Homework 3 Problems – Heat Engines and the 2nd Law

1. [10 points] A sample of an ideal gas goes through the process shown below. From A to B, the process is adiabatic; from B to C, it is isobaric with 100 kJ of energy flowing into the system by heat. From C to D, the process is isothermal; from D to A, it is isobaric with 150 kJ of energy flowing out of the system by heat.

   a. Determine the difference in internal energy between A and B.
   b. Determine the net work done in the entire cycle.

Note that 1 atm = 101325 Pa.

Answer:

   a. Since \( dU \) is an exact differential, this implies that \( \Delta U = 0 \) for the entire thermodynamic cycle. Therefore, based on the above figure, we can write \( \Delta U = U_B - U_A \) as

   \[
   \Delta U = U_B - U_A = -(U_C - U_B) + (U_D - U_C) + (U_A - U_D)
   \]

   Since the path from C to D is isothermal, then \( (U_D - U_C) = 0 \). Using the first law of thermodynamics, the change in internal energy from B to C is

   \[
   (U_C - U_B) = Q_{BC} + W_{BC} = 100 \text{ kJ} - P_B(V_C - V_B) \\
   = 100 \text{ kJ} - (303975 \text{ Pa})(0.40 \text{ m}^3 - 0.090 \text{ m}^3) = 5.8 \text{ kJ}
   \]

   Using the first law of thermodynamics, the change in internal energy from D to A is

   \[
   (U_A - U_D) = Q_{AD} + W_{AD} = -150 \text{ kJ} - P_D(V_A - V_D) \\
   = -150 \text{ kJ} - (101325 \text{ Pa})(0.20 \text{ m}^3 - 1.2 \text{ m}^3) = -48.7 \text{ kJ}
   \]

   Therefore, we have
b. The net work is given by

\[
W_{\text{net}} = W_{AB} + W_{BC} + W_{CD} + W_{DA} = \Delta U_{AB} - P_B(V_C - V_B) - NkT \ln \left( \frac{V_D}{V_C} \right) - P_D(V_A - V_D)
\]

\[
= \Delta U_{AB} - P_B(V_C - V_B) + P_C V_C \ln \left( \frac{V_C}{V_D} \right) - P_D(V_A - V_D)
\]

\[
= 42.9 \text{ kJ} - (303975 \text{ Pa})(0.40 \text{ m}^3 - 0.090 \text{ m}^3)
\]

\[
+ (303975 \text{ Pa})(0.40 \text{ m}^3) \ln \left( \frac{0.40 \text{ m}^3}{1.2 \text{ m}^3} \right) - (101325 \text{ Pa})(0.20 \text{ m}^3 - 1.2 \text{ m}^3)
\]

\[
= -83.8 \text{ kJ}
\]

2. [10 points] A reversible heat engine operates between two reservoirs, \( T_C = 250 \text{ K} \) and \( T_H = 400 \text{ K} \). The cold reservoir can be considered to have infinite mass, but the hot reservoir consists of a finite amount of gas at constant volume (1 mole with a heat capacity \( C_V = 3200 \text{ J/}^\circ \text{C} \)). As a result, the temperature of the hot reservoir decreases with time, and after the heat engine has operated for some long period of time, the temperature \( T_H \) is lowered to \( T_C \). Calculate the heat extracted from the hot reservoir during this period and the amount of work performed by the engine during this period.

Answer:

The heat extracted during this period will be

\[
Q_H = \int_{T_C}^{T_H} C_V \, dT = C_V(T_H - T_C) = \left( 3200 \frac{J}{^\circ \text{C}} \right) (400 \text{ K} - 250 \text{ K}) = 4.8 \times 10^5 \text{ J}
\]

The work evaluated during this period can be determined based on the efficiency of the cycle. For one cycle, we have

\[
\delta W = e(T) \delta Q_H = \left( 1 - \frac{T_C}{T_H} \right) \delta Q_H = -C_V \left( 1 - \frac{T_C}{T_H} \right) dT_H
\]

The total work done on the system is given by

\[
W = - \int_{T_H}^{T_C} \left( 1 - \frac{T_C}{T_H} \right) C_V dT_H = \int_{T_C}^{T_H} \left( 1 - \frac{T_C}{T_H} \right) C_V dT_H = C_V(T_H - T_C) - C_V T_C \ln \left( \frac{T_H}{T_C} \right)
\]

\[
= \left( 3200 \frac{J}{^\circ \text{C}} \right) \left[ (400 \text{ K} - 250 \text{ K}) - (250 \text{ K}) \ln \left( \frac{400 \text{ K}}{250 \text{ K}} \right) \right] = 1.04 \times 10^5 \text{ J}
\]
3. The idealized Diesel cycle is given below

![Diagram of the Diesel cycle]

a. Show that efficiency of the Diesel cycle can be written in terms of the compression ratio \( b = V_A/V_B \) and the cut-off ratio \( c = V_C/V_B \) as

\[
e_h = 1 - \frac{b^{1-\gamma}(c^{\gamma} - 1)}{\gamma(c - 1)}
\]

b. Show that for a given compression ratio, the Diesel cycle is less efficient than the Otto cycle. (This proves, of course, that the Diesel cycle is less efficient than the Carnot cycle)

c. In a Diesel engine, atmospheric air is quickly compressed to about 1/20 of its original volume. Estimate the temperature of the air after compression, and explain why a Diesel engine does not require spark plugs.

Answer:

a. The efficiency of a heat engine may be written as

\[
e_h = 1 - \frac{|Q_c|}{Q_h}
\]

Since legs \( a \to b \) and \( c \to d \) are adiabatic, no heat is exchanged during each leg. Heat is absorbed during the isobaric process and released during the isochoric process. Thus, we have

\[
Q_h = C_p(T_c - T_b), \quad |Q_c| = C_v(T_d - T_a)
\]

so the efficiency of the cycle is

\[
e_h = 1 - \frac{C_v(T_d - T_a)}{C_p(T_c - T_b)} = 1 - \frac{1}{\gamma} \frac{T_d - T_a}{T_c - T_b}
\]
The ratio of the temperatures can be given in terms of the compression ratio and cutoff ratio if we recognize that legs $a \rightarrow b$ and $c \rightarrow d$ are adiabatic. Using the ideal gas law, we can write the adiabatic equation of state as

$$TV^{\gamma-1} = \text{const.}$$

Therefore,

$$T_a V_1^{\gamma-1} = T_b V_2^{\gamma-1} \quad T_c V_3^{\gamma-1} = T_d V_1^{\gamma-1}$$

The above equations imply that

$$(T_a - T_d) V_1^{\gamma-1} = T_b V_2^{\gamma-1} - T_c V_3^{\gamma-1}$$

Simplifying gives

$$(T_a - T_d) \left(\frac{V_1}{V_2}\right)^{\gamma-1} = T_b - T_c \left(\frac{V_3}{V_2}\right)^{\gamma-1}$$

Since leg $1 \rightarrow 2$ is isobaric, then the ideal gas law reduces down to $T/V = \text{const}$. This implies that $T_b/V_2 = T_c/V_3 \rightarrow T_c = c T_b$ where $c$ is the cutoff ratio. Rewriting this in terms of the cut-off ratio and the compression ratio gives

$$(T_a - T_d) b^{\gamma-1} = T_b - T_c c^{\gamma-1} = T_b (1 - c^{\gamma}) \Rightarrow T_d - T_a = T_b (c^{\gamma} - 1)$$

Therefore, we can write the efficiency as

$$e_h = 1 - \frac{1}{\gamma} \frac{T_d - T_a}{T_c - T_b} = 1 - \frac{1}{\gamma} \frac{T_b (c^{\gamma} - 1)}{T_b (c - 1)} = 1 - \frac{b^{1-\gamma} (c^{\gamma} - 1)}{\gamma (c - 1)} = 1 - b^{1-\gamma} \left[\frac{(c^{\gamma} - 1)}{\gamma (c - 1)}\right]$$

b. Note that if we ignore the term in the bracket, then this expression is the same after the Otto cycle. Thus, all we have to show is the term in the bracket is greater than one. This can be shown by simply making a plot of the correction factor versus the cutoff ratio $c$. This graph is given below.
Since the correction term is always greater than 1, this implies that efficiency of the Diesel cycle is less than the efficiency of the Otto cycle.

c. A rapid compression of air implies an adiabatic compression. In other words, if a process occurs rapidly enough, there is not enough time for heat to be exchanged through the system’s boundaries. Thus, by the adiabatic equation of state, we have

\[ T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1} \Rightarrow T_f = T_i \left( \frac{V_i}{V_f} \right)^{\gamma-1} = T_i (20)^{\gamma-1} \approx (300 \, K)(20)^{0.4} = 994 \, K \]

Since the ignition temperature of Diesel fuel tends to be much lower than this temperature, the air after compression is hot enough to ignite without a spark.
4. [10 points] In this problem, you are going to examine the properties of the Otto engine. The idealized Otto cycle is given below.

![Otto cycle diagram]

a. Show that the efficiency of the Otto cycle can be written solely in terms of the compression ratio \( b = V_1/V_2 \) as

\[
e_h = 1 - b^{1-\gamma}
\]

b. Do you think that the efficiency of the cycle depends only on the compression ratio when other effects such as friction are taken into account? Would you expect a real engine to be most efficient when operating at high power or at low power? Defend your answer.

c. Show that the efficiency of the Otto cycle is less than a corresponding Carnot cycle acting between the highest and lowest temperatures.

d. Suppose that the compression ratio of the Otto cycle is \( b = 8.00 \). At the beginning of the compression process, 500 cm\(^3\) of gas is at 100 kPa and 20.0°C. At the beginning of the power stroke, the temperature is 750°C. Modeling the working substance as a diatomic ideal gas, determine the energy input, energy exhaust, and net work of the cycle.

Answer:

a. The efficiency of any cycle is given by

\[
e_h = \frac{W}{Q_H} = 1 - \frac{Q_c}{Q_H}
\]
The heat input and output for the Otto cycle occur during the isochoric legs from 2 → 3 and from 4 → 1. Therefore, we have

\[ |Q_C| = |\Delta U_{41}| = C_V(T_4 - T_1), \quad Q_H = \Delta U_{23} = C_V(T_3 - T_2) \]

Therefore, the efficiency of the Otto cycle is given by

\[ e_{otto} = 1 - \frac{|Q_C|}{Q_H} = 1 - \frac{T_4 - T_1}{T_3 - T_2} \]

The ratio of the temperatures in the square brackets can be given in terms of the compression ratio if we recognize that legs 3 → 4 and 1 → 2 are adiabatic. Using the ideal gas law, we can write the adiabatic equation of state as

\[ PV^\gamma = \text{const.} \rightarrow TV^{\gamma - 1} = \text{const.} \]

Therefore,

\[ T_3 V_2^{\gamma - 1} = T_4 V_1^{\gamma - 1} \Rightarrow T_4 = T_3 \left( \frac{V_2}{V_1} \right)^{\gamma - 1} = T_3 \left( \frac{V_1}{V_2} \right)^{1 - \gamma} = T_3 b^{1 - \gamma} \]

\[ T_1 V_2^{\gamma - 1} = T_2 V_1^{\gamma - 1} \Rightarrow T_1 = T_2 \left( \frac{V_2}{V_1} \right)^{\gamma - 1} = T_2 \left( \frac{V_1}{V_2} \right)^{1 - \gamma} = T_2 b^{1 - \gamma} \]

This implies that

\[ e_{otto} = 1 - \frac{T_4 - T_1}{T_3 - T_2} = 1 - b^{1 - \gamma} \frac{T_3 - T_2}{T_3 - T_2} = 1 - b^{\gamma - 1} \]

b. Other factors that would affect the efficiency of an automobile engine include friction, conductive heat loss, and incomplete combustion of the fuel. The energy loss due to friction of the pistons with the cylinder walls should be roughly the same per stroke regardless of the power and the amount of fuel consumed. As a fraction of the total energy produced, it is most probable that this loss will be the smallest when the engine is operating at high power, producing as much work per stroke as possible.

c. For the adiabatic legs of the Otto cycle, we note that

\[ T_1 V_1^{\gamma - 1} = T_2 V_2^{\gamma - 1} \Rightarrow \left( \frac{V_2}{V_1} \right)^{\gamma - 1} = \frac{T_1}{T_2} \]

Therefore, the efficiency of the Otto cycle in terms of temperature is given by
Thus, the efficiency of the Otto cycle is less than the efficiency of the Carnot cycle.

d. The mass of the air-fuel mixture is given by

\[
m = \frac{P_1 V_1}{R_d T_1} = \frac{(100 \text{ kPa})(0.500 \times 10^{-3} \text{ m}^3)}{(287 \text{ Pa m}^3/\text{kg K})} = 5.95 \times 10^{-4} \text{ kg}
\]

Since the compression stage is adiabatic, we have

\[
P_1 V_1^\gamma = P_2 V_2^\gamma \Rightarrow P_2 = P_1 \left(\frac{V_1}{V_2}\right)^\gamma = P_2 = (100 \text{ kPa})(8.00)^{1.4} = 1838 \text{ kPa}
\]

The temperature after the compression stage is given by

\[
T_2 = \frac{P_2 V_2}{m R_d} = \frac{(1838 \text{ kPa})(0.625 \times 10^{-4} \text{ m}^3)}{(5.95 \times 10^{-4} \text{ kg})(287 \text{ Pa m}^3/\text{kg K})} = 673 \text{ K}
\]

The pressure after the ignition stage is given by

\[
P_3 = \frac{m R_d T_3}{V_3} = \frac{m R_d T_3}{V_2} = \frac{(5.95 \times 10^{-4} \text{ kg})(287 \text{ Pa m}^3/\text{kg K})(1023 \text{ K})}{(0.625 \times 10^{-4} \text{ m}^3)} = 2795 \text{ kPa}
\]

Since the expansion stage is adiabatic, this means that \(PV^\gamma = \text{constant}\). We note that for a diatomic ideal gas,

\[
\gamma = \frac{C_p}{C_v} = \frac{f}{2Nk} = \frac{5}{2Nk} = \frac{5}{2Nk} = 1.4
\]

Thus, we have

\[
P_3 V_2^\gamma = P_4 V_1^\gamma \Rightarrow P_4 = P_3 \left(\frac{V_2}{V_1}\right)^\gamma = P_1 b^{-\gamma} = (2793 \text{ kPa})(8.00)^{-1.4} = 152 \text{ kPa}
\]

The temperature after the expansion is given by
Now that we have the temperature at the beginning and end of each process, we can calculate the net energy transfer and the net work done by engine. The net energy input in the cycle occurs during the ignition phase and it is given by

\[ Q_h = m c_v (T_3 - T_2) = \left( 5.95 \times 10^{-4} \text{ kg} \right) \left( 718 \frac{J}{\text{kg} \cdot \text{K}} \right) (1023 \text{ K} - 673 \text{ K}) = 149.5 \text{ J} \]

The net energy output in the cycle occurs during the exhaust phase and it is given by

\[ Q_c = m c_v (T_4 - T_1) = \left( 5.95 \times 10^{-4} \text{ kg} \right) \left( 718 \frac{J}{\text{kg} \cdot \text{K}} \right) (445 \text{ K} - 293 \text{ K}) = 64.9 \text{ J} \]

The net work done by the engine is given by

\[ W_{\text{net}} = Q_h - |Q_c| = 149.5 \text{ J} - 64.9 \text{ J} = 84.6 \text{ J} \]
5. In this question, you will analyze the thermodynamics of the Stirling heat engine.
   a. Based on the description of the Stirling heat engine in class, draw a PV diagram for this idealized Stirling cycle. Label each step in the engine’s cycle.
   b. Suppose that, during stage 2 of the Stirling cycle, the gas gives up heat to the cold reservoir instead of the regenerator and, during stage 4 of the Stirling cycle, the gas absorbs heat from the hot reservoir. Assuming an ideal gas, show that the efficiency of this engine can be written as
      \[ \frac{1}{e} = \frac{T_H}{T_H - T_C} + \frac{f}{2 \ln(V_2/V_1)} \]
      where \( f \) is the number of degrees of freedom per molecule.
   c. Suppose that you place the regenerator back into the system. Argue that, if it works perfectly, the efficiency of a Stirling engine is the same as that of a Carnot engine.

Answer:

a. The Stirling cycle can be expressed as two isothermal processes and two isochoric processes, as given below

\[ \begin{align*}
  & W_{net} = W_{12} + W_{34} = -\int_{V_1}^{V_2} P \, dV - \int_{V_2}^{V_1} P \, dV = -N k (T_H - T_C) \ln \left( \frac{V_2}{V_1} \right)
\end{align*} \]

The net heat input into the gas during the power stroke and during the transfer to the hot cylinder is given by
Therefore, the efficiency is given by

$$\frac{1}{e} = \frac{Q_H}{W} = \frac{NkT_H \ln \left( \frac{V_2}{V_1} \right) + \frac{f}{2} Nk(T_H - T_C)}{-Nk(T_H - T_C) \ln \left( \frac{V_2}{V_1} \right)} = \frac{T_H}{T_H - T_C} + \frac{f}{2 \ln(V_2/V_1)}$$

c. With an ideal regenerator, the heat input during step 4→1 comes for free because it’s exactly the same as the heat output during step 2→3. Therefore, only $Q_{12}$ should be counted as part of $Q_H$ when computing the efficiency. Thus, in the derivation in part (a), we have

$$\frac{1}{e} = \frac{Q_H}{W} = \frac{NkT_H \ln \left( \frac{V_2}{V_1} \right)}{-Nk(T_H - T_C) \ln \left( \frac{V_2}{V_1} \right)} = \frac{T_H}{T_H - T_C} \Rightarrow e = 1 - \frac{T_C}{T_H} = e_c$$
**Bonus:** [5 points] Let $C$ be a Carnot engine operating between the temperatures $T_h$ and $T_l$ ($T_h > T_l$) and let $M$ represent any other heat engine operating between the same two temperatures as shown below:

Let $Q_1$ ($Q_1'$) refer to the energy exchanged as heat with the reservoir at $T_h$ and let $Q_2$ ($Q_2'$) refer to the energy exchanged as heat with the reservoir at $T_l$. Consider a process consisting of $n'$ cycles of $M$ and $n$ cycles of $C$ in reverse.

a. Show that the net external work done by our system is $W_{total} = n'|W'| - n|W| = Q_h - Q_1$, where

\[
Q_h = n'|Q_1'| - n|Q_1| \\
Q_l = n'|Q_2'| - n|Q_2|
\]

b. Use (a) and (b) and the Kelvin-Planck statement to conclude that $n'|Q_2'| \geq n|Q_2|$. 
   Hint: Note that to as good an approximation we desire,

\[
\frac{|Q_1|}{|Q_1'|} = \frac{n'}{n}
\]

c. Use (b) to conclude that

\[
\frac{|Q_1|}{|Q_2|} \geq \frac{|Q_1'|}{|Q_2'|}
\]

What does this derivation prove?

**Answer:**

While operating in reverse, $C$ absorbs an amount of energy $Q_2$ in the form of heat from the low temperature reservoir and an amount of work $W$, depositing $Q_1$ into the reservoir at the high temperature reservoir. Meanwhile, $M$ absorbs an amount of energy $Q_1'$ from the high temperature reservoir, does work $W'$, and rejects $Q_2'$ into the low temperature reservoir. Therefore, the net external work done by our system is then
\[ W_{total} = n'W' + nW = n'|W'| - n|W| \]

The heat removed from the high temperature reservoir is

\[ Q_h = n'|Q_1'| + nQ_1 = n'|Q_1'| - n|Q_1| \]

The heat added to the high temperature reservoir is

\[ Q_l = -n'|Q_2'| - nQ_2 = n'|Q_2'| - n|Q_2| \]

By the first law, we have

\[ W_{total} = n'|W'| - n|W| = -Q_h + Q_l \]

Since

\[ \left| \frac{Q_1}{Q_1'} \right| = \frac{n'}{n} \Rightarrow |Q_1| = \frac{n'}{n} |Q_1'| \]

This means that the net heat extracted from the high temperature is zero. Thus, we have

\[ W_{total} = Q_l = n'|Q_2'| - n|Q_2| \]

Now the Kelvin-Planck statement ensures us that \( W_{total} \geq 0 \), for if \( W_{total} < 0 \), we would have a process which extracts a certain amount of heat from the low temperature reservoir and performs useful work with no other effect. Thus, we have

\[ W_{total} \geq 0 \Rightarrow n'|Q_2'| \geq n|Q_2| \]

Thus, replacing

\[ n' = n \left| \frac{Q_1}{Q_1'} \right| \]

We have our desired result

\[ \left| \frac{Q_1}{Q_2} \right| \geq \left| \frac{Q_1'}{Q_2'} \right| \]

Thus, the Carnot engine has the best efficiency of all engines operating between two temperatures. Since this engine can also be run in reverse to produce a refrigerator, this means also means that the Carnot refrigerator has the best efficiency of all refrigerators operating between two temperatures. The corollary to this proof is that every reversible engine that runs between two temperatures must also be a Carnot engine.