > b > 0.$$

ANS. $\pi(a-b)$.

7.2.11 Prove that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Hint. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

7.2.12 A quantum mechanical calculation of a transition probability leads to the function $f(t,\omega) = 2(1-\cos\omega t)/\omega^2$. Show that

$$\int_{-\infty}^{\infty} f(t,\omega)\, d\omega = 2\pi t.$$

7.2.13 Show that ($a > 0$)

(a) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-a}.$$

How is the right side modified if $\cos x$ is replaced by $\cos kx$?

(b) $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}.$$

How is the right side modified if $\sin x$ is replaced by $\sin kx$?

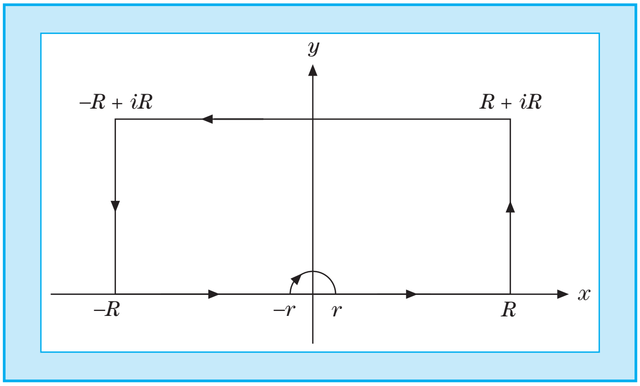
These integrals may also be interpreted as Fourier cosine and sine transforms (Chapter 15).

7.2.14 Use the contour shown in Fig. 7.11 with $R \to \infty$ to prove that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

Figure 7.11

Contour



7.2.15 In the quantum theory of atomic collisions we encounter the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin t}{t} e^{ipt} dt,$$

in which p is real. Show that

$$I = 0, \quad |p| > 1$$
$$I = \pi, \quad |p| < 1.$$

What happens if $p = \pm 1$?

7.2.16 Evaluate

$$\int_0^{\infty} \frac{(\ln x)^2}{1 + x^2} dx.$$

(a) by appropriate series expansion of the integrand to obtain

$$4 \sum_{n=0}^{\infty} (-1)^n (2n + 1)^{-3},$$

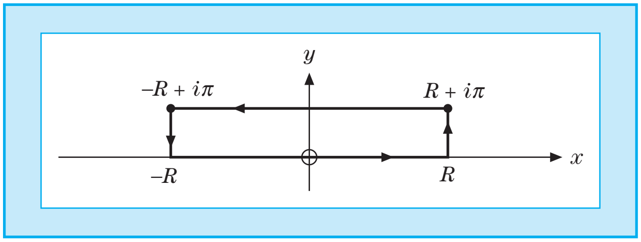
(b) and by contour integration to obtain

$$\frac{\pi^3}{8}.$$

Hint. $x \to z = e^t$. Try the contour shown in Fig. 7.12, letting $R \to \infty$.

Figure 7.12

Contour



7.2.17 Show that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \qquad a > 0.$$

7.2.18 Evaluate

$$\int_{-\infty}^\infty \frac{x^2}{1+x^4} dx.$$

ANS. $\pi/\sqrt{2}$.

7.2.19 Show that

$$\int_0^\infty \cos(t^2)dt = \int_0^\infty \sin(t^2)dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Hint. Try the contour shown in Fig. 7.13.
Note. These are the Fresnel integrals for the special case of infinity as the upper limit. For the general case of a varying upper limit, asymptotic expansions of the Fresnel integrals are the topic of Exercise 5.10.2. Spherical Bessel expansions are the subject of Exercise 12.4.13.

7.2.20 Several of the Bromwich integrals (Section 15.12) involve a portion that may be approximated by

$$I(y) = \int_{a-iy}^{a+iy} \frac{e^{zt}}{z^{1/2}} dz,$$

where a and t are positive and finite. Show that

$$\lim_{y\to\infty} I(y) = 0.$$

7.2.21 Show that

$$\int_0^\infty \frac{1}{1+x^n} dx = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 7.14.

Figure 7.13

Contour

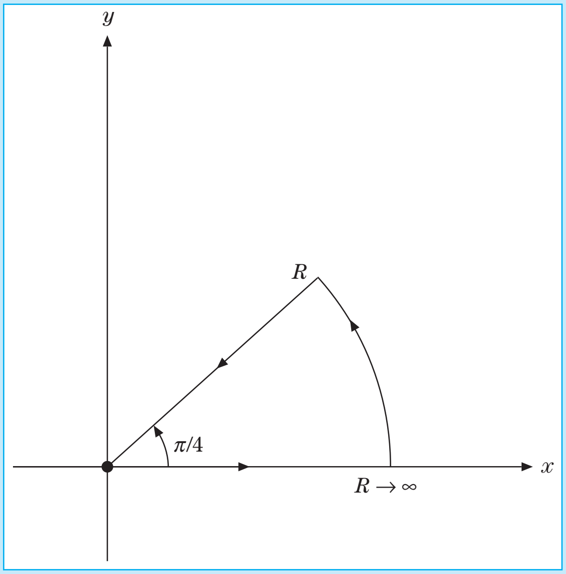
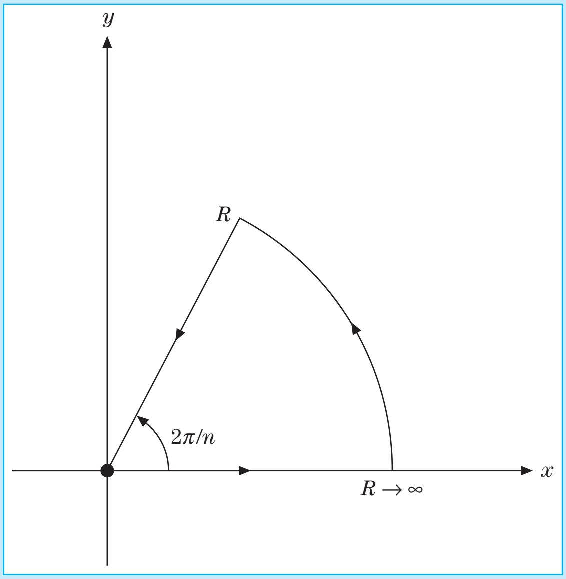


Figure 7.14

Contour



7.2.22 (a) Show that

$$f(z) = z^4 - 2\cos 2\theta z^2 + 1$$

has zeros at $e^{i\theta}$, $e^{-i\theta}$, $-e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 - 2\cos 2\theta x^2 + 1} = \frac{\pi}{2\sin\theta} = \frac{\pi}{2^{1/2}(1-\cos 2\theta)^{1/2}}.$$

Exercise 7.2.21 ($n = 4$) is a special case of this result.

7.2.23 Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 - 2\cos 2\theta x^2 + 1}$$

by contour integration.

7.2.24 The integral in Exercise 7.2.16 may be transformed into

$$\int_0^{\infty} e^{-y} \frac{y^2}{1+e^{-2y}} dy = \frac{\pi^3}{16}.$$

Evaluate this integral by the Gauss–Laguerre quadrature and compare your numerical result with $\pi^3/16$.

ANS. Integral = 1.93775 (10 points).

7.3 Method of Steepest Descents

Analytic Landscape

In analyzing problems in mathematical physics, one often finds it desirable to know the behavior of a function for large values of the variable or some parameter s, that is, the asymptotic behavior of the function. Specific examples are furnished by the gamma function (Chapter 10) and various Bessel functions (Chapter 12). All these analytic functions $[I(s)]$ are defined by integrals

$$I(s) = \int_C F(z, s)\, dz, \tag{7.64}$$

where F is analytic in z and depends on a real parameter s. We write $F(z)$ simply whenever possible.

So far, we have evaluated such definite integrals of analytic functions along the real axis (the initial path C) by deforming the path C to C' in the complex plane so that $|F|$ becomes small for all z on C'. [See Example 7.2.3 for $I(a)$.] This method succeeds as long as only isolated singularities occur in the area between C and C'. Their contributions are taken into account by applying the residue theorem of Section 7.2. The residues (from the simple pole part) give a measure of the singularities, where $|F| \to \infty$, which usually dominate and determine the value of the integral.

The behavior of an integral as in Eq. (7.64) clearly depends on the absolute value $|F|$ of the integrand. Moreover, the contours of $|F|$ at constant steps $\Delta|F|$ often become more closely spaced as s becomes large. Let us focus on a plot of $|F(x+iy)|^2 = U^2(x, y) + V^2(x, y)$ rather than the real part $\Re F = U$ and the imaginary part $\Im F = V$ separately. Such a plot of $|F|^2$ over the complex plane is called the **analytic landscape** after Jensen, who, in 1912, proved that it has **only saddle points and troughs, but no peaks**. Moreover, the troughs reach down all the way to the complex plane, that is, go to $|F| = 0$. **In the absence of singularities, saddle points** are next in line to **dominate the integral** in Eq. (7.64). Jensen's theorem explains why only saddle points (and singularities) of the integrand are so important for integrals. Hence the name saddle point method for finding the asymptotic behavior of $I(s)$ for $s \to \infty$ that we describe now. At a saddle point the real part U of F has a local maximum, for example, which implies that

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0,$$

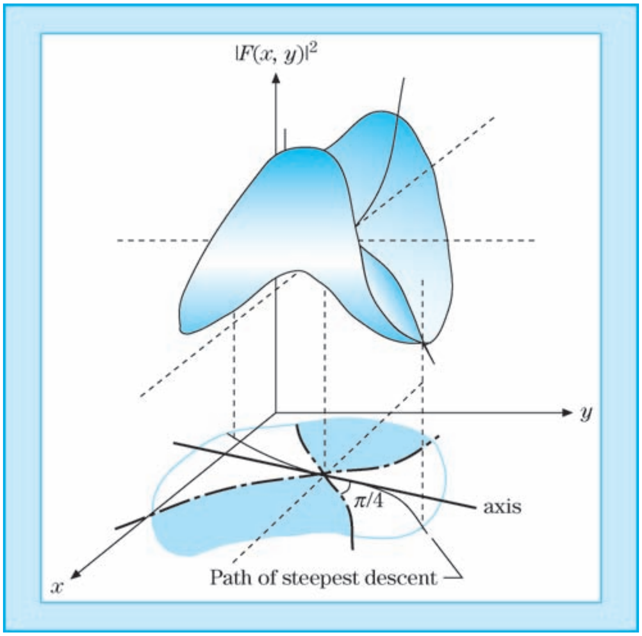
and therefore, by the use of the Cauchy–Riemann conditions of Section 6.2,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0,$$

so that V has a minimum or vice versa, and $F'(z) = 0$. **Jensen's theorem prevents** U **and** V **from having both a maximum or minimum**. See Fig. 7.15 for a typical shape (and Exercises 6.2.3 and 6.2.4). We will **choose the path** C **so that it runs over the saddle point and in the valleys elsewhere**

Figure 7.15

A Saddle Point



so that the saddle point dominates the value of $I(s)$. This deformation of the path is analogous to the applications of the residue theorem to integrals over functions with poles. In the rare case that there are several saddle points, we treat each alike, and their contributions will add to $I(s)$ for large s.

To prove that there are no peaks, assume there is one at z_0. That is, $|F(z_0)|^2 > |F(z)|^2$ for all z of a neighborhood $|z - z_0| \leq r$. If

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is the Taylor expansion at z_0, the mean value $m(F)$ on the circle $z = z_0 + r\exp(i\varphi)$ becomes

$$\begin{aligned} m(F) &\equiv \frac{1}{2\pi} \int_0^{2\pi} |F(z_0 + re^{i\varphi})|^2 d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{m,n=0}^{\infty} a_m^* a_n r^{m+n} e^{i(n-m)\varphi} d\varphi \\ &= \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \geq |a_0|^2 = |F(z_0)|^2, \end{aligned} \tag{7.65}$$

using orthogonality, $\frac{1}{2\pi}\int_0^{2\pi} \exp[i(n-m)\varphi]d\varphi = \delta_{nm}$. Since $m(F)$ is the mean value of $|F|^2$ on the circle of radius r, there must be a point z_1 on it so that $|F(z_1)|^2 \geq m(F) \geq |F(z_0)|^2$, which contradicts our assumption. Hence, there can be no such peak.

Next, let us assume there is a minimum at z_0 so that $0 < |F(z_0)|^2 < |F(z)|^2$ for all z of a neighborhood of z_0. In other words, the dip in the valley does not go down to the complex plane. Then $|F(z)|^2 > 0$ and since $1/F(z)$ is analytic there, it has a Taylor expansion and z_0 would be a peak of $1/|F(z)|^2$, which is impossible. This proves Jensen's theorem. We now turn our attention to the integral in Eq. (7.64).

Saddle Point Method

Since a saddle point z_0 of $|F(z)|^2$ lies above the complex plane, that is, $|F(z_0)|^2 > 0$, so $F(z_0) \neq 0$, we write F in exponential form, $F(z) = e^{f(z,s)}$, in its vicinity without loss of generality. Note that having no zero in the complex plane is a characteristic property of the exponential function. Moreover, any saddle point with $F(z) = 0$ becomes a trough of $|F(z)|^2$ because $|F(z)|^2 \geq 0$. A case in point is the function z^2 at $z = 0$ where $2z = 0$. Here $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$, and $2xy$ has a saddle point at $z = 0$, as well as at $x^2 - y^2$, but $|z|^4$ has a trough there. The phase of $F(z_0)$ is given by $\Im f(z_0)$. At z_0 the tangential plane is horizontal; that is,

$$\left.\frac{\partial F}{\partial z}\right|_{z=z_0} = 0, \quad \text{or equivalently} \quad \left.\frac{\partial f}{\partial z}\right|_{z=z_0} = 0.$$

This condition locates the saddle point.

Our next goal is to determine the **direction of steepest descent**, the heart of the saddle point method. To this end, we use the power-series expansion of f at z_0,

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \cdots, \tag{7.66}$$

or

$$f(z) = f(z_0) + \frac{1}{2}(f''(z_0) + \varepsilon)(z - z_0)^2, \tag{7.67}$$

upon collecting all higher powers in the (small) ε. Let us take $f''(z_0) \neq 0$ for simplicity. Then

$$f''(z_0)(z - z_0)^2 = -t^2, \quad t \quad \text{real} \tag{7.68}$$

defines a line through z_0 (**saddle point axis** in Fig. 7.15). At z_0, $t = 0$. Along the axis $\Im f''(z_0)(z - z_0)^2$ is zero and $v = \Im f(z) \approx \Im f(z_0)$ is constant if ε in Eq. (7.67) is neglected. Thus, F has **constant phase along the axis**. Equation (7.68) can also be expressed in terms of angles,

$$\arg(z - z_0) = \frac{\pi}{2} - \frac{1}{2} \arg f''(z_0) = \text{constant}. \tag{7.69}$$

Since $|F(z)|^2 = \exp(2\Re f)$ varies monotonically with $\Re f$, $|F(z)|^2 \sim \exp(-t^2)$ falls off exponentially from its maximum at $t = 0$ along this axis. Hence the name **steepest descent** for this method of extracting the asymptotic behavior of $I(s)$ for $s \to \infty$. The line through z_0 defined by

$$f''(z_0)(z - z_0)^2 = +t^2 \tag{7.70}$$

is orthogonal to this axis (**dashed** line in Fig. 7.15), which is evident from its angle

$$\arg(z - z_0) = -\frac{1}{2} \arg f''(z_0) = \text{constant}, \tag{7.71}$$

when compared with Eq. (7.69). Here, $|F(z)|^2$ grows exponentially.

The curves $\Re f(z) = \Re f(z_0)$ go through z_0 so that $\Re[(f''(z_0) + \varepsilon)(z - z_0)^2] = 0$, or $(f''(z_0) + \varepsilon)(z - z_0)^2 = it$ for real t. Expressing this in angles as

$$\arg(z - z_0) = \frac{\pi}{4} - \frac{1}{2} \arg(f''(z_0) + \varepsilon), \ \ t > 0 \tag{7.72a}$$

$$\arg(z - z_0) = -\frac{\pi}{4} - \frac{1}{2} \arg(f''(z_0) + \varepsilon), \ \ t < 0, \tag{7.72b}$$

and comparing with Eqs. (7.69) and (7.71), we note that these curves (**dot-dashed** line in Fig. 7.15) divide the saddle point region into four sectors—two with $\Re f(z) > \Re f(z_0)$ (hence $|F(z)| > |F(z_0)|$) shown shaded in Fig. 7.15 and two with $\Re f(z) < \Re f(z_0)$ (hence $|F(z)| < |F(z_0)|$). They are at $\pm\frac{\pi}{4}$ angles from the axis. Thus, the integration path has to avoid the shaded areas where $|F|$ rises. If a path is chosen to run up the slopes above the saddle point, large imaginary parts of $f(z)$ are involved that lead to rapid oscillations of $F(z) = e^{f(z)}$ and cancelling contributions to the integral. So far, our treatment has been general, except for $f''(z_0) \neq 0$, which can be relaxed.

Now we are ready to **specialize the integrand** F further in order to tie up the path selection with the **asymptotic behavior** as $s \to \infty$. We assume that the parameter s appears linearly in the exponent; that is, we replace $\exp f(z, s) \to \exp(sf(z))$. This dependence on s often occurs in physics applications and ensures that the saddle point at z_0 grows with $s \to \infty$ [if $\Re f(z_0) > 0$]. In order to account for the region far away from the saddle point that is not influenced by s, we include another analytic function $g(z)$ that varies slowly near the saddle point and is independent of s. Altogether, our integral has the more appropriate and **specific form**

$$I(s) = \int_C g(z) e^{sf(z)} dz. \tag{7.73}$$

Our **goal now is to estimate the integral** $I(s)$ near the saddle point. **The path of steepest descent is the saddle point axis** when we neglect the higher order terms, ε, in Eq. (7.67). With ε, the path of steepest descent is the curve close to the axis within the unshaded sectors, where the phase $v = \Im f(z)$ is strictly constant, whereas $\Im f(z)$ is only approximately constant on the axis. We approximate $I(s)$ by the integral along the piece of the axis inside the patch in Fig. 7.15, where [compare with Eq. (7.68)]

$$z = z_0 + xe^{i\alpha}, \quad \alpha = \frac{\pi}{2} - \frac{1}{2} \arg f''(z_0), \quad a \le x \le b. \tag{7.74}$$

Here, the interval $[a, b]$ will be given below. We find

$$I(s) \approx e^{i\alpha} \int_a^b g(z_0 + xe^{i\alpha}) \exp[sf(z_0 + xe^{i\alpha})] dx, \tag{7.75a}$$

and the omitted part is small and can be estimated because $\Re(f(z) - f(z_0))$ has an upper negative bound (e.g., $-R$) that depends on the size of the saddle point patch in Fig. 7.15 [i.e., the values of a, b in Eq. (7.74)] that we choose. In Eq. (7.75a) we use the Taylor expansions

$$\begin{aligned} f(z_0 + xe^{i\alpha}) &= f(z_0) + \frac{1}{2} f''(z_0) e^{2i\alpha} x^2 + \cdots, \\ g(z_0 + xe^{i\alpha}) &= g(z_0) + g'(z_0) e^{i\alpha} x + \cdots, \end{aligned} \tag{7.75b}$$

and recall from Eq. (7.74) that

$$\frac{1}{2} f''(z_0) e^{2i\alpha} = -\frac{1}{2} |f''(z_0)| < 0.$$

We find for the leading term

$$I(s) = g(z_0) e^{sf(z_0) ti\alpha} \int_a^b e^{-\frac{1}{2} s|f''(z_0)| x^2} dx. \tag{7.76}$$

Since the integrand in Eq. (7.76) is essentially zero when x departs appreciably from the origin, we let $b \to \infty$ and $a \to -\infty$. The small error involved is straightforward to estimate. Noting that the remaining integral is just a Gauss error integral,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}c^2x^2}\,dx = \frac{1}{c}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \frac{\sqrt{2\pi}}{c},$$

we obtain the asymptotic formula

$$I(s) = \frac{\sqrt{2\pi}\,g(z_0)e^{sf(z_0)}e^{i\alpha}}{|sf''(z_0)|^{1/2}} \tag{7.77}$$

for the case in which one saddle point dominates. Here, the phase α was introduced in Eqs. (7.74) and (7.69).

One note of warning: We assumed that the only significant contribution to the integral came from the immediate vicinity of the saddle point $z = z_0$. This condition must be checked for each new problem (Exercise 7.3.3).

EXAMPLE 7.3.1

Asymptotic Form of the Hankel Function, $H_\nu^{(1)}(s)$ In Section 12.3, it is shown that the Hankel functions, which satisfy Bessel's equation, may be defined by

$$H_\nu^{(1)}(s) = \frac{1}{\pi i}\int_{C_1(0)}^{\infty e^{i\pi}} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}}, \tag{7.78}$$

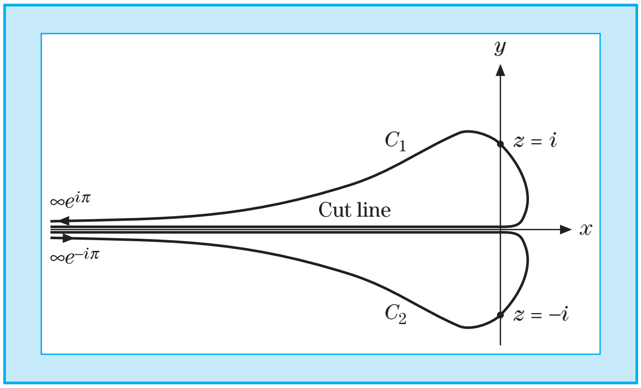
$$H_\nu^{(2)}(s) = \frac{1}{\pi i}\int_{C_2(\infty e^{-i\pi})}^{0} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}}, \tag{7.79}$$

where $\infty e^{i\pi} = -\infty$ and the contour C_1 is the curve in the upper half-plane of Fig. 7.16 that starts at the origin and ends at $-\infty$. The contour C_2 is in the lower half-plane, starting at $\infty e^{-i\pi}$ and ending at the origin. We apply the method of steepest descents to the first Hankel function, $H_\nu^{(1)}(s)$, which is conveniently in the form specified by Eq. (7.73), with $g(z) = 1/(i\pi z^{\nu+1})$ and $f(z)$ given by

$$f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right). \tag{7.80}$$

Figure 7.16

Hankel Function Contours



By differentiating, we obtain

$$f'(z) = \frac{1}{2} + \frac{1}{2z^2}, \qquad f''(z) = -\frac{1}{z^3}. \tag{7.81}$$

Setting $f'(z) = 0$, we obtain

$$z = i, -i. \tag{7.82}$$

Hence, there are saddle points at $z = +i$ and $z = -i$. At $z = i$, $f''(i) = -i$, or $\arg f''(i) = -\pi/2$, so that the saddle point direction is given by Eq. (7.74) as $\alpha = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3}{4}\pi$. For the integral for $H_\nu^{(1)}(s)$ we must choose the contour through the point $z = +i$ so that it starts at the origin, moves out tangentially to the positive real axis, and then moves around through the saddle point at $z = +i$ in the direction given by the angle $\alpha = 3\pi/4$ and then on out to minus infinity, asymptotic with the negative real axis. The path of steepest ascent that we must avoid has the phase $-\frac{1}{2}\arg f''(i) = \frac{\pi}{4}$ according to Eq. (7.71) and is orthogonal to the axis, our path of steepest descent.

Direct substitution into Eq. (7.77) with $\alpha = 3\pi/4$ yields

$$\begin{aligned} H_\nu^{(1)}(s) &= \frac{1}{\pi i}\frac{\sqrt{2\pi i^{-\nu-1}}e^{(s/2)(i-1/i)}e^{3\pi i/4}}{|(s/2)(-2/i^3)|^{1/2}} \\ &= \sqrt{\frac{2}{\pi s}}e^{(i\pi/2)(-\nu-2)}e^{is}e^{i(3\pi/4)}. \end{aligned} \tag{7.83}$$

By combining terms, we finally obtain

$$H_\nu^{(1)}(s) \approx \sqrt{\frac{2}{\pi s}}e^{i(s-\nu(\pi/2)-\pi/4)} \tag{7.84}$$

as the leading term of the asymptotic expansion of the Hankel function $H_\nu^{(1)}(s)$. Additional terms, if desired, may be picked up from the power series of f and g in Eq. (7.75b). The other Hankel function can be treated similarly using the saddle point at $z = -i$. ■

EXAMPLE 7.3.2

Asymptotic Form of the Factorial Function In many physical problems, particularly in the field of statistical mechanics, it is desirable to have an accurate approximation of the gamma or factorial function of very large numbers. As developed in Section 10.1, the factorial function may be defined by the Euler integral

$$s! = \int_0^\infty \rho^s e^{-\rho} d\rho = s^{s+1}\int_0^\infty e^{s(\ln z - z)} dz. \tag{7.85}$$

Here, we have made the substitution $\rho = zs$ in order to put the integral into the form required by Eq. (7.73). As before, we assume that s is real and positive, from which it follows that the integrand vanishes at the limits 0 and ∞. By differentiating the z dependence appearing in the exponent, we obtain

$$f'(z) = \frac{d}{dz}(\ln z - z) = \frac{1}{z} - 1, \qquad f''(z) = -\frac{1}{z^2}, \tag{7.86}$$

which shows that the point $z = 1$ is a saddle point, and $\arg f''(1) = \arg(-1) = \pi$. According to Eq. (7.74) we let

$$z - 1 = xe^{i\alpha}, \qquad \alpha = \frac{\pi}{2} - \frac{1}{2}\arg f''(1) = \frac{\pi}{2} - \frac{\pi}{2}, \tag{7.87}$$

with x small to describe the contour in the vicinity of the saddle point. From this we see that the direction of steepest descent is along the real axis, a conclusion that we could have reached more or less intuitively.

Direct substitution into Eq. (7.77) with $\alpha = 0$ gives

$$s! \approx \frac{\sqrt{2\pi}\, s^{s+1} e^{-s}}{|s(-1^{-2})|^{1/2}}. \tag{7.88}$$

Thus, the first term in the asymptotic expansion of the factorial function is

$$s! \approx \sqrt{2\pi s}\, s^s e^{-s}. \tag{7.89}$$

This result is the first term in Stirling's expansion of the factorial function. The method of steepest descent is probably the easiest way of obtaining this first term. If more terms in the expansion are desired, then the method of Section 10.3 is preferable. ■

In the foregoing example the calculation was carried out by assuming s to be real. This assumption is not necessary. We may show that Eq. (7.89) also holds when s is replaced by the complex variable w, provided only that the real part of w is required to be large and positive.

SUMMARY

Asymptotic limits of integral representations of functions are extremely important in many approximations and applications in physics:

$$\int_C g(z) e^{sf(z)} dz \sim \frac{\sqrt{2\pi}\, g(z_0) e^{sf(z_0)} e^{i\alpha}}{\sqrt{|sf''(z_0)|}}.$$

The saddle point method is one method of choice for deriving them and belongs in the tool kit of every physicist and engineer.

EXERCISES

7.3.1 Using the method of steepest descents, evaluate the second Hankel function given by

$$H_\nu^{(2)}(s) = \frac{1}{\pi i} \int_{-\infty C_2}^{0} e^{(s/2)(z-1/z)} \frac{dz}{z^{\nu+1}},$$

with contour C_2 as shown in Fig. 7.16.

ANS. $H_\nu^{(2)}(s) \approx \sqrt{\frac{2}{\pi s}} e^{-i(s-\pi/4-\nu\pi/2)}$.

7.3.2 Find the steepest path and leading asymptotic expansion for the Fresnel integrals $\int_0^s \cos x^2 dx$, $\int_0^s \sin x^2 dx$.
Hint. Use $\int_0^1 e^{isz^2} dz$.

7.3.3 (a) In applying the method of steepest descent to the Hankel function $H_\nu^{(1)}(s)$, show that

$$\Re[f(z)] < \Re[f(z_0)] = 0$$

for z on the contour C_1 but away from the point $z = z_0 = i$.

(b) Show that

$$\Re[f(z)] > 0 \qquad \text{for} \quad 0 < r < 1, \quad \begin{cases} \frac{\pi}{2} < \theta \le \pi \\ -\pi \le \theta < \frac{\pi}{2} \end{cases}$$

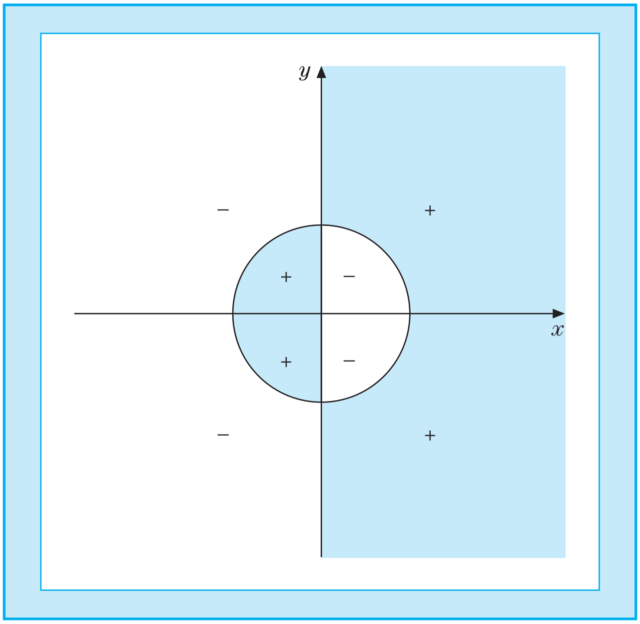
and

$$\Re[f(z)] < 0 \qquad \text{for} \quad r > 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

(Fig. 7.17). This is why C_1 may not be deformed to pass through the second saddle point $z = -i$. Compare with and verify the dot-dashed lines in Fig. 7.15 for this case.

Figure 7.17

Saddle Points of Hankel Function



7.3.4 Show that Stirling's formula

$$s! \approx \sqrt{2\pi s} s^s e^{-s}$$

holds for complex values of s [with $\Re(s)$ large and positive].

Hint. This involves assigning a phase to s and then demanding that $\Im[sf(z)] =$ constant in the vicinity of the saddle point.

7.3.5 Assume $H_\nu^{(1)}(s)$ to have a negative power-series expansion of the form

$$H_\nu^{(1)}(s) = \sqrt{\frac{2}{\pi s}} e^{i(s-\nu(\pi/2)-\pi/4)} \sum_{n=0}^{\infty} a_{-n} s^{-n},$$

with the coefficient of the summation obtained by the method of steepest descent. Substitute into Bessel's equation and show that you reproduce the asymptotic series for $H_\nu^{(1)}(s)$ given in Section 12.3.

Additional Reading

Wyld, H. W. (1976). *Mathematical Methods for Physics*. Benjamin/Cummings, Reading, MA. Reprinted, Perseus, Cambridge, MA. (1999). This is a relatively advanced text.