Synoptic Meteorology II: Quasi-Geostrophic Frontal Dynamics

24-26 March 2015

Readings: Section 6.3 of *Midlatitude Synoptic Meteorology*.

Introduction

When we examined frontogenesis and frontolysis, we examined frontal kinematics; e.g., the influence of the wind field upon the temperature field. To large extent, this explains much (~75%) of the frontal development process. However, in the real atmosphere, the wind field and temperature fields are linked; each evolves in concert with the other, particularly through ageostrophic (or secondary) circulations. As a result, we cannot describe frontal development, evolution, and dynamics using the frontogenetical function alone; something more comprehensive is necessary in order to do so.

To begin to address this, we first reconsider the basic structure of a front. In "polar front" theory, a front can be described as a boundary between two different air masses. If an air mass is characterized by its density, this implies that density is discontinuous across the front; in other words, there is a "jump" in density along the front. From the ideal gas law, since pressure must be continuous across the front in order for pressure gradient force-induced parcel accelerations to be finite (i.e., reasonable), there must be a "jump" in temperature along the front.

In reality, pressure is continuous across a front; the isobars "kink" with cyclonic curvature along and across the front, with the front itself being located along a minimum in pressure. However, density and temperature are both also continuous across a front. A front is not a sharp boundary between air masses but rather a zone of finite width (from ~1 km to ~100 km or more), such that it is more accurately characterized by a discontinuous temperature *gradient*. Consequently, the two-dimensional structure of a front is not accurately depicted in "polar front" theory. There are, of course, other shortcomings.

That said, "polar front" theory does a reasonable job of specifying the vertical slope of a front, as specified through the Margules' frontal slope equation. Briefly, Margules' frontal slope equation states that the slope of a frontal boundary is primarily related to the ratio between the across-front vorticity and across-front difference in temperature; there is also a latitudinal-dependence that arises due to the Coriolis parameter f.

The above-stated insight into "polar front" theory arises from a basic consideration of frontal dynamics using, nominally, quasi-geostrophic theory. As stated above, there are some positive and some negative aspects to its depiction of frontal dynamics. This suggests that quasi-geostrophic theory, though not entirely correct, may be at least partially correct in how it depicts frontal dynamics. In this lecture, we strive to more comprehensively describe frontal dynamics in the quasi-geostrophic system in order to better highlight its strengths and limitations. Many of its limitations are addressed by what is known as the *semigeostrophic approximation*; however, a treatment of the semigeostrophic approximation is beyond the scope of this class.

Preliminary Remarks

Figure 1 below depicts an idealized two-dimensional view of isentropes through a frontal zone. We scale $x \sim l$, where l is the cross-front scale, and $y \sim L$, where L is the along-front scale. This represents what is known as anisotropic scaling; in other words, the two horizontal length scales are not the same. We state that $l << L$, such that cross-front distance is much less than the along-front distance, as we discussed when we described the observational characteristics of fronts in a previous lecture. Properties vary rapidly along l whereas they do not vary so rapidly along L.

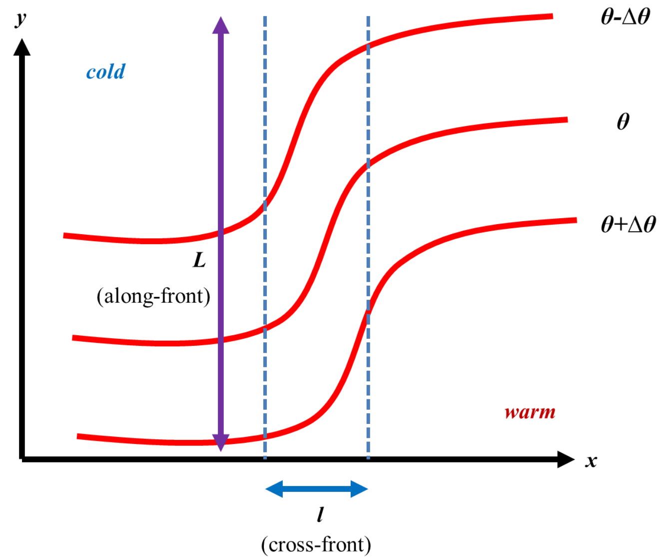

Figure 1. Idealized schematic of isentropes (red lines) through a cold frontal zone. Note that the along-front and cross-front directions and scales are defined specifically with respect to the northeast-southwest sloping cold frontal zone itself.

The wind in the x-direction scales as $u \sim U$, whereas that in the y-direction scales as $v \sim V$. To examine the relative values of U and V, we invoke continuity on a constant height surface, i.e.,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

While we do not explicitly know the scale of $\partial w/\partial z$, we do know that we want the scales of the other two terms to be approximately equal to one another. Therefore, we require that $U/l \sim V/L$, or $U/V \sim l/L$. If $l << L$, then $U << V$ for this relationship to be satisfied. Time scales as $t \sim U/l$ in the x-direction and as $t \sim V/L$ in the y-direction.

The horizontal momentum equations on a constant height surface, incorporating geostrophic balance plus the acceleration terms, can be expressed as:

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi}{\partial x} \tag{2a}$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi}{\partial y} \tag{2b}$$

Recall from that same lecture that a general Rossby number can be defined as:

$$Ro = \frac{U}{f_0 L} \tag{3}$$

More generally, the Rossby number reflects the ratio of the acceleration term to that of the Coriolis term. With this in mind, the Rossby numbers in the x- and y-directions are given by:

$$Ro_x = \left| \frac{\frac{Du}{Dt}}{fv} \right| \tag{4a}$$

$$Ro_y = \left| \frac{\frac{Dv}{Dt}}{fu} \right| \tag{4b}$$

Let us first examine the scale of the total derivatives in (4), making use of the relationship between *U/V* and *l/L* expressed above:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{U^2}{l} + U\frac{U}{l} + V\frac{U}{L} \rightarrow \frac{UV}{L} \tag{5a}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{V^2}{L} + U\frac{V}{l} + V\frac{V}{L} \rightarrow \frac{UV}{l} \tag{5b}$$

The scale of fv is simply fV, whereas that of fu is simply fU. Thus, plugging this and (5) into (4), we obtain:

$$Ro_x = \left| \frac{\frac{UV}{L}}{fV} \right| = \frac{U}{fL} \tag{6a}$$

$$Ro_y = \left| \frac{\frac{UV}{l}}{fU} \right| = \frac{V}{fl} \tag{6b}$$

Above, we noted that $U << V$ and $l << L$. Thus, for Ro_x, the numerator is relatively small while the denominator is relatively large. Conversely, for Ro_y, the numerator is relatively large while the denominator is relatively small. Thus, Ro_x is relatively small while Ro_y is relatively large. Numerically, if we let $U \sim 5$ m s^{-1}, $V \sim 25$ m s^{-1}, $l \sim 200$ km, and $L \sim 1000$ km, then $Ro_x \sim 0.05$ and $Ro_y \sim 1.25$.

As a result, in the x-direction (i.e., the cross-front direction), the flow is approximately geostrophic given the very small value of the Rossby number. This is known as *cross-front geostrophy*, where the *along-front wind* is nearly geostrophic (i.e., $v \sim v_g$). Note that it is the along-front wind that is defined as geostrophic here because that is the wind which appears on the right-hand side of the x-momentum equation.

In the y-direction (i.e., the along-front direction), the flow is not at all geostrophic given the very large value of the Rossby number. Thus, we cannot make the geostrophic approximation for the *cross-front wind* and must assume that the cross-front geostrophic and cross-front ageostrophic wind are of similar magnitude (e.g., $u_g \sim u_{ag}$).

This consideration in and of itself begins to illustrate why quasi-geostrophic theory is in sufficient to describe frontal dynamics: the cross-front wind is not quasi-geostrophic, and consequently we should thus have no impression that insight drawn from quasi-geostrophic theory will be qualitatively or quantitatively accurate. Thus, having illustrated (in part) ***why*** quasi-geostrophic theory fails, we want to illustrate ***how*** quasi-geostrophic theory fails to describe frontal dynamics.

The Quasi-Geostrophic System of Equations for Frontal Dynamics

We now wish to state the primitive equations applicable to the quasi-geostrophic system. In an earlier lecture, we stated the quasi-geostrophic form of the primitive equations on isobaric surfaces; however, we now wish to state them as applicable on constant height surfaces for simplicity. In addition, we will write them in slightly different terms than we did before, albeit without derivation as we do so.

We start with the x-momentum equation. Because we stated that the x-direction, or the along-front direction, is in approximate geostrophic balance, the x-momentum equation can be reduced to its geostrophic form, i.e.,

$$f v_g = \frac{\partial \Phi}{\partial x} \tag{7a}$$

Next, we consider the y-momentum equation. Because the y-direction, or the cross-front direction, is not in approximate geostrophic balance, we cannot reduce the y-momentum equation to its geostrophic form. (We could, if we desired a fully geostrophic depiction of frontal dynamics. Instead, however, we do not so as to obtain a quasi-geostrophic depiction of frontal dynamics.) Consequently, the y-momentum equation is given by:

$$\frac{D_g v_g}{Dt} = -fu - \frac{\partial \Phi}{\partial y} \tag{7b}$$

But, from geostrophic balance, we know that:

$$f u_g = -\frac{\partial \Phi}{\partial y}$$

Thus, substituting into (7b), where $u = u_g + u_{ag}$, then (7b) becomes:

$$\frac{D_g v_g}{Dt} = -f u_{ag} \tag{7c}$$

The hydrostatic equation on a constant height surface can be expressed as:

$$\frac{\partial \Phi}{\partial z} = b \tag{8}$$

Where b in (8) is given by:

$$b = \frac{g}{\theta_{00}}(\theta_0 - \theta)$$

where θ_{00} is a reference-state potential temperature and θ_0 is the surface potential temperature.

The thermodynamic equation on a constant height surface can be expressed as:

$$\frac{D_g b}{Dt} = -N^2 w \tag{9}$$

where N^2, a measure of the static stability, is given by:

$$N^2 = \frac{g}{\theta_{00}}\frac{d\theta_0}{dz}$$

Note that the total derivative in (7c) and (9) is of the form:

$$\frac{D_g(\;)}{Dt} = \frac{\partial(\;)}{\partial t} + u_g\frac{\partial(\;)}{\partial x} + v_g\frac{\partial(\;)}{\partial y} \tag{10}$$

In defining the total derivative, we only include the geostrophic component of the along-front wind *u*. However, earlier in this lecture, we stated that the along-front wind was not geostrophic. Why, then, do we consider it to be geostrophic here, in contrast to the above? This is done to remain faithful to the idea that this is a *quasi-geostrophic* (and not geostrophic or semigeostrophic) system while still keeping in mind the basic distinctions between the cross-front and along-front directions and scales. There are important physical considerations here as well: expressing (10) in terms of purely geostrophic quantities does not allow for the possibility of advection by vertical motion w or the ageostrophic flow u_{ag}. These quantities define the secondary (ageostrophic, vertical, transverse, etc.) circulation of a front.

Thus, while we allow for there to be a secondary circulation to the front (e.g., via continuity, as expressed by (11) below), we do not allow for secondary circulation-related processes to impact frontal dynamics in this quasi-geostrophic framework. Doing so would take the cross-front and along-front distinctions noted above into full account, which is known as the *semigeostrophic approximation*. As stated above, however, examination of the semigeostrophic approximation and how its depiction of frontal dynamics differs from that obtained from the quasi-geostrophic system is beyond the scope of this class.

The continuity equation valid for this set of equations is given by:

$$\frac{\partial u_{ag}}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{11}$$

The system of equations given by (7), (8), (9), and (11), along with the ideal gas law, form the quasi-geostrophic primitive equation set that we will use to study frontal dynamics.

Further Examination: Toward the Sawyer-Eliassen Equation

When we developed the quasi-geostrophic omega equation, we stated that the role of the geostrophic wind was to destroy geostrophic balance. In other words, all of the forcing terms for omega, an inherently ageostrophic quantity, are a function of purely geostrophic forcings. The role of the ageostrophic wind (and, consequently, omega) is to restore geostrophic balance. We find that this is true in the quasi-geostrophic depiction of frontal dynamics as well, though it takes a fair bit of mathematical manipulation to be able to make this clear.

If we take ∂/∂z of (7a) so as to form a thermal wind equation for this system, we obtain:

$$f\frac{\partial v_g}{\partial z} = \frac{\partial}{\partial z}\frac{\partial \Phi}{\partial x} \tag{12a}$$

If we commute the derivatives on the right-hand side of (12a) and substitute from the hydrostatic equation, we obtain:

$$f\frac{\partial v_g}{\partial z} = \frac{\partial b}{\partial x} \tag{12b}$$

In so much as (12b) holds, then we can also state that the following must be true:

$$\frac{D_g}{Dt}\left(f\frac{\partial v_g}{\partial z}\right) = \frac{D_g}{Dt}\left(\frac{\partial b}{\partial x}\right) \tag{13}$$

We wish to be able to re-write the expression given by (13). To do so, we need to obtain expressions for the two partial derivatives that it contains. These can be obtained by expanding and operating upon expanded forms of (7c) and (9).

Taking *f*∂/∂z* of (7c), we obtain:

$$f\frac{\partial}{\partial z}\left(\frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y}\right) = -f^2\frac{\partial u_{ag}}{\partial z} \tag{14}$$

If we expand (14) by use of the chain rule, commuting derivatives as necessary, and group like terms, we obtain:

$$\frac{D_g}{Dt}\left(f\frac{\partial v_g}{\partial z}\right) = -f^2\frac{\partial u_{ag}}{\partial z} - f\frac{\partial u_g}{\partial z}\frac{\partial v_g}{\partial x} - f\frac{\partial v_g}{\partial z}\frac{\partial v_g}{\partial y} \tag{15}$$

Likewise, if we take *∂/∂x* of (9), we obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial b}{\partial t}+u_g\frac{\partial b}{\partial x}+v_g\frac{\partial b}{\partial y}\right)=-N^2\frac{\partial w}{\partial x} \tag{16}$$

If we expand (16) by use of the chain rule, commuting derivatives as necessary, and group like terms, we obtain:

$$\frac{D_g}{Dt}\left(\frac{\partial b}{\partial x}\right)=-N^2\frac{\partial w}{\partial x}-\frac{\partial u_g}{\partial x}\frac{\partial b}{\partial x}-\frac{\partial v_g}{\partial x}\frac{\partial b}{\partial y} \tag{17}$$

We wish to re-write select terms in (15) in terms of *b* or, more specifically, partial derivatives of *b*. To do so, we can make use of the hydrostatic equation. If we take $\partial/\partial y$ of the hydrostatic equation (8), we obtain:

$$\frac{\partial b}{\partial y}=\frac{\partial}{\partial y}\frac{\partial \Phi}{\partial z}=\frac{\partial}{\partial z}\frac{\partial \Phi}{\partial y}=-f\frac{\partial u_g}{\partial z} \tag{18a}$$

Likewise, if we take $\partial/\partial y$ of the hydrostatic equation, we obtain:

$$\frac{\partial b}{\partial x}=\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial z}=\frac{\partial}{\partial z}\frac{\partial \Phi}{\partial x}=f\frac{\partial v_g}{\partial z} \tag{18b}$$

Furthermore, recall that the geostrophic wind is, by definition, non-divergent. This allows us to state that:

$$\frac{\partial u_g}{\partial x}=-\frac{\partial v_g}{\partial y} \tag{19}$$

If we substitute (18) and (19) into (15), we obtain:

$$\frac{D_g}{Dt}\left(f\frac{\partial v_g}{\partial z}\right)=-f^2\frac{\partial u_{ag}}{\partial z}+\frac{\partial b}{\partial y}\frac{\partial v_g}{\partial x}+\frac{\partial b}{\partial x}\frac{\partial u_g}{\partial x} \tag{20}$$

We define a quantity Q_1 as:

$$Q_1=-\frac{\partial b}{\partial y}\frac{\partial v_g}{\partial x}-\frac{\partial b}{\partial x}\frac{\partial u_g}{\partial x} \tag{21}$$

Note that Q_1 is a purely geostrophic quantity.

Substituting into (20) and (17) with (21), we obtain:

$$\frac{D_g}{Dt}\left(f\frac{\partial v_g}{\partial z}\right)=-f^2\frac{\partial u_{ag}}{\partial z}-Q_1 \quad (22a)$$

$$\frac{D_g}{Dt}\left(\frac{\partial b}{\partial x}\right)=-N^2\frac{\partial w}{\partial x}+Q_1 \quad (22b)$$

If we substitute (22) into the relationship given by (13) and solve for Q_1, we obtain:

$$N^2\frac{\partial w}{\partial x}-f^2\frac{\partial u_{ag}}{\partial z}=2Q_1 \quad (23)$$

Equation (23) can be viewed as akin to a quasi-geostrophic omega equation. It illustrates that purely geostrophic forcing, as manifest through Q_1, is responsible for the destruction of geostrophic balance, as manifest by the presence of non-zero w and u_{ag} on the left-hand side of (23). Stated differently, geostrophic forcing results in departures from thermal wind balance (e.g., the balance between the vertical wind shear and the horizontal temperature gradient that defines a frontal zone) that, ultimately, the ageostrophic circulation attempts to restore.

We can manipulate the continuity equation, given by (11), to show that:

$$\frac{\partial u_{ag}}{\partial x}=-\frac{\partial w}{\partial z} \quad (24)$$

From (24), we can define a streamfunction ψ that satisfies the equality in (24). This streamfunction is defined in terms of u_{ag} and w by the following:

$$u_{ag}=\frac{\partial \psi}{\partial z},\ w=-\frac{\partial \psi}{\partial x} \quad (25)$$

If you plug (25) into (24), you see that the equality holds. The streamfunction is simply a representation of streamlines. In this case, it represents the streamlines of the secondary circulation given by the cross-front ageostrophic wind u_{ag} and the vertical motion w. The streamfunction is defined in the northern hemisphere as positive for counterclockwise turning and as negative for clockwise turning.

In the *x-z* plane, for the isentrope configuration depicted in Figure 1, a positive streamfunction ($\psi > 0$) is depicted in Figure 2. Conversely, a negative streamfunction ($\psi < 0$) in the *x-z* plane for this same isentrope configuration is depicted in Figure 3. A positive streamfunction is characterized by ascent where it is warm and descent where it is cold and is thus a direct circulation. A negative streamfunction is characterized by ascent where it is cold and descent where it is warm and is thus an indirect circulation.

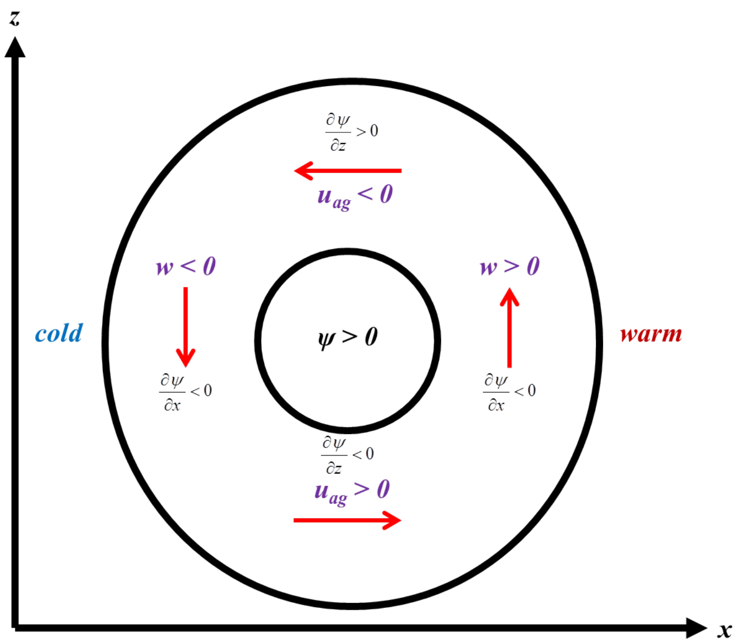

Figure 2. A positive streamfunction, maximized at the center of the two circles, in the *x*-*z* plane. The signs on w and u_{ag} are determined by the horizontal (x) and vertical (z) derivatives of the streamfunction ψ respectively.

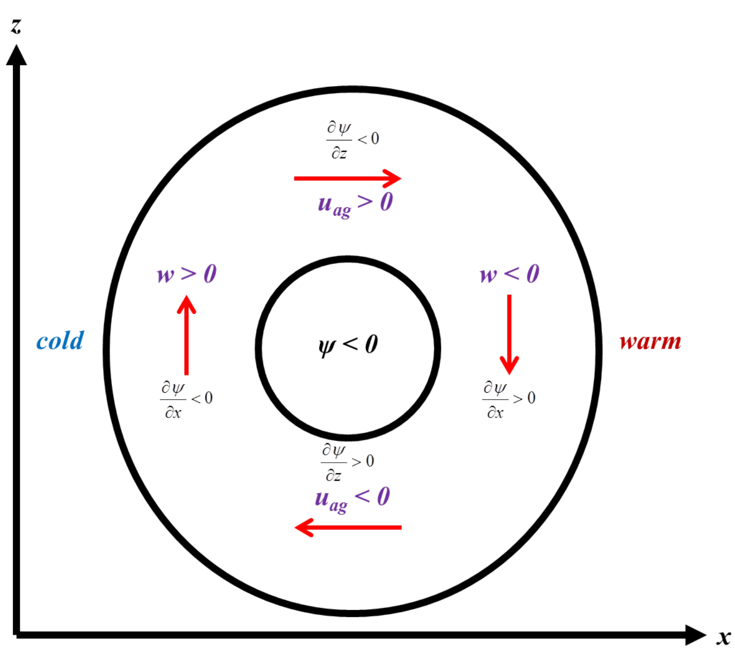

Figure 3. As in Figure 2, except for a negative streamfunction maximized at the center of the two circles.

The Sawyer-Eliassen Equation

Geostrophic and Ageostrophic Frontogenesis and Frontolysis

If we plug (25) into (23), we obtain:

$$N^2 \frac{\partial^2 \psi}{\partial x^2} + f^2 \frac{\partial^2 \psi}{\partial z^2} = -2Q_1 \quad (26)$$

Equation (26) is what is known as the quasi-geostrophic form of the *Sawyer-Eliassen equation*.

Earlier in the semester, we stated that the second derivative of a quantity is proportional to the negative of that quantity. Therefore, for N^2 and f^2 both positive-definite, the left-hand side of (26) is proportional to $-\psi$. Consequently, given the leading negative on both sides, ψ is proportional to Q_1; i.e., where Q_1 is positive, so too is the streamfunction. Naturally, this invites a further examination of the forcing expressed by Q_1.

First, let us recall (22). Q_1 appears as a negative value on the right-hand side of (22a) whereas it appears as a positive value on the right-hand side of (22b). Equation (22a) describes the evolution of the along-front vertical geostrophic wind shear following the motion, whereas equation (22b) describes the evolution of the cross-front buoyancy gradient following the motion. It can be shown that the latter is proportional to the evolution of the cross-front potential temperature gradient following the motion.

Consequently, positive Q_1 increases the cross-front potential temperature gradient (i.e., ***is frontogenetic***) while decreasing the along-front vertical geostrophic wind shear. Inherently, this counteracts thermal wind balance, which states that the cross-front potential temperature gradient is in balance with (i.e., is directly proportional to) the along-front vertical geostrophic wind shear. Thus, this provides another illustrative example of how purely geostrophic forcing acts to destroy geostrophic and thermal wind balance. For positive Q_1, the resultant ageostrophic circulation attempts to counteract this forcing, reducing the cross-front potential temperature gradient and increasing the along-front vertical geostrophic wind shear.

Above, we stated that a positive streamfunction is characterized by a thermally direct circulation, with ascent in the warm air and descent in the cold air. We also stated that a positive streamfunction is associated with a positive value of Q_1. The ***geostrophic*** forcing provided by $Q_1 > 0$ acts to strengthen the cross-front potential temperature gradient and weaken the along-front vertical geostrophic wind shear. The ***ageostrophic*** forcing that it drives, thus, acts to weaken the cross-front potential temperature gradient and increase the along-front vertical geostrophic wind shear in order to restore thermal wind balance. Thus, the thermally direct ageostrophic circulation is said to be frontolytic.

Conversely, a negative streamfunction is characterized by a thermally indirect circulation, with descent in the warm air and ascent in the cold air. We also stated that a negative streamfunction is associated with a negative value of Q_1. The ***geostrophic*** forcing provided by $Q_1 < 0$ acts to weaken the cross-front potential temperature gradient and strengthen the along-front vertical geostrophic wind shear. The ***ageostrophic*** forcing that it drives, thus, acts to strengthen the cross-front potential temperature gradient and decrease the along-front vertical geostrophic wind shear in order to restore thermal wind balance. Thus, the thermally indirect ageostrophic circulation is said to be frontogenetic.

The above-stated insight is in agreement with the analysis of the tilting term to the frontogenetical function examined in the last lecture. Ascent brings about adiabatic cooling via expansion whereas descent brings about adiabatic warming via compression. Ascent in the cold air and descent in the warm air thus cools where it is cold and warms where it is warm, strengthening the horizontal potential temperature gradient.

Care must be taken, however, to keep the ageostrophic and geostrophic forcing separate in one's mind when assessing frontogenetic and frontolytic situations. The entrance and exit regions of jet streaks highlight this concern nicely. The entrance region of a jet, as we demonstrated in our lecture on Q-vectors, is generally frontogenetic whereas the exit region of a jet is generally frontolytic. However, this is a *geostrophic* forcing; the horizontal geostrophic flow is confluent in the jet entrance region and diffluent in the jet exit region. Meanwhile, the entrance region of a jet is characterized by a thermally direct ageostrophic circulation, which is frontolytic, whereas the exit region of a jet is characterized by a thermally indirect ageostrophic circulation, which is frontogenetic. This, inherently, is an *ageostrophic* forcing.

Energetics

Above, we stated that the role of the ageostrophic circulation is to restore thermal wind balance. For a thermally direct ageostrophic circulation, where $Q_1 > 0$ and $\psi > 0$, this is accomplished by a reduction in the cross-front potential temperature gradient and an increase in the along-front vertical geostrophic wind shear. For a thermally indirect ageostrophic circulation, where $Q_1 < 0$ and $\psi < 0$, this is accomplished by an increase in the cross-front potential temperature gradient and a reduction in the along-front vertical geostrophic wind shear.

It is thus natural to ask, how is this accomplished in terms of atmospheric energetics, namely available potential energy and kinetic energy? In the case of the thermally direct ageostrophic circulation, available potential energy is extracted into kinetic energy, as reflected by the increased along-front vertical geostrophic wind shear. In the case of the thermally indirect ageostrophic circulation, kinetic energy is lost as it is converted back to available potential energy, as reflected by the decreased along-front vertical geostrophic wind shear.

Further Interrogation of the Sawyer-Eliassen Equation

For convenience, we restate Q_1:

$$Q_1 = -\frac{\partial b}{\partial y}\frac{\partial v_g}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial u_g}{\partial x} \tag{27}$$

There are two forcing terms to Q_1: one involving the along-front geostrophic wind v_g and one involving the cross-front geostrophic wind u_g. The term involving the along-front geostrophic wind is known as the *horizontal shear* forcing term, as it reflects the horizontal shear of the along-front geostrophic wind, nominally across the frontal zone. The term involving the cross-front geostrophic wind is known as the *confluence* forcing term, as it reflects the combined effects of convergence and deformation along the frontal zone.

To examine the effects of confluence and horizontal shear upon Q_1, let us consider an illustrative example of a northwest-to-southeast oriented horizontal potential temperature gradient. Cold air is found to the northwest while warm air is found to the southeast. Consequently, for this example, $\partial b/\partial x > 0$ (warmer/more buoyant along the positive x-axis, colder/less buoyant along the negative x-axis) and $\partial b/\partial y < 0$ (colder/less buoyant along the positive y-axis, warmer/more buoyant along the negative y-axis).

Let us impose a confluent flow upon this horizontal potential temperature gradient, as illustrated in Figure 4. In this example, the axis of contraction is aligned with the x-axis whereas the axis of dilatation is aligned with the y-axis. From visual inspection, we presume that this flow should be frontogenetic: the streamlines act to bring the isentropes closer together, particularly toward the center of the diagram. We can confirm this mathematically by evaluating $\partial u_g/\partial x$ across the frontal zone. To the east, along the positive x-axis, $u_g < 0$ (i.e., directed from east to west); conversely, to the west, along the negative x-axis, $u_g > 0$ (i.e., directed from west to east). Thus, $\partial u_g/\partial x < 0$. Since $\partial b/\partial x > 0$, and given the leading negative on the confluence term, $Q_1 > 0$. From (22b), $Q_1 > 0$ is frontogenetic, confirming our visual inspection.

Now, let us impose a cyclonic horizontal shear flow upon this horizontal potential temperature gradient, as illustrated in Figure 5. The horizontal shear is entirely in the meridional direction given the convention on the horizontal shear term. From visual inspection, we presume that this flow will at least result in the isentropes being rotated cyclonically. Mathematically, we find that this flow is also frontogenetic. To the east, along the positive x-axis, $v_g > 0$ (i.e., directed from south to north); conversely, to the west, along the negative x-axis, $v_g < 0$ (i.e., directed from north to south). Thus, $\partial v_g/\partial x > 0$. Since $\partial b/\partial y < 0$, and given the leading negative on the horizontal shear term, $Q_1 > 0$, again a frontogenetic situation.

Of course, not all flows will be as easy to visually inspect as these two flows. Indeed, most atmospheric flows contain rotational, divergent, and deformation components, and it is often difficult to visually separate them from each other. Likewise, it goes without saying that not all flows across horizontal potential temperature gradients will be frontogenetic. Thus, it is extremely helpful to know the mathematical framework described above!

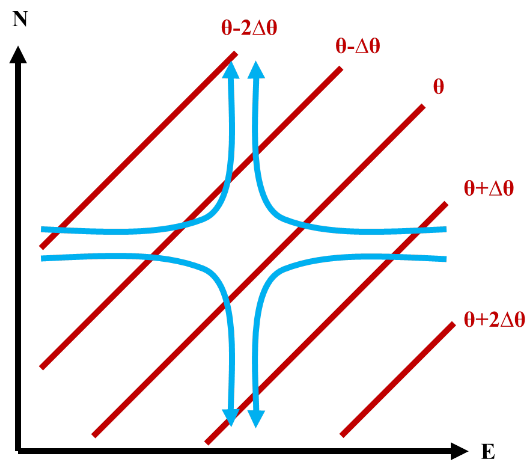

Figure 4. Idealized schematic of a northwest-to-southeast oriented horizontal potential temperature gradient with cold air to the northwest and warm air to the southeast. Isentropes are depicted by the labeled maroon lines while the confluent flow is depicted by the light blue streamlines.

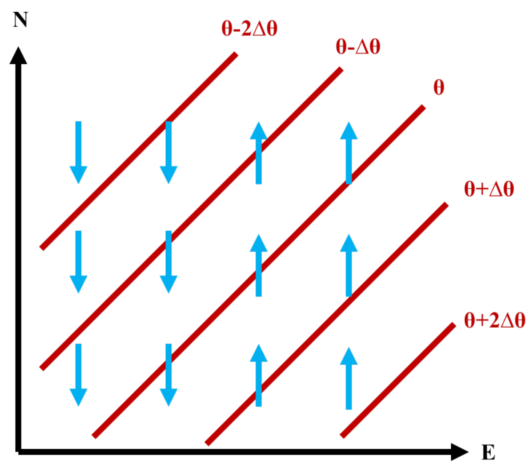

Figure 5. As in Figure 4, except for a cyclonic horizontal shear flow, as depicted by the light blue vectors.

We now wish to examine the vertical structure of quasi-geostrophic frontal dynamics in some detail. We noted above that the thermally direct circulation is frontolytic, though the geostrophic flow that forces it is frontogenetic. At the surface, where $w = 0$ by definition, the frontolytic effect of the thermally direct circulation is negligible. As you ascend into the middle troposphere, w typically increases in magnitude, thereby increasing the frontolytic effect of the thermally direct circulation. (This can alternatively but equivalently be viewed in the context of the tilting term to the frontogenetical function.)

Thus, geostrophic frontogenesis occurs largely unabated at the surface but is mitigated to some extent by the ageostrophic circulation aloft. Isentropes at the surface become tightly packed whereas they become less tightly packed aloft. The inverse is true for a frontogenetic thermally indirect circulation. The geostrophic flow that forces the indirect circulation is frontolytic. Such frontolysis is strongest at the surface and decays upward as the influence of the ageostrophic circulation increases with height. Isentropes at the surface become less tightly packed whereas they do so to a lesser extent aloft.

Consequently, a front in the quasi-geostrophic system looks like that depicted in Figure 6 below: strongest near the ground and less intense with height. This in and of itself is not necessarily a problem, but there are two other interesting elements highlighted by Figure 6 that are problems. There is no tilt of the front with height, whereas we know that fronts in the real world tilt with height. Furthermore, on the warm side of the front, lower tropospheric static instability (i.e., potential temperature decreasing with height) of a physically-unrealistic sense may be found. These unrealistic elements to quasi-geostrophic frontal structure arise because the definition of the total derivative given by (10) neglects advection by the ageostrophic component of the cross-front wind u_{ag}. Including this in the definition of the total derivative addresses these issues to large extent.

This shortcoming also impacts the rate at which a front develops within the quasi-geostrophic system. To illustrate this, let us express (17) in terms of the confluence term alone:

$$\frac{D_g}{Dt}\left(\frac{\partial b}{\partial x}\right) = -\frac{\partial u_g}{\partial x}\frac{\partial b}{\partial x} \tag{28}$$

Let $\alpha = -\partial u_g/\partial x$, such that:

$$\frac{D_g}{Dt}\left(\frac{\partial b}{\partial x}\right) = \alpha\frac{\partial b}{\partial x} \tag{29}$$

Equation (29) is an ordinary differential equation with a basic solution of the form:

$$\frac{\partial b}{\partial x} = \left.\frac{\partial b}{\partial x}\right|_{t=0} e^{\alpha t} \tag{30}$$

subject to the condition that the magnitude of the horizontal buoyancy (or potential temperature) gradient approaches infinity as time t approaches infinity. Equation (30) describes exponential growth of the cross-front horizontal potential temperature gradient as a function of time and the magnitude of the horizontal confluence. However, for realistic values of α, frontal development is too slow compared to observations. This illustrates yet another shortcoming of quasi-geostrophic frontal dynamics.

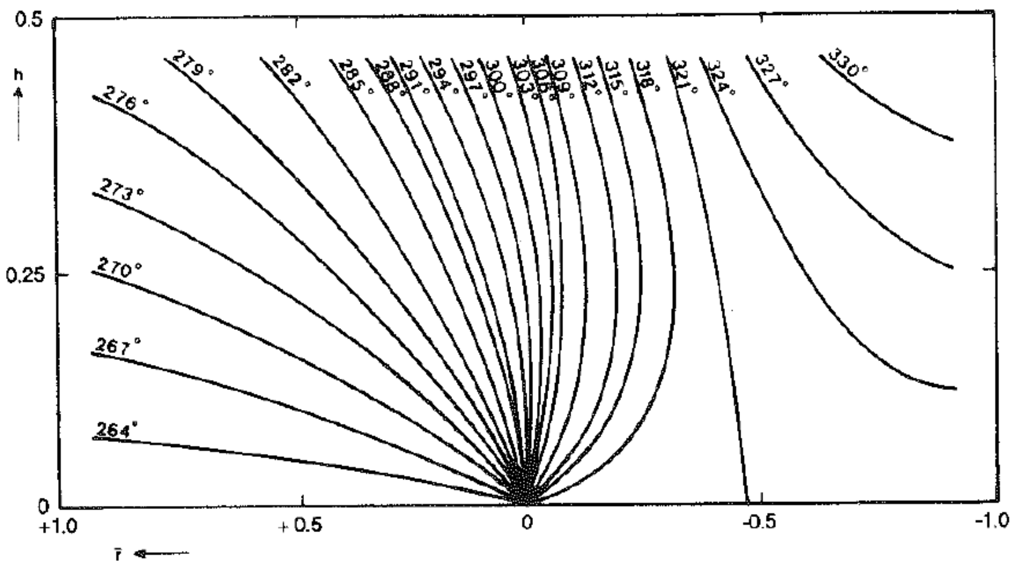

Figure 6. Vertical cross-section (x-z) depiction of isentropes (labeled every 3 K) associated with a frontal zone under the constraints of quasi-geostrophic dynamics. Reproduced from *Mid-Latitude Synoptic-Dynamic Meteorology (Vol. II)* by H. Bluestein, their Figure 2.52.

Finally, through the quasi-geostrophic vorticity equation, it can be shown that lower tropospheric convergence beneath the ascending branch of an ageostrophic circulation and lower tropospheric divergence beneath the descending branch of an ageostrophic circulation result in the spin-up of cyclonic and anticyclonic geostrophic relative vorticity, respectively, due to the stretching of planetary vorticity. In the quasi-geostrophic system, such convergence and divergence are of equal magnitude, such that regions of both large cyclonic and anticyclonic geostrophic relative vorticity form in the vicinity of the frontal zone. In reality, however, cyclonic geostrophic relative vorticity is dominant. This shortcoming arises because the stretching term acts only upon planetary vorticity when, instead, it should most accurately operate upon both planetary and geostrophic relative vorticity.