# Maxwell's Equations

**Overview and Motivation**: Maxwell's (M's) equations, along with the Lorentz force law constitute essentially all of classical electricity and magnetism (E and M). One phenomenon that arises from M's equations is electromagnetic radiation, that is, electromagnetic waves. Here we introduce M's equations, discuss how they should be viewed, and see how M's equations imply the wave equation. We will look at a plane-wave solution to M's equations. We will also see that the conservation of electric charge is a direct result of Maxwell's equations.

**Key Mathematics**: We will gain some practice with "del" ($\nabla$) used in calculating the divergence and the curl of the electric and magnetic vector fields.

## I. Maxwell's Equations

The basic Maxwell's equations are typically written, in SI units[1], as

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\varepsilon_0}, \tag{1}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \tag{2}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0\, \mathbf{j}(\mathbf{r},t) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}. \tag{4}$$

These are coupled first-order, linear, partial differential equations for the electric and magnetic vector fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, respectively. The two constants in the equations, $\varepsilon_0$ and $\mu_0$, are the fundamental constants of E and M, respectively. You probably first encountered these two constants in your introductory physics class when you studied the electric force from a point charge and the magnetic force form a long, straight wire carrying a constant current. The other two quantities in these equations are the electric charge density $\rho(\mathbf{r},t)$ and electric-charge current density $\mathbf{j}(\mathbf{r},t)$.

[1] There have been no fewer than 5 systems of units traditionally used for E and M: electrostatic (esu), electromagnetic (emu), Gaussian (cgs), Heaviside-Lorentz, and Rationalized MKSA (now known as SI). Beware when reading the literature!

So what is the meaning of Eqs. (1) – (4)? The way these equations are written you should think of the quantity on the rhs as giving rise to the quantity on the lhs of each equation. So Eq. (1) tells us that electric charge density is the source of electric field. Similarly, Eq. (2) tells us that there is no such corresponding magnetic charge density. Equation (3) tells us that a time varying magnetic field can produce an electric field, and Eq. (4) says that both an electric-charge current density and a time varying electric field can produce a magnetic field. Maxwell was actually only responsible for the last term in Eq. (4), but the term is key to E and M because without it there would be no electromagnetic radiation.

Often one considers the charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)$ to be given quantities. That is, one assumes that there is something external to the problem that controls $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ and so they are simply treated as given source terms for the equations. However, in some problems the dynamics of $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are determined by the fields themselves through the Lorentz force equation

$$\mathbf{F} = q\left[\mathbf{E}(\mathbf{r},t) + \left(\vec{v} \times \mathbf{B}(\mathbf{r},t)\right)\right], \tag{5}$$

which describes the force on a particle with charge $q$ and velocity $v$. In such cases Eqs. (1) – (5) must be solved self consistently for both the time varying fields and charge and current distributions.

## II. The Conservation of Electric Charge

One of the consequences of M's equations is the conservation of electric charge. If we start with Eq. (1) and takes its time derivative, we obtain

$$\nabla \cdot \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial \rho(\mathbf{r},t)}{\partial t}, \tag{6}$$

after switching the order of the divergence and time derivative on the lhs. Equation (4) can be rearranged as

$$\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \frac{1}{\mu_0 \varepsilon_0} \nabla \times \mathbf{B}(\mathbf{r},t) - \frac{1}{\varepsilon_0} \mathbf{j}(\mathbf{r},t). \tag{7}$$

Using this equation to substitute for $\partial \mathbf{E}/\partial t$ in Eq. (6) then yields

$$\nabla \cdot \left[\frac{1}{\mu_0 \varepsilon_0} \nabla \times \mathbf{B}(\mathbf{r},t) - \frac{1}{\varepsilon_0} \mathbf{j}(\mathbf{r},t)\right] = \frac{1}{\varepsilon_0} \frac{\partial \rho(\mathbf{r},t)}{\partial t}. \tag{8}$$

But this simplifies considerably because $\nabla\cdot(\nabla\times\mathbf{V})=0$ for any vector field $\mathbf{V}$. Thus we have

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t}+\nabla\cdot\mathbf{j}(\mathbf{r},t)=0, \tag{9}$$

the continuity equation for the charge density and the charge current density. As we discussed in an earlier lecture, the continuity equation is the local form of the statement of the conservation of the particular quantity (in this case electric charge) that corresponds to the given density and current density.

## III. Wave Equations for the Electric and Magnetic Fields

As we now demonstrate, M's equations imply waves equations for both the electric and magnetic fields. To keep things simple we consider M's equations in a charge-free, current-free region of space. Then Eq. (1) – (4) become

$$\nabla\cdot\mathbf{E}(\mathbf{r},t)=0, \tag{10}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=0, \tag{11}$$

$$\nabla\times\mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{12}$$

$$\nabla\times\mathbf{B}(\mathbf{r},t)=\mu_0\varepsilon_0\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t}. \tag{13}$$

These are know as the homogeneous M's equations because either **E** or **B** appear linearly in every (nonzero) term in the equations. Let's derive the wave equation for the magnetic field. To do this we first take the curl of Eq. (13) to produce

$$\nabla\times[\nabla\times\mathbf{B}(\mathbf{r},t)]=\mu_0\varepsilon_0\nabla\times\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} \tag{14}$$

Using the identity $\nabla\times(\nabla\times\mathbf{V}(\mathbf{r}))=\nabla(\nabla\cdot\mathbf{V}(\mathbf{r}))-\nabla^2\mathbf{V}(\mathbf{r})$ on the lhs and switching the order of the curl and time derivative on the rhs then produces

$$\nabla[\nabla\cdot\mathbf{B}(\mathbf{r},t)]-\nabla^2\mathbf{B}(\mathbf{r},t)=\mu_0\varepsilon_0\frac{\partial[\nabla\times\mathbf{E}(\mathbf{r},t)]}{\partial t}. \tag{15}$$

Now using Eq. (11) to substitute for $\nabla\cdot\mathbf{B}(\mathbf{r},t)$ on the lhs and Eq. (12) to substitute for $\nabla\times\mathbf{E}(\mathbf{r},t)$ on the rhs yields

$$\nabla^2 \mathbf{B}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}(\mathbf{r},t)}{\partial t^2}, \tag{16}$$

the wave equation for the magnetic field $\mathbf{B}(\mathbf{r},t)$, where the standard constant $c^2$ in the wave equation is equal to $1/\mu_0\varepsilon_0$. One can similarly derive the corresponding wave equation for the electric field

$$\nabla^2 \mathbf{E}(\mathbf{r},t) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}(\mathbf{r},t)}{\partial t^2}. \tag{17}$$

Notice that each of these wave equations is for a vector quantity, and so in essence each of these equations is really three wave equations, one for each component of the electric or magnetic field. Another thing to note is that while M's equations imply the wave equations for $\mathbf{E}$ and $\mathbf{B}$, the two fields are not independent. That is, all solutions to Eqs. (16) and (17) will not necessarily satisfy M's equations. Another way to think about this is that by taking a derivative (the curl) of Eq. (13) (to derive the $\mathbf{B}$-field wave equation) we lost some information originally contained in that equation.

## IV. Plane Wave Solutions to M's Equations

We already know that a plane wave is one possible type of solution to the 3D wave equation. So let's assume that we have an electric field of the form

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi). \tag{18}$$

Remember, the wave vector $\mathbf{k}$ points in the direction of propagation and $k \equiv |\mathbf{k}| = 2\pi/\lambda$, where $\lambda$ is the wavelength. Substituting Eq. (18) into Eq. (17) gives us, as usual, the dispersion relation, which in this case is $\omega = ck$ where $c = 1/\sqrt{\mu_0\varepsilon_0}$ is known as the speed of light. As far as the wave equation is concerned, the dispersion relation is the only constraint that needs to be satisfied in order for Eq. (18) to be a solution.

However, M's equations put a further constraint on the electric field. Let's substitute Eq. (18) into Eq. (10). Then we obtain

$$\nabla\cdot[\mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)] = 0 \tag{19}$$

To see what this equation implies for our plane-wave solution let's rewrite it using Cartesian coordinates,

$$\left(\frac{\partial}{\partial x}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\hat{\mathbf{z}}\right)\cdot\left[\left(E_{0x}\hat{\mathbf{x}}+E_{0y}\hat{\mathbf{y}}+E_{0z}\hat{\mathbf{z}}\right)\cos\left(k_x x+k_y y+k_z z-\omega t+\phi\right)\right]=0\,. \tag{20}$$

Calculating the derivatives produces

$$\left(E_{0x}k_x+E_{0y}k_y+E_{0z}k_z\right)\sin\left(k_x x+k_y y+k_z z-\omega t+\phi\right)=0 \tag{21}$$

which can be written in coordinate-independent notation as

$$\left(\mathbf{E}_0\cdot\mathbf{k}\right)\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi\right)=0\,. \tag{22}$$

For this to be true for all values of $\mathbf{r}$ and $t$, this then implies

$$\mathbf{E}_0\cdot\mathbf{k}=0\,. \tag{23}$$

So what does this tell us? Because $\mathbf{k}$ points in the direction of the wave's propagation, this tells us that the electric field must be perpendicular (or transverse) to the direction of propagation. That is, there is no longitudinal component to the electric field for a plane-wave solution to M's equations.

So what about the magnetic field? Somewhere we should have learned that electromagnetic radiation consists of both propagating electric and magnetic fields. In order to see what else M's tell us let's assume that the magnetic field associated with the electric field given by Eq. (18) is of the form

$$\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0\cos\left(\mathbf{k}'\cdot\mathbf{r}-\omega' t+\phi'\right). \tag{24}$$

Let's now see what M's equations tell us about $\mathbf{B}_0$, $\mathbf{k}'$, $\omega'$, and $\phi'$. As with the electric field, we can use Eq. (11) to tell us that $\mathbf{B}_0$ is also perpendicular to the direction of propagation. We can learn more by substituting Eqs. (18) and (24) into Eq. (13). After a bit of algebra and differentiation we end up with the result

$$\left(\mathbf{B}_0\times\mathbf{k}'\right)\sin\left(\mathbf{k}'\cdot\mathbf{r}-\omega' t+\phi'\right)=\frac{\omega}{c^2}\mathbf{E}_0\sin\left(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi\right), \tag{25}$$

where we have used $\mu_0\varepsilon_0=1/c^2$. For this to hold for all values of $\mathbf{r}$ and $t$ we must have the following relationships: $\mathbf{k}'=\mathbf{k}$, $\omega'=\omega$, $\phi'=\phi$, and $\left(\mathbf{B}_0\times\mathbf{k}\right)=\left(\omega/c^2\right)\mathbf{E}_0$. The first three relationships tell us that the electric and magnetic fields have the same wavelength, frequency, and phase, and propagate in the same direction. The last relationship is a bit more interesting. The last relationship tells us that $\mathbf{E}_0$ is perpendicular to both $\mathbf{k}$ (which we knew already) and $\mathbf{B}_0$. Thus, because both $\mathbf{E}_0$ and $\mathbf{B}_0$ are perpendicular to $\mathbf{k}$ all three vector are perpendicular to each other. Furthermore, because $(\mathbf{B}_0 \times \mathbf{k}) \| \mathbf{E}_0$ we must also have $(\mathbf{E}_0 \times \mathbf{B}_0) \| \mathbf{k}$ and $(\mathbf{k} \times \mathbf{E}_0) \| \mathbf{B}_0$. And lastly, because $\mathbf{B}_0$ and $\mathbf{k}$ are perpendicular, the relationship $(\mathbf{B}_0 \times \mathbf{k}) = (\omega/c^2)\mathbf{E}_0$ tells us that $B_0 k = (\omega/c^2)E_0$ or, because $\omega = ck$, $B_0 = E_0/c$. Thus, the magnetic field can be expressed as

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}(\hat{\mathbf{k}} \times \mathbf{E}_0)\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi), \tag{26}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. Or we can simply write, for our plane wave solution to M's equations,

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}(\vec{r},t).$$

## Exercises

***31.1**. As was done in the notes for the magnetic field, derive the wave equation for the electric field.

***31.2**. The product $\mu_0\varepsilon_0$ appears in the wave equations where $1/c^2$ traditionally appears. Look up $\mu_0$ and $\varepsilon_0$ and calculate $c = 1/\sqrt{\mu_0\varepsilon_0}$. What do you get?

***31.3**. Derive the dispersion relation $\omega = ck$ by substituting Eq. (18) into Eq. (17).

****31.4**. **Conditions on $\vec{E}$ and $\vec{B}$ for a plane wave**
(**a**) Substitute Eqs. (18) and (24) into Eq. (13) and derive Eq. (25).
(**b**) Substitute Eqs. (18) and (24) instead into Eq. (12) and again derive Eq. (25). This shows that in this situation the information in Eq. (12) is redundant.

****31.5**. **A traveling-wave solution to Maxwell's equations**. Show that the traveling wave fields

$$\mathbf{E}(\mathbf{r},t) = E_0\hat{\mathbf{x}}\cos(kz - \omega t) \text{ and } \mathbf{B}(\mathbf{r},t) = \frac{E_0}{c}\hat{\mathbf{y}}\cos(kz - \omega t)$$

satisfy all four homogeneous Maxwell equations.

# Energy Density and the Poynting Vector

**Overview and Motivation**: We saw in the last lecture that electromagnetic waves are one consequence of Maxwell's (M's) equations. With electromagnetic waves, as with other waves, there is an associated energy density and energy flux. Here we introduce these electromagnetic quantities and discuss the conservation of energy in the electromagnetic fields. Further, we see how the expressions for the energy density and energy flux can be put into a form that is similar to expressions for the same quantities for waves on a string.

**Key Mathematics**: We will gain some more practice with the "del" operator $\nabla$ . We will also discuss what is meant by a time-averaged quantity.

## I. Energy Density and Energy-current Density in EM Waves

Recall from the last lecture the basic Maxwell's equations,

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\varepsilon_0}, \tag{1}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \tag{2}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_0\, \mathbf{j}(\mathbf{r},t) + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}. \tag{4}$$

As we discussed last time, for $\rho(\mathbf{r},t)=0$ and $\mathbf{j}(\mathbf{r},t)=0$, M's equations imply the wave equation for both $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. We know that waves transport energy. So how is the energy in an electromagnetic wave expressed? Well, you should have learned in your introductory physics course that the energy density contained in the electric field is given by[1]

$$u_{el}(\mathbf{r},t) = \frac{\varepsilon_0}{2}\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = \frac{\varepsilon_0}{2}\left[\mathbf{E}(\mathbf{r},t)\right]^2. \tag{5}$$

Typically this energy density is introduced in a discussion of the energy required to charge up a capacitor (which produces an electric field between the plates). Similarly, the energy density contained in the magnetic field is given by

[1] In keeping with standard EM notation, we use $u$ for the energy density and $\mathbf{S}$ for the energy flux.

$$u_{mag}(\mathbf{r},t) = \frac{1}{2\mu_0}\mathbf{B}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t) = \frac{1}{2\mu_0}[\mathbf{B}(\mathbf{r},t)]^2. \tag{6}$$

Typically this relationship is introduced in a discussion of the energy required to establish a current in a toroid (which produces a magnetic field inside the toroid). Notice again that the two fundamental constants of E and M, $\varepsilon_0$ and $\mu_0$, appear in Eq. (5) and Eq. (6), respectively. Thus the total energy $u(\mathbf{r},t)$ contained in a region of space with both electric and magnetic fields is

$$u(\mathbf{r},t) = \frac{1}{2}\left\{\varepsilon_0[\mathbf{E}(\mathbf{r},t)]^2 + \frac{1}{\mu_0}[\mathbf{B}(\mathbf{r},t)]^2\right\}. \tag{7}$$

Because $c^2 = 1/(\mu_0\varepsilon_0)$, this can also be written as

$$u(\mathbf{r},t) = \frac{1}{2\mu_0}\left\{\left[\frac{\mathbf{E}(\mathbf{r},t)}{c}\right]^2 + [\mathbf{B}(\mathbf{r},t)]^2\right\}. \tag{8}$$

Recall, for a traveling EM wave in vacuum the electric and magnetic field amplitudes are related by $B = E/c$. Equation (8) thus shows that equal amounts of energy are contained in the electric and magnetic fields in such a wave.

What about the energy current density (also known as the energy flux)? Well, another basic fact about electromagnetic radiation (that you may or may not have learned in your introductory physics course) is that the energy flux in a particular region of space is equal to

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{\mu_0}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) \tag{9}$$

As we learned in the last lecture, the direction of propagation of an electromagnetic plane wave is in the direction of $\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)$. As expected, Eq. (9) indicates that the energy flux points in this same direction. In E and M the energy flux is known as the **Poynting vector** (convenient because it *points* in the direction of the energy flow).

## II. Continuity Equation for $u$ and $\vec{S}$

If $u$ and $\vec{S}$ are indeed the energy and energy-current densities, respectively, then we expect that they should be related by the continuity equation

$$\frac{\partial u(\mathbf{r},t)}{\partial t}+\nabla\cdot\mathbf{S}(\mathbf{r},t)=0\,. \tag{10}$$

Let's see if M's equations indeed imply Eq. (10). We first start with Eq. (8) and calculate its time derivative, which gives us

$$\frac{\partial u}{\partial t}=\frac{1}{\mu_0}\left(\frac{1}{c^2}\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t}+\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\right). \tag{11}$$

In deriving Eq. (11) we must remember, for example, that $\mathbf{E}^2$ is really shorthand for $\mathbf{E}\cdot\mathbf{E}$ [see Eq. (5)]. Starting with Eq. (9) we can also calculate $\nabla\cdot\mathbf{S}$, which gives us, after using the vector identity $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$,

$$\nabla\cdot\mathbf{S}=\frac{1}{\mu_0}\left[\mathbf{B}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{B})\right]. \tag{12}$$

We can now use M's Eqs. (3) and (4) to replace the curls in Eq. (12), which produces, after a bit of manipulation and the use of $\mu_0\varepsilon_0=1/c^2$,

$$\nabla\cdot\mathbf{S}=-\frac{1}{\mu_0}\left(\frac{1}{c^2}\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t}+\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\right)-\mathbf{E}\cdot\mathbf{j}. \tag{13}$$

Comparing Eqs. (11) and (13) we see that

$$\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=-\mathbf{E}\cdot\mathbf{j} \tag{14}$$

So what happened? Why didn't M's equations gives us Eq. (10)? Well, there is a very good reason. The energy density $u$ and energy-current density $\mathbf{S}$ are densities associated with the fields only. But energy can also exist in the (kinetic) energy of the charge density. The term $\mathbf{E}\cdot\mathbf{j}$ is known as **Joule heating**; it expresses the rate of energy transfer to the charge carriers from the fields. This is the (spatially) local version of an equation with which you are already familiar, $P=VI$. Notice that this term only contains the electric field because the magnetic field can do no work on the charges. The term appears with a negative sign in Eq. (14) because an increase in energy of the charge carriers contributes to a decrease in energy in the fields.

Obviously, if the homogeneous M's equations apply [ $\rho(\mathbf{r},t)=0$ and $\mathbf{j}(\mathbf{r},t)=0$ ], then Eq. (10), the standard continuity equation is indeed valid.

**III. The densities $u$ and $\mathbf{S}$ for an EM Plane Wave**

In the last lecture we looked at the plane-wave solution

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi) \tag{15}$$

$$\mathbf{B}(\mathbf{r},t)=\frac{1}{c}\left(\hat{\mathbf{k}}\times\mathbf{E}_0\right)\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi) \tag{16}$$

to the homogeneous Maxwell's equations. Let's calculate $u$ and $\vec{S}$ for these fields. Substituting Eqs. (15) and (16) into Eqs. (8) and (9) produces

$$u(\mathbf{r},t)=\frac{1}{c}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left[E_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi)\right]^2, \tag{17}$$

and

$$\mathbf{S}(\mathbf{r},t)=\sqrt{\frac{\varepsilon_0}{\mu_0}}\left[E_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi)\right]^2\hat{\mathbf{k}}, \tag{18}$$

respectively. Comparing Eqs. (17) and (18) we see that

$$\mathbf{S}(\mathbf{r},t)=c\,u(\mathbf{r},t)\hat{\mathbf{k}}. \tag{19}$$

The agrees with the general expectation for a traveling wave that the energy current flux $\mathbf{j}_\varepsilon(\mathbf{r},t)$ is related to its associated energy density $\rho_\varepsilon(\mathbf{r},t)$ via $\mathbf{j}_\varepsilon(\mathbf{r},t)=\rho_\varepsilon(\mathbf{r},t)\mathbf{v}$, where $\mathbf{v}$ is the velocity of $\rho_\varepsilon(\mathbf{r},t)$.

The Poynting vector expressed in Eq. (18) is a space and time dependent quantity. Often, however, often we are more interested in the time-averaged value of this quantity. In general, the time-averaged value of a periodic function with period $T$ is given by

$$\langle A(t)\rangle_t=\frac{1}{T}\int_0^T A(t)\,dt. \tag{20}$$

With this definition the time-averaged value of $\mathbf{S}$ is

$$\left\langle \mathbf{S}(t) \right\rangle_t = E_0^2 \sqrt{\frac{\varepsilon_0}{\mu_0}} \hat{\mathbf{k}} \left\{ \frac{\omega}{2\pi} \int_0^{\omega/2\pi} \left[\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)\right]^2 dt \right\} \tag{21}$$

Because the average value of any harmonic function squared is simply $1/2$, we have

$$\left\langle \mathbf{S}(t) \right\rangle_t = \frac{E_0^2}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \hat{\mathbf{k}} . \tag{22}$$

On last remark about $\left\langle \mathbf{S}(\mathbf{r},t) \right\rangle_t$. In the optics world $\left\langle \mathbf{S}(\mathbf{r},t) \right\rangle_t$ is known as the **intensity** associated with the electromagnetic wave. Its dot product with a normal vector to some surface gives the average power per unit area incident on that surface.

## IV. An Analogy Between Mechanical and EM Waves

We previously studied the energy contained in mechanical waves. In particular, we looked at transverse waves on a string, which have an energy density and energy-current density that were essentially expressed as

$$\rho_\varepsilon(x,t) = \frac{\tau}{2}\left\{ \left[\frac{1}{c}\frac{\partial q(x,t)}{\partial t}\right]^2 + \left[\frac{\partial q(x,t)}{\partial x}\right]^2 \right\}, \tag{23}$$

$$j_\varepsilon(x,t) = -\tau \left[\frac{\partial q(x,t)}{\partial t}\right]\left[\frac{\partial q(x,t)}{\partial x}\right]. \tag{24}$$

As they stand, these equations do not look particularly like Eqs. (8) and (9) for the corresponding electromagnetic quantities.

The mechanical-waves expressions are written in terms of derivatives of the displacement while the electromagnetic quantities are written in terms of the fields. However, in the theory of electricity and magnetism we can introduce a quantity known as the vector potential $\mathbf{A}(\mathbf{r},t)$ that, in the absence of $\rho(\vec{r},t)$ and $\mathbf{j}(\mathbf{r},t)$, can be defined such that it is related to the electric and magnetic fields via

$$\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \tag{25}$$

and

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t). \tag{26}$$

Substituting these expressions into Eqs. (8) and (9) then gives us two equations that now look quite similar to Eqs. (23) and (24),

$$u(\mathbf{r},t)=\frac{1}{2\mu_0}\left\{\left[\frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right]^2+\left[\nabla\times\mathbf{A}(\mathbf{r},t)\right]^2\right\}, \tag{27}$$

$$\mathbf{S}(\mathbf{r},t)=-\frac{1}{\mu_0}\left(\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\times\left[\nabla\times\mathbf{A}(\mathbf{r},t)\right]\right). \tag{28}$$

For a more exact analogy, let's go back to Eqs. (15) and (16), the plane-wave solution to M's equations. Let's simplify things by choosing the coordinate system so that $\mathbf{k}$ points in the $+x$ direction, $\mathbf{E}_0$ points along the $+y$ direction, which leaves $\mathbf{B}_0$ to point along the $+z$ direction. The electric and magnetic fields for the plane wave can then be written as

$$\mathbf{E}(x,t)=E_0\hat{\mathbf{y}}\cos(kx-\omega t+\phi), \tag{29}$$

$$\mathbf{B}(x,t)=\frac{E_0}{c}\hat{\mathbf{z}}\cos(kx-\omega t+\phi). \tag{30}$$

These two fields are consistent with the vector potential

$$\mathbf{A}(x,t)=\frac{E_0}{\omega}\hat{\mathbf{y}}\sin(kx-\omega t+\phi)=A_y(x,t)\hat{\mathbf{y}}. \tag{31}$$

With this vector potential the time derivative and curl of $\mathbf{A}$ simplify to $\partial\mathbf{A}/\partial t=(\partial A_y/\partial t)\hat{\mathbf{y}}$ and $\nabla\times\mathbf{A}=(\partial A_y/\partial x)\hat{\mathbf{z}}$ so that

$$\left(\frac{\partial\mathbf{A}}{\partial t}\right)^2=\left(\frac{\partial A_y}{\partial t}\right)^2, \tag{32}$$

$$(\nabla\times\mathbf{A})^2=\left(\frac{\partial A_y}{\partial x}\right)^2, \tag{33}$$

and

$$\frac{\partial \mathbf{A}}{\partial t} \times (\nabla \times \mathbf{A}) = \frac{\partial A_y}{\partial t}\frac{\partial A_y}{\partial x}\hat{\mathbf{x}}. \tag{34}$$

With these last three expressions we can express $u$ and $\vec{S}$ for our plane wave solution [Eqs. (31) and (32)] as

$$u(x,t) = \frac{1}{2\mu_0}\left\{\left[\frac{1}{c}\frac{\partial A_y}{\partial t}\right]^2 + \left[\frac{\partial A_y}{\partial x}\right]^2\right\}, \tag{35}$$

$$\mathbf{S}(x,t) = -\frac{1}{\mu_0}\frac{\partial A_y}{\partial t}\frac{\partial A_y}{\partial x}\hat{\mathbf{x}}. \tag{36}$$

These expression are now essentially identical to Eqs. (23) and (24), the analogous expressions for mechanical waves on a string if the following correspondences are made: $q \leftrightarrow A_y$ and $\tau \leftrightarrow 1/\mu_0$.

## Exercises

***32.1** Show that Eqs. (29) and (30) follow from Eq. (31).

****32.2 A traveling-wave solution to Maxwell's equations**. Consider the electric field $\mathbf{E}(r,t) = E_0\hat{\mathbf{x}}\cos(kz - kct)$

(**a**) What is the corresponding magnetic field?
(**b**) Calculate the energy density $u(z,t)$ associated with each of these fields.
(**c**) Calculate the Poynting vector $\mathbf{S}(z,t)$ associated with these fields.
(**d**) Show that $u(z,t)$ and $\mathbf{S}(z,t)$ satisfy the appropriate continuity equation.

***32.3** Show in the absence of charge and current densities that – in general – the vector potential $\mathbf{A}(\mathbf{r},t)$ satisfies the wave equation. In addition to equations in the notes, you will need to use the fact that the vector potential satisfies $\nabla \cdot \mathbf{A} = 0$.