Synoptic Meteorology II: The Q-Vector Form of the Omega Equation

3-5 March 2015

Readings: Section 2.3 of *Midlatitude Synoptic Meteorology*.

Motivation: Why Do We Need Another Omega Equation?

The quasi-geostrophic omega equation is an excellent diagnostic tool and has served as the one of the foundations for introductory synoptic-scale weather analysis for fifty or more years. However, it is not an infallible tool. Rather, it has two primary shortcomings:

- The two primary forcing terms in the quasi-geostrophic omega equation, the differential geostrophic vorticity advection and Laplacian of the potential temperature advection terms, often have different signs from one another. Without computing the actual magnitude of each term, it is difficult if not impossible to assess which one is larger (and thus exerts a primary control upon the synoptic-scale vertical motion).

- The quasi-geostrophic omega equation is sensitive to the reference frame – stationary or moving with the flow – in which it is computed. In other words, different results are obtained if the reference frame is changed, even if the meteorological features are the same! This is not a primary concern for us in this class, however, as we are primarily considering stationary reference frames (e.g., as depicted on standard weather charts).

These shortcomings motivate a desire to obtain a new equation for synoptic-scale vertical motions that does not have these problems. This equation, known as the *Q-vector* form of the quasi-geostrophic omega equation, is derived below.

Obtaining The Q-Vector Form of the Quasi-Geostrophic Omega Equation

To obtain the Q-vector form of the quasi-geostrophic omega equation, rather than start with the quasi-geostrophic vorticity and thermodynamic equations, we will start with the quasi-geostrophic horizontal momentum and thermodynamic equations. For simplicity, we will assume that the Coriolis parameter f is constant, i.e., $f = f_0$, such that all terms involving β, the meridional variability in f, are zero. We will also neglect diabatic heating.

Thus, our basic equation set is given by:

$$\frac{\partial u_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla u_g - f_0 v_{ag} = 0 \quad (1)$$

$$\frac{\partial v_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla v_g + f_0 u_{ag} = 0 \quad (2)$$

$$\frac{\partial T}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla T - S_p \omega = 0 \tag{3}$$

First, find $p*\partial/\partial p$ of (1):

$$p\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t}\right) + p\frac{\partial}{\partial p}\left(u_g \frac{\partial u_g}{\partial x}\right) + p\frac{\partial}{\partial p}\left(v_g \frac{\partial u_g}{\partial y}\right) = f_0 p \frac{\partial v_{ag}}{\partial p} \tag{4}$$

Next, find $R/f_0*\partial/\partial y$ of (3):

$$\frac{R}{f_0}\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial t}\right) + \frac{R}{f_0}\frac{\partial}{\partial y}\left(u_g \frac{\partial T}{\partial x}\right) + \frac{R}{f_0}\frac{\partial}{\partial y}\left(v_g \frac{\partial T}{\partial y}\right) = \frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} \tag{5}$$

We need to utilize the chain rule on the second and third partial derivatives on the left-hand sides of both (4) and (5). Where applicable, we also wish to commute (or change the order of) the partial derivatives; note that this applies all terms on the left-hand sides of (4) and (5). This enables us to write as many terms as possible in terms of $\partial u_g/\partial p$ and $\partial T/\partial y$. The reasons for doing so will become clear shortly.

Apply the chain rule, commute the partial derivatives as applicable, and compute (4) – (5), grouping like terms where possible to obtain:

$$\begin{aligned} \frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} - f_0 p \frac{\partial v_{ag}}{\partial p} &= -\left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}\right)\left(p\frac{\partial u_g}{\partial p} - \frac{R}{f_0}\frac{\partial T}{\partial y}\right) \\ &- p\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right] + \frac{R}{f_0}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right] \end{aligned} \tag{6}$$

For further simplification, recall the thermal wind relationship, introduced in an earlier lecture:

$$p\frac{\partial v_g}{\partial p} = -\frac{R}{f_0}\frac{\partial T}{\partial x} \tag{7a}$$

$$p\frac{\partial u_g}{\partial p} = \frac{R}{f_0}\frac{\partial T}{\partial y} \tag{7b}$$

Note that in (7), we have substituted f_0 for f given that we are currently assuming that f = constant. By application of (7b) into (6), the first set of terms on the right-hand side of (6) goes away. This is why we rewrote the terms of (4) and (5) as described above – to make use of the thermal wind relationship to simplify the result! Likewise, we can apply both (7a) and (7b) to rewrite the second set of terms on the right-hand side of (6) in terms of R/f_0. Doing so, we obtain:

$$\frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} - f_0 p \frac{\partial v_{ag}}{\partial p} = -\frac{R}{f_0}\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial y} + \frac{R}{f_0}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y} + \frac{R}{f_0}\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{R}{f_0}\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} \tag{8}$$

Recall that, by definition, for f = constant, the divergence of the geostrophic wind is zero. This is equivalent to stating that:

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y} \tag{9}$$

If we substitute (9) into the first term on the right-hand side of (8), we find that it is equal to the second term on the right-hand side of (8). Likewise, it is apparent that the last two terms on the right-hand side of (8) are equivalent. Thus,

$$\frac{RS_p}{f_0}\frac{\partial \omega}{\partial y} - f_0 p \frac{\partial v_{ag}}{\partial p} = 2\frac{R}{f_0}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) \tag{10}$$

Finally, noting that $S_p = \sigma p/R$, if we substitute for S_p, divide (10) by p, and then multiply by f_0, we obtain:

$$\sigma\frac{\partial \omega}{\partial y} - f_0^2\frac{\partial v_{ag}}{\partial p} = 2\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) = 2\frac{R}{p}\left(\frac{\partial \vec{\mathbf{v}}_g}{\partial y}\cdot\nabla T\right) \tag{11}$$

In (11), note that:

$$Q_2 = -\frac{R}{p}\left(\frac{\partial \vec{\mathbf{v}}_g}{\partial y}\cdot\nabla T\right) \tag{12}$$

Such that:

$$\sigma\frac{\partial \omega}{\partial y} - f_0^2\frac{\partial v_{ag}}{\partial p} = -2Q_2 \tag{13}$$

Now, we wish to find $p*\partial/\partial p$ of (2):

$$p\frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial t}\right) + p\frac{\partial}{\partial p}\left(u_g\frac{\partial v_g}{\partial x}\right) + p\frac{\partial}{\partial p}\left(v_g\frac{\partial v_g}{\partial y}\right) = -f_0 p\frac{\partial u_{ag}}{\partial p} \tag{14}$$

Next, find $R/f_0*\partial/\partial x$ of (3):

$$\frac{R}{f_0}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right)+\frac{R}{f_0}\frac{\partial}{\partial x}\left(u_g\frac{\partial T}{\partial x}\right)+\frac{R}{f_0}\frac{\partial}{\partial x}\left(v_g\frac{\partial T}{\partial y}\right)=\frac{RS_p}{f_0}\frac{\partial \omega}{\partial x} \tag{15}$$

As before, apply the chain rule, commute the partial derivatives as applicable, and compute (14) + (15), grouping like terms where possible to obtain:

$$\begin{aligned}\frac{RS_p}{f_0}\frac{\partial \omega}{\partial x}-f_0p\frac{\partial u_{ag}}{\partial p}&=\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)\left(p\frac{\partial v_g}{\partial p}+\frac{R}{f_0}\frac{\partial T}{\partial x}\right)\\&+p\left[\frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x}+\frac{\partial v_g}{\partial p}\frac{v_g}{\partial y}\right]+\frac{R}{f_0}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right]\end{aligned} \tag{16}$$

By application of (7a), the first set of terms on the right-hand side of (16) goes away. Likewise, application of (7a) and (7b) allows us to rewrite the second set of terms on the right-hand side of (16) in terms of R/f_0. Doing so, we obtain:

$$\frac{RS_p}{f_0}\frac{\partial \omega}{\partial x}-f_0p\frac{\partial u_{ag}}{\partial p}=-\frac{R}{f_0}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial x}+\frac{R}{f_0}\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{R}{f_0}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}+\frac{R}{f_0}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} \tag{17}$$

By application of (9), the first term on the right-hand side of (17) is equal to the second term on the right-hand side of (17). Note also that the last two terms on the right-hand side of (17) are also equal to one another. Simplifying (17) with this while also substituting for S_p, dividing by p, and multiplying by f_0, we obtain:

$$\sigma\frac{\partial \omega}{\partial x}-f_0^2\frac{\partial u_{ag}}{\partial p}=2\frac{R}{p}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)=2\frac{R}{p}\left(\frac{\partial \vec{\mathbf{v}}_g}{\partial x}\cdot\nabla T\right) \tag{18}$$

In (18), note that:

$$Q_1=-\frac{R}{p}\left(\frac{\partial \vec{\mathbf{v}}_g}{\partial x}\cdot\nabla T\right) \tag{19}$$

Such that:

$$\sigma\frac{\partial \omega}{\partial x}-f_0^2\frac{\partial u_{ag}}{\partial p}=-2Q_1 \tag{20}$$

Equations (13) and (20) contain forcing terms related to both the vertical motion ω and ageostrophic wind $\mathbf{v}_{ag}$. We wish to eliminate the latter. We do so by computing $\partial/\partial x$ of (20) +

$\partial/\partial y$ of (13). If we do so, commuting the derivatives on the ageostrophic wind terms in so doing, we obtain:

$$\sigma \frac{\partial^2 \omega}{\partial x^2} + \sigma \frac{\partial^2 \omega}{\partial y^2} - f_0^2 \frac{\partial}{\partial p}\left(\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} \right) = -2\left(\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} \right) \tag{21}$$

The last term on the left-hand side of (21), representing the divergence of the ageostrophic wind, can be substituted for by making use of the form of the continuity equation applicable in the quasi-geostrophic system,

$$\nabla \cdot \vec{\mathbf{v}}_{ag} + \frac{\partial \omega}{\partial p} = 0 \tag{22}$$

Substituting (22) into (21) and simplifying the equation, we obtain:

$$\sigma \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -2\nabla \cdot \mathbf{Q} \tag{23}$$

Where $\mathbf{Q} = (Q_1, Q_2)$, with Q_1 and Q_2 as defined in (19) and (12), respectively.

<u>Basic Interpretation and Application of the Q-Vector Equation</u>

Equation (23) gives the *Q-vector form of the quasi-geostrophic omega equation*. The quasi-geostrophic vertical motion is due entirely to <u>**Q-vector divergence**</u> (where $\nabla \cdot (\ )$ is the divergence operator), which we can readily compute and/or estimate. Unlike with the regular form of the quasi-geostrophic omega equation, there are not multiple forcing terms that may conflict with one another, a significant advantage! Likewise, there are no partial derivatives with respect to pressure, meaning that we only need data on one isobaric level to diagnose the vertical motion – another advantage!

As with the regular form of the quasi-geostrophic omega equation, we apply this equation for the *diagnosis* of middle tropospheric vertical motions. Note, however, that we still do not actually 'solve' for the vertical motion, given the second-order partial derivative operators on the left-hand side of (23) that require iterative methods to solve, as previously described.

The basic interpretation of (23) is straightforward. Recalling that $\nabla^2 \omega \propto -\omega$, $\omega \propto \nabla \cdot \mathbf{Q}$. Thus,

- Synoptic-scale ascent ($\omega < 0$) is found where there is Q-vector ***convergence*** ($\nabla \cdot \mathbf{Q} < 0$).
- Synoptic-scale descent ($\omega > 0$) is found where there is Q-vector ***divergence*** ($\nabla \cdot \mathbf{Q} > 0$).

Without actually *computing* **Q**, its components (namely, the horizontal partial derivatives of $\mathbf{v}_g$ and T), and its divergence, how can we best *estimate* **Q** and its divergence from a weather map? We utilize a minor coordinate transformation, into a coordinate system akin to a *natural coordinate system*, in order to aid in estimating **Q** and its divergence.

Let us define the x-axis to be along, or parallel to, an isotherm, **with warm air to the right of the positive x-axis**. The y-axis is defined perpendicular to the x-axis. An idealized schematic of this is provided in Figure 1 below.

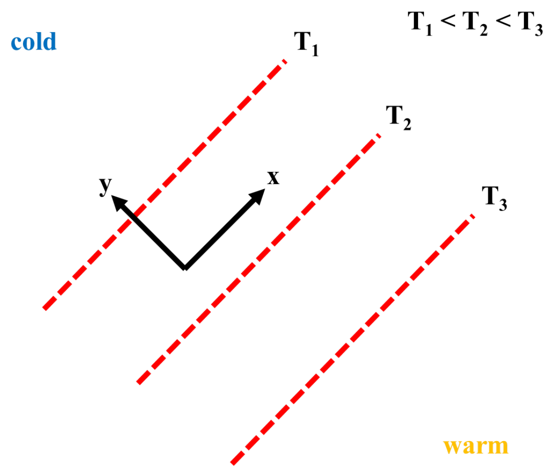

Figure 1. Idealized depiction of the coordinate transformation described in the text above.

This is akin to placing the x-axis along the direction in which the thermal wind blows, although it should be noted that we are only considering temperature on one isobaric level and not a layer-mean temperature.

In this coordinate system, $\partial T/\partial x$, or the change in temperature along the isotherm, is inherently zero. Thus, the terms in the definitions of Q_1 and Q_2 in (19) and (12) above, respectively, that involve $\partial T/\partial x$ are 0. As a result, **Q** becomes:

$$\mathbf{Q} = -\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x}\mathbf{i} + \frac{\partial v_g}{\partial y}\mathbf{j}\right) \tag{24}$$

If we make use of (9), we can rewrite the second term of (24), such that:

$$\mathbf{Q} = -\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x}\mathbf{i} - \frac{\partial u_g}{\partial x}\mathbf{j}\right) \tag{25}$$

And, by vector identity, the term in the parentheses of (25) can be rewritten, such that:

$$\mathbf{Q} = -\frac{R}{p}\frac{\partial T}{\partial y}\left(\mathbf{k} \times \frac{\partial \vec{\mathbf{v}}_g}{\partial x}\right) \tag{26}$$

Thus, to evaluate **Q**, we first want to find the *vector change* in $\mathbf{v}_g$ *along the isotherm*. The **k** x () operator, geometrically, is manifest like the *"right hand rule"* and signifies a 90° clockwise rotation of the just-determined change vector. Next, we multiply this vector by the *magnitude* of $\partial T/\partial y$, or the magnitude of the temperature gradient. It is not explicitly necessary to multiply by R/p since, on an isobaric surface, both R and p are constant in space. If we follow this procedure at several locations on a weather map, we can estimate the divergence of **Q** and, thus, estimate at least the sign of the vertical motion.

Let us now consider three examples of such application. As we do so, we will demonstrate how the same answer is given by the Q-vector analysis as would be obtained from the regular form of the quasi-geostrophic omega equation.

Example 1: Idealized Trough/Ridge Pattern

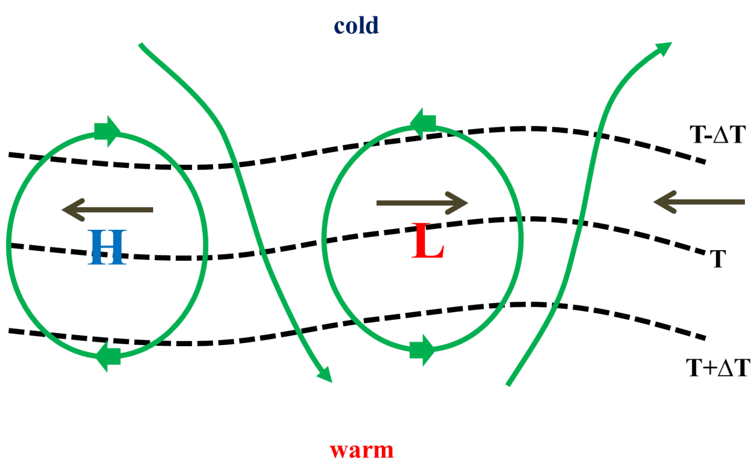

Figure 2. **Q** vectors (solid grey arrows) for an idealized trough/ridge pattern. Isotherms are depicted in black dashed lines, with cold air to the north, and streamlines depicting the geostrophic flow are depicted in solid green lines.

In the vicinity of the areas of high pressure, the geostrophic wind is northerly along the positive x-axis (to the east) and southerly along the negative x-axis (to the west). In the vicinity of the area of low pressure, the geostrophic wind is southerly along the positive x-axis (to the east) and northerly along the negative x-axis (to the west).

In each case, we want to subtract the geostrophic wind vector along the negative x-axis from the geostrophic wind vector along the positive x-axis. To do so, it is helpful to recall principles of vector subtraction. To subtract two vectors, take the vector being subtracted, flip it 180°, and add it to the first vector. Vectors are added by placing the origin of the second vector at the tip/end of the first vector, then by drawing a new vector from the origin of the first vector to the tip/end of the second vector. This is depicted in Figure 3 below for both situations described above.

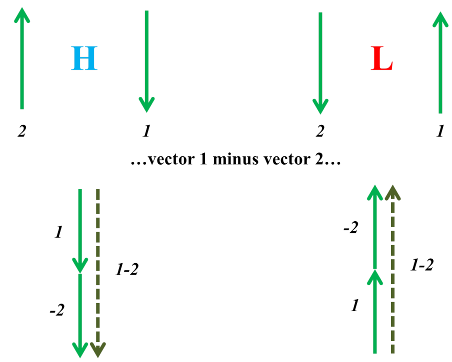

Figure 3. Illustration of the vector subtraction operations described in the text above.

After subtracting the vectors, application of the **k** x () operator necessitates rotating the new vector 90° to the right. This results in a vector pointing from east to west with the areas of high pressure and a vector pointing from west to east with the area of low pressure. The precise length of each vector can then be determined by multiplying the vector by the magnitude of the temperature gradient.

The pattern of vertical motion can thus be determined by evaluating the divergence of the Q vectors. To the west of the area of low pressure, there is divergence of the Q vector. This implies descent. To the east of the area of low pressure, there is convergence of the Q vector. This implies ascent.

For a sanity check, let us compare this evaluation with that which can be obtained from the quasi-geostrophic omega equation. To the west of the area of low pressure, there is cold geostrophic temperature advection, typically associated with descent. Conversely, to the east of the area of low pressure, there is warm geostrophic temperature advection, typically associated with ascent. Both of these findings are consistent with our Q vector-based interpretation.

Since the magnitude of the geostrophic relative vorticity ζ_g is maximized at the base of troughs and apex of ridges, we can infer cyclonic geostrophic relative vorticity advection to the east of the area of low pressure and anticyclonic geostrophic relative vorticity advection to the west the area of low pressure. If we presume that the geostrophic relative vorticity advection is relatively small in the lower troposphere, this implies cyclonic geostrophic relative vorticity advection increasing with height to the east of the area of low pressure and anticyclonic geostrophic relative vorticity advection increasing with height to the west the area of low pressure. This implies ascent east and descent west of the area of low pressure, again consistent with our Q vector-based interpretation.

Example 2: Idealized Trough/Ridge Pattern With No Temperature Advection

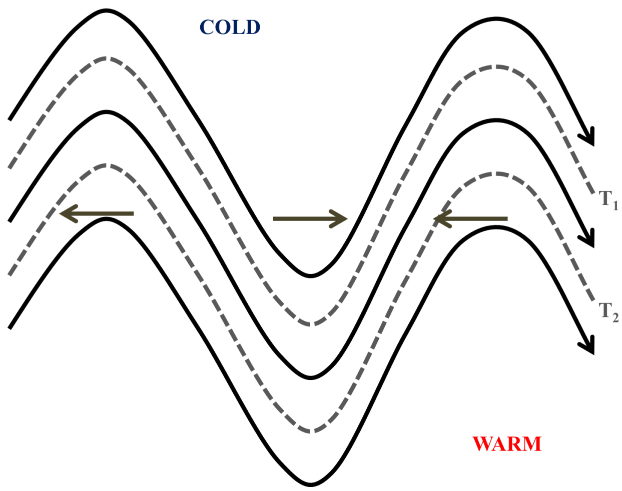

Figure 4. **Q** vectors (solid grey arrows) for an idealized trough/ridge pattern. Isotherms are depicted in grey dashed lines, with cold air to the north, and streamlines depicting the geostrophic flow are depicted in solid black lines.

In many ways, this example is similar to the one presented above. However, in this case, the isotherms are parallel to the geostrophic wind which, by the definition of the geostrophic wind, means that the isotherms are parallel to the contours of constant geopotential height. In the following, as before, "east" refers to the positive x-axis along an isotherm while "west" refers to the negative x-axis along an isotherm.

In the base of the trough, the geostrophic wind is from the southwest to the east and from the northwest to the west. Subtracting the latter vector from the former results in a vector pointing from south to north. Applying the **k** x () operator rotates this vector 90° to the right, such that the Q vector points from west to east. In the apex of the ridges, the geostrophic wind is from the northwest to the east and from the southwest to the west. Subtracting the latter vector from the former results in a vector pointing from north to south. Applying the **k** x () operator rotates this vector 90° to the right, such that the Q vector points from east to west. The precise magnitude of each Q vector can be obtained by multiplying each by the magnitude of the temperature gradient.

In Figure 4, it is clear that Q vectors converge to the east of the trough and diverge to the west. As before, this signifies ascent and descent, respectively.

Let us again interpret the scenario depicted in Figure 4 in terms of the quasi-geostrophic omega equation. With no geostrophic temperature advection, forcing is exclusively due to differential geostrophic vorticity advection. The pattern of differential geostrophic vorticity advection is identical to that described in our first example for the same physical reasons: cyclonic geostrophic relative vorticity advection increasing with height to the east of the trough and anticyclonic geostrophic relative vorticity advection increasing with height to the west. As before, this implies ascent and descent, respectively, consistent with the Q vector interpretation.

Example 3: Confluent Flow in the Entrance Region of a Jet Streak

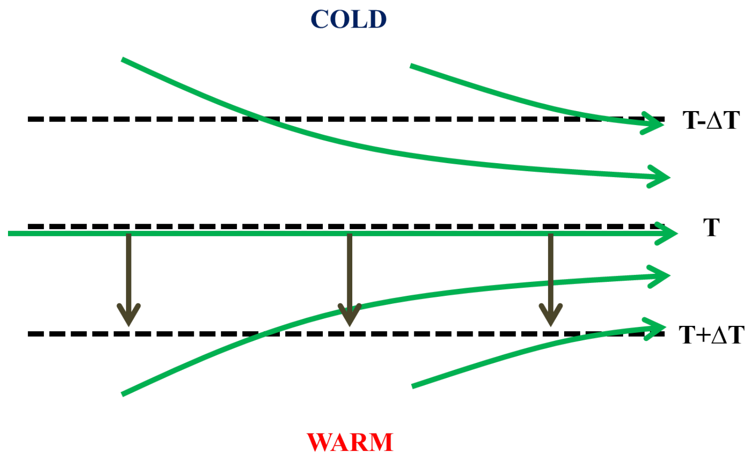

Figure 5. Orientation of Q vectors (solid grey arrows) for confluent flow (inferred from the green streamlines obtained from the geostrophic flow) associated with a westerly jet streak. Isotherms are depicted by the dashed black lines with cold air to the north.

In the confluent flow scenario depicted above, the geostrophic wind accelerates (becomes larger) to the east. Thus, along an isotherm, the magnitude of the geostrophic wind – primarily westerly – is always larger to the east. Vector subtraction results in a relatively short vector pointing from west to east. Applying the **k** x () operator to this vector rotates it to the right by 90°, such that it points from north to south, as depicted in Figure 5 above.

As depicted in Figure 5, there is no explicit Q vector convergence or divergence. Thus, what do the Q vectors look like further to the north and south? Visually, we can see that along the isotherms, the streamlines are not much different in direction or in packing/tightness. This implies no meaningful change in the direction or magnitude of the geostrophic wind along these isotherms, further implying that the magnitude of the Q vectors is relatively small.

Thus, Q vectors are divergent to the north, in the colder air, and convergent to the south, in the warmer air. Thus, we see ascent to the south and descent to the north. We will revisit this and related concepts when we study jet streaks later in this course.

Again, we wish to confirm our evaluation by utilizing arguments associated with the quasi-geostrophic omega equation. There is implied warm geostrophic temperature advection (and thus implied ascent) to the south and cold geostrophic temperature advection (and thus implied descent) to the north. The interpretation in terms of differential geostrophic vorticity advection is slightly more nuanced. The streamlines imply the presence of a trough to the southwest and a ridge to the northwest. This implies cyclonic geostrophic relative vorticity advection on the southern side of the jet streak and anticyclonic geostrophic relative vorticity advection on the northern side of the jet streak. Again presuming that geostrophic relative vorticity advection is weak near the surface, this pattern implies ascent to the south and descent to the north. Once again, this is consistent with our interpretation from the Q vector analysis.

More Examples

We will work through more examples of how the Q-vector form of the quasi-geostrophic omega equation may be applied to diagnose synoptic-scale vertical motion both in class, utilizing the NCAR/MMM Real-Time Diagnostics page linked from the course website, and in the second laboratory assignment of the semester. However, please do make use of the NCAR/MMM website outside of class to further aid your own study and interpretation of these concepts!

Advanced Interpretation, Geostrophic Balance, and Application to Frontogenesis

Advanced Interpretation: Relation to Horizontal Temperature Gradient

To this point, we have discussed how the **Q** vector may be computed and/or estimated. Likewise, we have shown how its divergence can be used to infer the direction and magnitude of the synoptic-scale vertical motion. However, we have yet to describe the physical meaning of the components of the **Q** vector. That is the focus of this section.

Equations (19) and (12) give our expressions for Q_1 and Q_2, respectively. Examining these expressions, we see that both look like temperature advection by the horizontal shear of the geostrophic wind. Alternatively, Q_1 and Q_2 could be interpreted as being related to the evolution of the horizontal temperature gradient. However, it is fair to ask whether these interpretations are the best possible interpretations for the **Q** vector.

To do so, let us return to the simplified form of the quasi-geostrophic thermodynamic equation posed in (3). Let's make this equation even simpler: let us state that the flow is purely geostrophic such that the vertical velocity term $S_p\omega$ vanishes. This allows us to write:

$$\frac{\partial T}{\partial t} + u_g \frac{\partial T}{\partial x} + v_g \frac{\partial T}{\partial y} = 0 \tag{27}$$

Note that we have expanded the advection term in (3) into its components in writing (27). We first wish to find $\partial/\partial x$ of (27). In so doing, we must make use of the product rule when taking the partial derivatives of the second and third terms on the left-hand side of (27). Doing so, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right)+\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+u_g\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}+v_g\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right)=0 \tag{28}$$

Next, we wish to commute the order of the partial derivatives in the first and last terms on the left-hand side of (28). Doing so and grouping like terms, we obtain:

$$\left(\frac{\partial}{\partial t}+\vec{\mathbf{v}}_g\cdot\nabla\right)\frac{\partial T}{\partial x}+\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}=0 \tag{29}$$

The last two terms on the right-hand side of (29) can be written in vector form, resulting in:

$$\left(\frac{\partial}{\partial t}+\vec{\mathbf{v}}_g\cdot\nabla\right)\frac{\partial T}{\partial x}+\frac{\partial\vec{\mathbf{v}}_g}{\partial x}\cdot\nabla T=0 \tag{30}$$

However, we know that this term is related to Q_1 – in fact, from (19), we know that it is equivalent to $-Q_1p/R$. Moving this term to the right-hand side of (30), we obtain:

$$\left(\frac{\partial}{\partial t}+\vec{\mathbf{v}}_g\cdot\nabla\right)\frac{\partial T}{\partial x}=\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right)=\frac{Q_1p}{R} \tag{31}$$

We now wish to find $\partial/\partial y$ of (27). In so doing, we must again make use of the product rule when taking the partial derivatives of the second and third terms on the left-hand side of (27). Doing so, we obtain:

$$\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial t}\right)+\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+u_g\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)+\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}+v_g\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)=0 \tag{32}$$

Next, we wish to commute the order of the partial derivatives in the first and third terms on the left-hand side of (32). Doing so and grouping like terms, we obtain:

$$\left(\frac{\partial}{\partial t}+\vec{\mathbf{v}}_g\cdot\nabla\right)\frac{\partial T}{\partial y}+\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}=0 \tag{33}$$

The last two terms on the right-hand side of (33) can be written in vector form, resulting in:

$$\left(\frac{\partial}{\partial t}+\vec{\mathbf{v}}_g\cdot\nabla\right)\frac{\partial T}{\partial y}+\frac{\partial\vec{\mathbf{v}}_g}{\partial y}\cdot\nabla T=0 \tag{34}$$

However, we know that this term is related to Q_2 – in fact, from (12), we know that it is equivalent to $-Q_2p/R$. Moving this term to the right-hand side of (34), we obtain:

$$\left(\frac{\partial}{\partial t}+\vec{\mathbf{v}}_g\cdot\nabla\right)\frac{\partial T}{\partial y}=\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right)=\frac{Q_2 p}{R} \tag{35}$$

Combining (31) and (35), we obtain:

$$\frac{D_g}{Dt}\left(\frac{R}{p}\nabla T\right)=Q_1\hat{\mathbf{i}}+Q_2\hat{\mathbf{j}} \tag{36}$$

Equation (36) demonstrates that the **Q** vector can be interpreted as being proportional to the *rate of change of the horizontal temperature gradient* as forced exclusively by the geostrophic motion. We will return to this interpretation momentarily.

Application to Geostrophic Balance

Equations (12) and (19) state the definitions of Q_2 and Q_1, respectively. The forcings upon Q_1 and Q_2 are entirely geostrophic in nature, whether directly ($\mathbf{v}_g$) or indirectly (T, related to geopotential height via the hydrostatic equation). Thus, as we discussed in the context of the quasi-geostrophic omega equation, *purely geostrophic flow is responsible for departures from geostrophy* (i.e., for ageostrophic flow). The resultant ageostrophic circulation, which is related to Q_1 and Q_2 by (11) and (18), works to restore geostrophic and thermal wind balance.

We can further demonstrate the concept of geostrophic flow being responsible for departures from geostrophy in the context of the thermal wind relationship. It should be noted, however, that we will likely not cover this portion of the material in-depth in class, nor will you be responsible for it on quizzes or exams.

First, we wish to use the thermal wind relationship posed in (7) to re-write (31) and (35) in terms of the vertical shear of the geostrophic wind. Substituting (7b) into (35) and (7a) into (31), we obtain:

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right)=Q_2 \tag{37a}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right)=-Q_1 \tag{37b}$$

For comparison, we now return to equations (1) and (2). As we did with the simplified form of the quasi-geostrophic thermodynamic equation, we wish to consider the special case where the flow is purely geostrophic. This allows us to state:

$$\frac{\partial u_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla u_g = 0 \tag{38a}$$

$$\frac{\partial v_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla v_g = 0 \tag{38b}$$

By finding ∂/∂p of (38a) and (38b), we can develop alternate expressions for the left-hand sides of (37a) and (37b). We make use of the product rule and commute the order of particular partial derivatives in obtaining these expressions, as we did in equations (28) through (30) when operating on the quasi-geostrophic thermodynamic equation. In mathematical form,

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla u_g\right) &= \frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t}\right) + \frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + u_g \frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial x}\right) + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y} + v_g \frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial y}\right) \\ &= \frac{D_g}{Dt}\left(\frac{\partial u_g}{\partial p}\right) + \frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\end{aligned} \tag{39a}$$

$$\begin{aligned}\frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial t} + \vec{\mathbf{v}}_g \cdot \nabla v_g\right) &= \frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial t}\right) + \frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + u_g \frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial x}\right) + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y} + v_g \frac{\partial}{\partial p}\left(\frac{\partial v_g}{\partial y}\right) \\ &= \frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right) + \frac{\partial u_g}{\partial p}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial v_g}{\partial y}\end{aligned} \tag{39b}$$

If we apply the thermal wind relationship given by (7) to re-write the second and third terms of (39a) and (39b), we obtain:

$$\frac{D_g}{Dt}\left(\frac{\partial u_g}{\partial p}\right) = \frac{R}{f_0 p}\left(\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x}\right) \tag{40a}$$

$$\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right) = \frac{R}{f_0 p}\left(-\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial v_g}{\partial y}\right) \tag{40b}$$

Because the divergence of the geostrophic wind for f = constant is zero, the following statement is true:

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y} \tag{41}$$

We can apply (41) to the second term on the right-hand sides of (40a) and (40b) to obtain:

$$\frac{D_g}{Dt}\left(\frac{\partial u_g}{\partial p}\right) = \frac{R}{f_0 p}\left(\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y} + \frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y}\right) \tag{42a}$$

$$\frac{D_g}{Dt}\left(\frac{\partial v_g}{\partial p}\right) = -\frac{R}{f_0 p}\left(\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial u_g}{\partial x}\right) \tag{42b}$$

Substituting from (12) and (19), the definitions of Q1 and Q2, we obtain:

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -Q_2 \tag{43a}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial p}\right) = Q_1 \tag{43b}$$

Compare (43) to (37). The left-hand sides of both equations are equivalent. However, while the right-hand sides of both equations are of equivalent magnitude, they are of opposite sign! This means that the forcing from the temperature gradient and vertical wind shear are not in balance.

If thermal wind balance were maintained, (37) and (43) would not be opposite in sign. Instead, because this sign discrepancy exists, geostrophic forcing manifest through the **Q** vector destroys thermal wind balance. The ageostrophic circulation and accompanying vertical motion also manifest through the **Q** vector, as described above, works to offset this sign discrepancy and thus attempt to restore thermal wind balance.

Application to Frontogenesis

We close by returning to (36), the relationship between the **Q** vector and the rate of change of the horizontal temperature gradient as forced by the geostrophic motion. Recall that we can analyze fronts in terms of horizontal temperature gradients. Cold fronts are typically found on the leading edge of cold air, while warm fronts are typically found on the leading edge of warm air. Across the front, there is a large horizontal gradient of temperature. How this gradient – or, more specifically, its magnitude – changes with time gives a measure of how the *intensity of the front* changes with time.

To explore this idea further, we want to develop a relationship for the rate of change of the magnitude of the horizontal temperature gradient:

$$\frac{D_g}{Dt}\left(\|\nabla T\|\right) = \frac{D_g}{Dt}\left(\sqrt{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2}\right) \tag{44}$$

We desire to expand the right-hand side of (44). In doing so, we make use of the following general relationship:

$$\frac{D_g}{Dt}\left(f^n\right)=nf^{n-1}\frac{D_g}{Dt}\left(f\right) \tag{45}$$

In (45), f is some arbitrary function and n is some arbitrary exponent or power. Making use of (45) as we expand the right-hand side of (44), we obtain:

$$\begin{aligned}
\frac{D_g}{Dt}\left(\left\|\nabla T\right\|\right) &= \frac{1}{2}\left(\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial x}\right)^2\right)^{-1/2}\frac{D_g}{Dt}\left(\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial x}\right)^2\right) \\
&= \frac{1}{2\left\|\nabla T\right\|}\left[2\frac{\partial T}{\partial x}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right)+2\frac{\partial T}{\partial y}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right)\right] \\
&= \frac{1}{\left\|\nabla T\right\|}\left[\frac{\partial T}{\partial x}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial x}\right)+\frac{\partial T}{\partial y}\frac{D_g}{Dt}\left(\frac{\partial T}{\partial y}\right)\right]
\end{aligned} \tag{46}$$

If we substitute the definition of **Q**, as given by (31) and (35), (46) becomes:

$$\frac{D_g}{Dt}\left(\left\|\nabla T\right\|\right)=\frac{1}{\left\|\nabla T\right\|}\left[\frac{\partial T}{\partial x}\frac{p}{R}Q_1+\frac{\partial T}{\partial y}\frac{p}{R}Q_2\right] \tag{47}$$

Or, in vector notation,

$$\frac{D_g}{Dt}\left(\left\|\nabla T\right\|\right)=\frac{1}{\left\|\nabla T\right\|}\frac{p}{R}\left(\nabla T\cdot\vec{\mathbf{Q}}\right) \tag{48}$$

What does (48) signify? This equation signifies that the rate of change of the *magnitude* of the horizontal temperature gradient by purely geostrophic processes, the left-hand side of (48), is proportional to how the horizontal temperature gradient (∇T) and **Q** vectors (**Q**) are oriented with respect to one another.

This latter remark about the orientation of the two vectors arises from the definition of the dot product contained within (48). To facilitate interpretation of (48), it is helpful to recall the properties of the dot product:

- For any two vectors **A** and **B**, if **A** is *perpendicular* to **B**, their dot product is zero.
- If **A** points in *the same direction* as **B**, their dot product is positive.
- If **A** points in *the opposite direction* as **B**, their dot product is negative.

As applied to (48), the rate of change of the magnitude of the horizontal temperature gradient by purely geostrophic processes is zero if the horizontal temperature gradient (always directed from cold toward warm air) and **Q** vectors are perpendicular to one another. If the horizontal temperature gradient and **Q** vectors point in the same direction, the rate of change of the magnitude of the horizontal temperature gradient is positive. Conversely, if the horizontal temperature gradient and **Q** vectors point in the opposite direction, the rate of change of the magnitude of the horizontal temperature gradient is negative.

Let us apply this concept to our "Example 3," that of confluent flow into a jet streak, from earlier in this lecture. The horizontal temperature gradient in that example points from north to south, from cold toward warm air. Likewise, in that example, we demonstrated that the **Q** vectors point from north to south. Thus, in this case, the horizontal temperature gradient and **Q** vectors point in the same direction. This means that the magnitude of the horizontal temperature gradient will grow larger with time, a frontogenetic situation.

We can confirm this by consider how the geostrophic wind, given by the streamlines in Figure 5, blows across the isotherms. To the northwest, the geostrophic wind blows from cold toward warm air. Conversely, to the southwest, the geostrophic wind blows from warm toward cold air. This pattern of geostrophic temperature advection acts to increase the magnitude of the horizontal temperature gradient to the west, as we deduced above.

This exercise can be repeated for diffluent flow in the exit region coming out of a jet streak. In that case, the **Q** vectors and horizontal temperature gradient point in opposite directions, causing the magnitude of the horizontal temperature gradient to become smaller with time, a frontolytic situation. To gain experience with these concepts, I encourage you to work through this exercise on your own and to ask me if you run into any trouble in so doing.

Thus, in the quasi-geostrophic system, the development and decay of fronts can be evaluated by considering how the **Q** vectors are oriented with respect to the isotherms! The dual utility of the **Q** vector – to evaluate both synoptic-scale vertical motion (and its associated ageostrophic flow) as well as the development and decay of fronts – illustrates yet another powerful advantage of the **Q** vector formulation over the normal form of the quasi-geostrophic omega equation! It also illustrates how ascent, clouds, and precipitation are often found in regions of frontogenesis, given the strong relationship between vertical motion and frontogenesis manifest by the **Q** vector.

It should be noted, however, that the above framework only considers how *geostrophic processes* act to change the magnitude of the horizontal temperature gradient. Ageostrophic processes may be – and often are – important contributors to this as well.

Advanced Application: The Four Quadrant Jet Model

Earlier, we evaluated the Q-vectors (and, thus, vertical motion) for the case of confluent flow in the entrance region of a jet streak with isotherms oriented parallel to the jet. There is Q-vector convergence in the right entrance region of the jet streak, indicating middle tropospheric ascent, and Q-vector divergence in the left entrance region of the jet streak, indicating middle tropospheric descent.

Similar arguments can be posed for the exit region of the jet streak. In that case, the Q-vectors are oriented from south to north, indicative of Q-vector convergence in the left exit region and Q-vector divergence in the right exit region of the jet. This indicates middle tropospheric ascent and descent, respectively.

One advantage of the Q-vector-based interpretation of the four quadrant jet model is its explicit consideration of the thermal gradient – or, in other words, the horizontal distribution of temperature. If the isotherms are not oriented parallel to the jet streak, the distribution of ascent and descent will be changed somewhat from that seen in the idealized example presented herein. The Q-vector-based interpretation accounts for this directly; the Q-G omega equation-based interpretation can do so as well if its thermal forcing term is evaluated. The parcel acceleration-based interpretation, however, cannot do so directly.