LECTURE NOTES 2

CONSERVATION LAWS (continued)

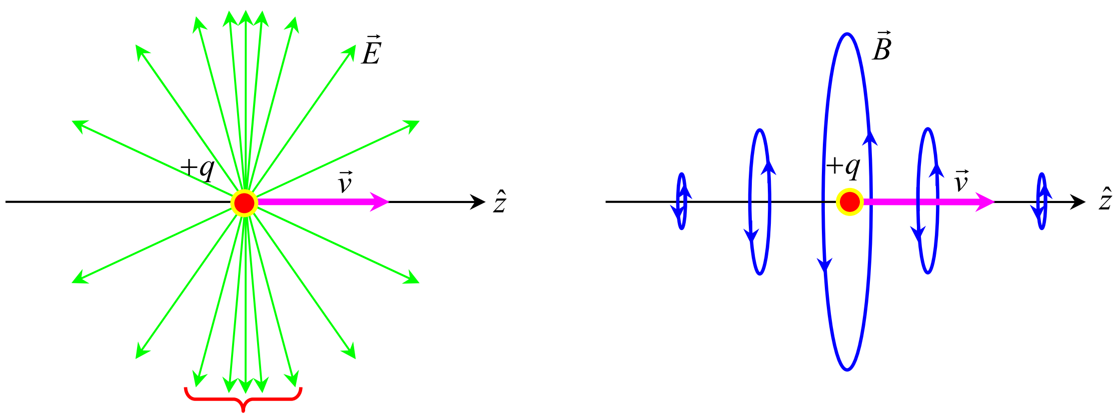
Conservation of Linear Momentum $\vec{p}$

Newton's 3rd Law in Electrodynamics – *"Every action has an equal and opposite reaction"*.

Consider a point charge $+q$ traveling along the $+\hat{z}$-axis with constant speed v $(\vec{v} = v\hat{z})$. Because the electric charge is moving (relative to the laboratory frame of reference), its electric field is not given perfectly/mathematically precisely as described by Coulomb's Law,

$$i.e.\ \vec{E}(\vec{r},t) \neq \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}.$$

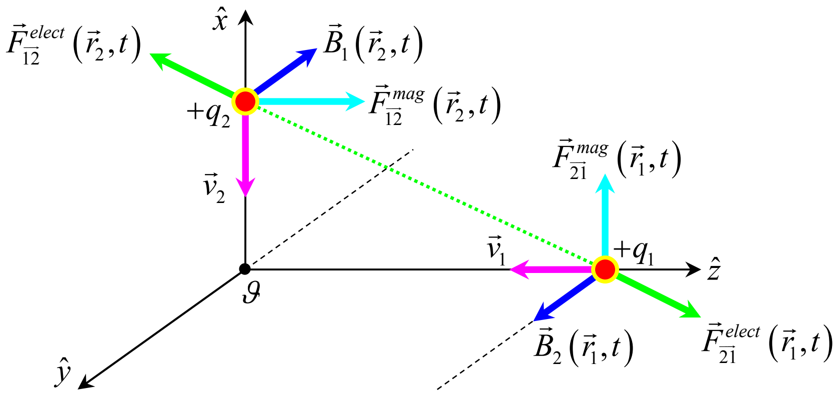
Nevertheless, $\vec{E}(\vec{r},t)$ does still point radially outward from its ***instantaneous*** position, – the location of the electric charge $q(\vec{r},t)$. {*n.b.* when we get to Griffiths Ch. 10 (relativistic electrodynamics, we will learn what the fully-correct form of $\vec{E}(\vec{r},t)$ is for a moving charge..}



n.b. The $\vec{E}$-field lines of a ***moving*** electric charge are ***compressed*** in the ***transverse*** direction!

Technically speaking, a moving single point electric charge does ***not*** constitute a steady/DC electrical current (as we have previously discussed in P435 Lecture Notes 13 & 14). Thus, the magnetic field associated with a moving point charge is ***not*** precisely mathematically correctly/properly given as described by the Biot-Savart Law. Nevertheless, $\vec{B}(\vec{r},t)$ still points in the azimuthal (*i.e.* $\hat{\varphi}$-) direction. (Again, we will discuss this further when we get to Griffiths Ch. 10 on relativistic electrodynamics…)

Let us now consider what happens when a point electric charge $+q_1$ traveling with constant velocity $\vec{v}_1 = -v_1\hat{z}$ encounters a second point electric charge $+q_2$, *e.g.* traveling with constant velocity $\vec{v}_2 = -v_2\hat{x}$ as shown in the figure below:



The two like electric charges obviously will ***repel*** each other (*i.e.* at the microscopic level, they will ***scatter*** off of each other – via exchange of one (or more) virtual photons).

As time progresses, the electromagnetic forces acting between them will drive them off of their initial axes as they repel / scatter off of each other. For simplicity's sake (here) let us assume that (by magic) the electric charges are mounted on straight tracks that prevent the electric charges from deviating from their initial directions.

Obviously, the ***electric*** force between the two electric charges (which acts on the line joining them together – see above figure) is repulsive and also manifestly obey's Newton's 3rd law:

$$\boxed{\vec{F}_{12}^{elect}(\vec{r}_2,t) = -\vec{F}_{21}^{elect}(\vec{r}_1,t)}$$

Is the ***magnetic*** force acting between the two charges also repulsive??

By the right-hand rule:

The magnetic field of q_1 at the position-location of q_2 points into the page: $\boxed{\vec{B}_1(\vec{r}_2,t) \parallel -\hat{y}}$

The magnetic field of q_2 at the position-location of q_1 points out of the page: $\boxed{\vec{B}_2(\vec{r}_1,t) \parallel +\hat{y}}$

Thus, the magnetic force $\boxed{\vec{F}_{12}^{mag}(\vec{r}_2,t) = q_2\vec{v}_2(\vec{r}_2,t)\times\vec{B}_1(\vec{r}_2,t) \;\; \parallel +\hat{z}}$ due to the effect of charge q_1's $\vec{B}$-field $\vec{B}_1(\vec{r}_2,t)$ ***at*** charge q_2's position $\vec{r}_2$ points in the $+\hat{z}$ direction (*i.e.* to the ***right*** in the above figure).

However the magnetic force $\boxed{\vec{F}_{21}^{mag}(\vec{r}_1,t) = q_1\vec{v}_1(\vec{r}_1,t)\times\vec{B}_2(\vec{r}_1,t) \;\; \parallel +\hat{x}}$ due to the effect of charge q_2's $\vec{B}$-field $\vec{B}_2(\vec{r}_1,t)$ at charge q_1's position $\vec{r}_1$ points in the $+\hat{x}$ direction (*i.e.* ***upward*** in the above figure). Thus we see that $\boxed{\vec{F}_{12}^{mag}(\vec{r}_2,t)\;\{\parallel +\hat{z}\} \neq -\vec{F}_{21}^{mag}(\vec{r}_1,t)\;\;\{\parallel +\hat{x}\}}$!!!

The electromagnetic force of q_1 acting on q_2 is equal in magnitude to, but is ***not*** opposite to the electromagnetic force of q_2 acting on q_1, in {apparent} violation of Newton's 3rd Law of Motion! Specifically, it is the magnetic interaction between two charges with relative motion between them that is responsible for this {apparent} violation of Newton's 3rd Law of Motion:

$$\boxed{\vec{F}_{12}^{elect}(\vec{r}_2,t) = -\vec{F}_{21}^{elect}(\vec{r}_1,t)} \quad \text{but:} \quad \boxed{\vec{F}_{12}^{mag}(\vec{r}_2,t) \neq -\vec{F}_{21}^{mag}(\vec{r}_1,t)}$$

But: $$\boxed{\left|\vec{F}_{12}^{elect}(\vec{r}_2,t)\right| = \left|\vec{F}_{21}^{elect}(\vec{r}_1,t)\right|} \quad \text{and:} \quad \boxed{\left|\vec{F}_{12}^{mag}(\vec{r}_2,t)\right| = \left|\vec{F}_{21}^{mag}(\vec{r}_1,t)\right|}$$

Then: $$\boxed{\vec{F}_{12}^{EMtot}(\vec{r}_2,t) = \vec{F}_{12}^{elect}(\vec{r}_2,t) + \vec{F}_{12}^{mag}(\vec{r}_2,t)} \quad \boxed{\vec{F}_{21}^{EMtot}(\vec{r}_1,t) = \vec{F}_{21}^{elect}(\vec{r}_1,t) + \vec{F}_{21}^{mag}(\vec{r}_1,t)}$$

$\therefore$ $$\boxed{\vec{F}_{12}^{EMtot}(\vec{r}_2,t) \neq \vec{F}_{21}^{EMtot}(\vec{r}_1,t)} \quad \text{but:} \quad \boxed{\left|\vec{F}_{12}^{EMtot}(\vec{r}_2,t)\right| = \left|\vec{F}_{21}^{EMtot}(\vec{r}_1,t)\right|}$$

n.b. In electro***statics*** and in magneto***statics***, Newton's 3rd Law of Motion ***always*** holds.

In electro***dynamics***, Newton's 3rd Law of Motion does ***not*** hold for the ***apparent*** ***relative*** motion of two electric charges! (*n.b.* Isaac Newton could not have forseen this {from an apple falling on his head} because gravito-magnetic forces associated with a falling apple are so vastly much feebler than the gravito-electric force (the "normal" gravity we know & love about in the "everyday" world).

Since Newton's 3rd Law is intimately connected/related to conservation of ***linear*** momentum, on the ***surface*** of this issue, it would seem that electrodynamical phenomena would then also seem to ***violate*** conservation of linear momentum – eeeeeEEEEKKKK!!!

If one considers only the ***relative*** motion of the (***visible***) electric charges (here q_1 and q_2) then, yes, it would ***indeed*** appear that ***linear*** momentum is ***not*** conserved!

However, the ***correct*** picture / ***correct*** reality is that the *EM* field(s) accompanying the (moving) charged particles ***also*** carry ***linear*** momentum $\vec{p}$ (as well as energy, E)!!!

Thus, in ***electrodynamics***, the electric charges and/or electric currents ***plus*** the electromagnetic fields accompanying the electric charges/currents ***together*** conserve total ***linear*** momentum $\vec{p}$. Thus, Newton's 3rd Law is ***not*** violated after all, when this broader / larger perspective on the nature of electrodynamics is properly/fully understood!!!

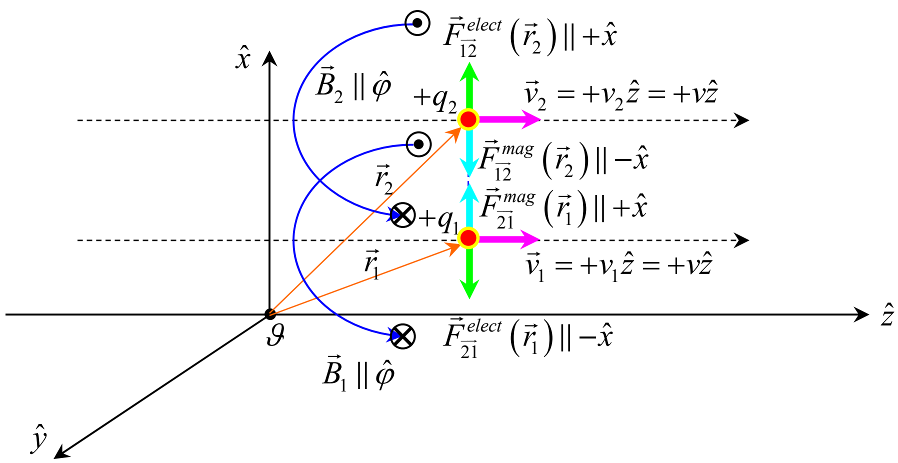
Microscopically, the virtual (and/or real) photons associated with the macroscopic / "mean field" electric and magnetic fields $\vec{E}_1(\vec{r}_2,t)$, $\vec{E}_2(\vec{r}_1,t)$ and $\vec{B}_1(\vec{r}_2,t)$, $\vec{B}_2(\vec{r}_1,t)$ do indeed carry / have associated with them linear momentum (and {kinetic} energy) {as well as ***angular*** momentum..}!!!

In the above example of two like-charged particles scattering off of each other, whatever momentum "lost" by the charged particles is gained by the *EM* field(s) associated with them!

Thus, Newton's 3rd Law ***is*** obeyed - total ***linear*** momentum ***is*** conserved when we consider the ***true*** ***total*** linear momentum of this system:

$$\boxed{\begin{aligned}\vec{p}_{tot} = \vec{p}_{mechanical} + \vec{p}_{EM} &= \vec{p}_1^{\,mech} + \vec{p}_2^{\,mech} + \vec{p}_1^{\,EM} + \vec{p}_2^{\,EM} \\ &= \underbrace{m_1\vec{v}_1 + m_2\vec{v}_2}_{\text{Non-relativistic case (here)!!!}} + \vec{p}_1^{\,EM} + \vec{p}_2^{\,EM}\end{aligned}}$$

Note that in the above example of two moving electrically-charged particles there is an interesting, special/limiting case when the two charged particles are moving ***parallel*** to each other, *e.g.* with equal constant velocities $\vec{v}_1 = \vec{v}_2 = +v\hat{z}$ relative to a fixed origin in the lab frame:



We can easily see here that the electric force on the two electrically-charged particles acts on the line joining the two charged particles is repulsive (due to the like-charges, here) and is such that $\vec{F}_{12}^{elect}(\vec{r}_2) = -\vec{F}_{21}^{elect}(\vec{r}_1) \parallel +\hat{x}$. Similarly, because the two like-charged particles are also traveling parallel to each other, the magnetic Lorentz force $\vec{F}^{mag} = q\vec{v} \times \vec{B}_{ext}$ on the two charged particles also acts along this same line joining the two charged particles and, by the right-hand rule is such that $\vec{F}_{12}^{mag}(\vec{r}_2) = -\vec{F}_{21}^{mag}(\vec{r}_1) \parallel -\hat{x}$.

Thus, we also see that:

$$\vec{F}_{12}^{EMtot}(\vec{r}_2) = \left[\vec{F}_{12}^{elect}(\vec{r}_2) + \vec{F}_{12}^{mag}(\vec{r}_2)\right]$$
$$= -\vec{F}_{21}^{EMtot}(\vec{r}_1) = -\left[\vec{F}_{21}^{elect}(\vec{r}_1) + \vec{F}_{21}^{mag}(\vec{r}_1)\right]$$

Thus, here, for this special case, Newton's 3rd law ***is*** obeyed simply by the mechanical linear momentum associated with this system – the linear momentum carried by the *EM* field(s) in this special-case situation is zero.

Note that this special-case situation is related to the case of parallel electric currents attracting each other – *e.g.* two parallel conducting wires carrying steady currents I_1 and I_2. It must be remembered that current-carrying wires remain overall electrically neutral, because real currents in real conducting wires are carried by negatively-charged "free" conduction electrons that are embedded in a three-dimensional lattice of positive-charged atoms. The positively-charged atoms screen out / cancel the electric fields associated with the "free" conduction electrons, thus only the (attractive) magnetic Lorentz force remains!

Yet another interesting aspect of this special-case situation is to go into the rest frame of the two electric charges, where for identical lab velocities $\vec{v}_1 = \vec{v}_2 = +v\hat{z}$, in the rest frame of the charges, the magnetic field(s) both vanish – *i.e.* an observer in the rest frame of the two charges sees *no* magnetic Lorentz force(s) acting on the like-charged particles!

Maxwell's Stress Tensor $\overleftrightarrow{T}$

Let us use the Lorentz force law to calculate the ***total*** electromagnetic force $\vec{F}_{Tot}^{EM}(t)$ due to the totality of the electric charges contained within a (source) volume v:

$$\vec{F}_{Tot}^{EM}(t) = \int_v \vec{f}_{Tot}^{EM}(\vec{r},t)\,d\tau = \int_v \left\{\vec{E}(\vec{r},t) + \vec{v}(\vec{r},t)\times\vec{B}(\vec{r},t)\right\}\rho(\vec{r},t)\,d\tau$$

where: $\vec{f}_{Tot}^{EM}(\vec{r},t)$ = *EM* force per unit volume (*aka* force "***density***") (SI units: N/m^3),

and: $\vec{J}(\vec{r},t) \equiv \rho(\vec{r},t)\vec{v}(\vec{r},t)$ electric volume current density (SI units: A/m^2).

Thus: $$\vec{F}_{Tot}^{EM}(t) = \int_v \left\{\rho(\vec{r},t)\vec{E}(\vec{r},t) + \vec{J}(\vec{r},t)\times\vec{B}(\vec{r},t)\right\}d\tau$$

$$\vec{f}_{Tot}^{EM}(\vec{r},t) = EM \text{ force per unit volume} = \rho(\vec{r},t)\vec{E}(\vec{r},t) + \vec{J}(\vec{r},t)\times\vec{B}(\vec{r},t)$$

n.b. If we talk about $\vec{f}_{Tot}^{EM}(\vec{r},t)$ in isolation (*i.e.* we do not have to do the integral), then, for transparency's sake of these lecture notes, we will (temporarily) ***suppress*** the $(\vec{r},t)$ arguments – however it is very important for the reader to keep this in mind (at all times) that these arguments ***are*** there in order to actually (properly/correctly) do any calculation!!!

Thus: $$\vec{f}_{Tot}^{EM} = \rho\vec{E} + \vec{J}\times\vec{B}$$

Maxwell's equations (in differential form) can now be used to eliminate ρ and $\vec{J}$:

Coulomb's Law: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_o \Rightarrow \rho = \varepsilon_o\vec{\nabla}\cdot\vec{E}$

Ampere's Law: (with Maxwell's displacement current correction term):

$$\vec{\nabla}\times\vec{B} = \mu_o\vec{J} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{J} = \frac{1}{\mu_o}\left(\vec{\nabla}\times\vec{B}\right) - \varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

Thus: $$\vec{f}_{Tot}^{EM} = \rho\vec{E} + \vec{J}\times\vec{B} = \varepsilon_o\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E} + \left(\frac{1}{\mu_o}\left(\vec{\nabla}\times\vec{B}\right) - \varepsilon_o\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}$$

Now: $$\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) = \left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) + \left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$$ (by the chain rule of differentiation)

$\therefore$ $$\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) = \frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) - \left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$$

Faraday's Law: $\left(\vec{\nabla}\times\vec{E}\right) = -\dfrac{\partial\vec{B}}{\partial t} \Rightarrow \dfrac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$ $\therefore$ $\left(\dfrac{\partial\vec{E}}{\partial t}\times\vec{B}\right) = \dfrac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) + \left(\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right)$

Thus: $$\vec{f}_{Tot}^{EM} = \varepsilon_o\left[\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E} - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right] - \frac{1}{\mu_o}\left[\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right] - \varepsilon_o\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

Without changing the physics in any way, we can add the term $(\vec{\nabla}\bullet\vec{B})\vec{B}$ to the above expression since $\vec{\nabla}\bullet\vec{B}=0$ (*i.e.* there exist no isolated *N*/*S* magnetic charges/magnetic monopoles in nature).

Then $\vec{f}_{Tot}^{EM}$ becomes *more symmetric* between the $\vec{E}$ and $\vec{B}$ fields (which is *aesthetically* pleasing):

$$\vec{f}_{Tot}^{EM}=\varepsilon_o\left[\left(\vec{\nabla}\bullet\vec{E}\right)\vec{E}-\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]+\frac{1}{\mu_o}\left[\left(\vec{\nabla}\bullet\vec{B}\right)\vec{B}-\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right]-\varepsilon_o\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

Now: $\vec{\nabla}\left(E^2\right)=2\left(\vec{E}\bullet\vec{\nabla}\right)\vec{E}+2\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)$ — Using Griffiths "Product Rule #4"

Or: $\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)=\tfrac{1}{2}\vec{\nabla}\left(E^2\right)-\left(\vec{E}\bullet\vec{\nabla}\right)\vec{E}$ — {*n.b.* is also applied similarly for $\vec{B}$ }

Thus:

$$\vec{f}_{Tot}^{EM}=\varepsilon_o\left[\left(\vec{\nabla}\bullet\vec{E}\right)\vec{E}+\left(\vec{E}\bullet\vec{\nabla}\right)\vec{E}\right]+\frac{1}{\mu_o}\left[\left(\vec{\nabla}\bullet\vec{B}\right)\vec{B}+\left(\vec{B}\bullet\vec{\nabla}\right)\vec{B}\right]-\frac{1}{2}\vec{\nabla}\left(\varepsilon_o E^2+\frac{1}{\mu_o}B^2\right)-\varepsilon_o\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

We now introduce ***Maxwell's stress tensor*** $\overleftrightarrow{T}$ (a 3×3 matrix), the nine elements of which are defined as:

$$T_{ij}\equiv\varepsilon_o\left(E_iE_j-\tfrac{1}{2}\delta_{ij}E^2\right)+\frac{1}{\mu_o}\left(B_iB_j-\tfrac{1}{2}\delta_{ij}B^2\right)$$

where: $i,j=1,2,3$ and: $i,j=1=x$ $\quad i,j=2=y$ $\quad i,j=3=z$ *i.e.* the i,j indices of Maxwell's stress tensor physically correspond to the *x*, *y*, *z* components of the *E* & *B*-fields.

and: δ_{ij} = Kroenecker δ-function: $\delta_{ij}=0$ for $i\neq j$, $\delta_{ij}=1$ for $i=j$

and: $E^2=\vec{E}\bullet\vec{E}=E_x^2+E_y^2+E_z^2$ $\qquad$ $B^2=\vec{B}\bullet\vec{B}=B_x^2+B_y^2+B_z^2$

T_{ij} ← Column index; ↑ Row index

$$\overleftrightarrow{T}=\begin{pmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{pmatrix}=\begin{pmatrix}T_{xx}&T_{xy}&T_{xz}\\T_{yx}&T_{yy}&T_{yz}\\T_{zx}&T_{zy}&T_{zz}\end{pmatrix}$$

Note that from the above definition of the elements of $\overleftrightarrow{T}$, we see that $\overleftrightarrow{T}$ is ***symmetric*** under the interchange of the indices $i\leftrightarrow j$ (with indices $i,j=1,2,3$ = spatial x,y,z), *i.e.* one of the ***symmetry properties*** of Maxwell's stress tensor is: $T_{ij}=+T_{ji}$

We say that $\overleftrightarrow{T}$ is a ***symmetric*** rank-2 tensor (*i.e.* ***symmetric*** 3×3 matrix) ***because***: $T_{ij}=+T_{ji}$.

Thus, Maxwell's stress tensor $\overleftrightarrow{T}$ actually has only **six** (6) ***independent*** components, not nine!!!

Note also that the 9 elements of $\overleftrightarrow{T}$ are actually "***double vectors***", *e.g.* $T_{ij}\hat{i}\hat{j}$, $T_{xy}\hat{x}\hat{y}$, $T_{zz}\hat{z}\hat{z}$, *etc.*

The six independent elements of the symmetric Maxwell's stress tensor are:

$$T_{11} = T_{xx} = \frac{1}{2}\varepsilon_o\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_o}\left(B_x^2 - B_y^2 - B_z^2\right)$$

$$T_{22} = T_{yy} = \frac{1}{2}\varepsilon_o\left(E_y^2 - E_z^2 - E_x^2\right) + \frac{1}{2\mu_o}\left(B_y^2 - B_z^2 - B_x^2\right)$$ generate by cyclic permutation

$$T_{33} = T_{zz} = \frac{1}{2}\varepsilon_o\left(E_z^2 - E_x^2 - E_y^2\right) + \frac{1}{2\mu_o}\left(B_z^2 - B_x^2 - B_y^2\right)$$ generate by cyclic permutation

$$T_{12} = T_{21} = T_{xy} = T_{yx} = \varepsilon_o\left(E_x E_y\right) + \frac{1}{\mu_o}\left(B_x B_y\right)$$

$$T_{13} = T_{31} = T_{xz} = T_{zx} = \varepsilon_o\left(E_x E_z\right) + \frac{1}{\mu_o}\left(B_x B_z\right)$$ generate by cyclic permutation

$$T_{23} = T_{32} = T_{yz} = T_{zy} = \varepsilon_o\left(E_y E_z\right) + \frac{1}{\mu_o}\left(B_y B_z\right)$$ generate by cyclic permutation

Note also that $\overleftrightarrow{T}$ contains no $\vec{E}\times\vec{B}$ (*etc.*) type cross-terms!!

Because $\overleftrightarrow{T}$ is a rank-2 tensor, it is represented by a 2-dimensional, 3×3 matrix.
{Higher rank tensors: *e.g.* T_{ijk} (= rank-3 / 3-D matrix), A_{ij}^{kl} (= rank-4 / 4-D matrix), *etc.*}

We can take the dot product of a vector $\vec{a}$ with a tensor $\overleftrightarrow{T}$ to obtain (another) vector $\vec{b}$: $\boxed{\vec{b} = \vec{a}\bullet\overleftrightarrow{T}}$

$$\boxed{b_j = \left(\vec{a}\bullet\overleftrightarrow{T}\right)_j = \sum_{i=1}^{3} a_i T_{ij} = a_i T_{ij}}$$

explicit summation over i-index; only index j remains.

implicit summation over i-index is implied, only index j remains.

n.b. this summation convention is very important:

$$b_j = \underbrace{a_i T_{ij}}_{} = \underbrace{\sum_{i=1}^{3} a_i T_{ij}}_{}$$

implicit sum over i *explicit* sum over i

Note also that another vector can be formed, *e.g.*: $\boxed{\vec{c} = \overleftrightarrow{T}\bullet\vec{a}}$ where (here): $\boxed{c_i = \sum_{j=1}^{3} T_{ij} a_j = T_{ij} a_j}$

Compare these two <u>types</u> of vectors side-by-side:

$\boxed{\vec{b} = \vec{a}\bullet\overleftrightarrow{T}}$: $\boxed{b_j = \left(\vec{a}\bullet\overleftrightarrow{T}\right)_j = \sum_{i=1}^{3} a_i T_{ij} = a_i T_{ij}}$ *n.b.* summation is over index i (*i.e.* **<u>rows</u>** in $\overleftrightarrow{T}$)!!

$\boxed{\vec{c} = \overleftrightarrow{T}\bullet\vec{a}}$: $\boxed{c_i = \left(\overleftrightarrow{T}\bullet\vec{a}\right)_i = \sum_{j=1}^{3} T_{ij} a_j = T_{ij} a_j}$ *n.b.* summation is over index j (*i.e.* **<u>columns</u>** in $\overleftrightarrow{T}$)!!

n.b. Additional info is given on tensors/tensor properties at the end of these lecture notes…

Now if the vector $\vec{a}$ in the dot product $\vec{b} = \vec{a} \bullet \overleftrightarrow{T}$ "happens" to be $\vec{a} = \vec{\nabla}$, then if:

$$T_{ij} \equiv \varepsilon_o \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_o}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

$$\left(\vec{\nabla} \bullet \overleftrightarrow{T}\right)_j = \varepsilon_o \left(\left(\vec{\nabla} \bullet \vec{E}\right) E_j + \left(\vec{E} \bullet \vec{\nabla}\right) E_j - \frac{1}{2}\nabla_j E^2 \right) + \frac{1}{\mu_o}\left(\left(\vec{\nabla} \bullet \vec{B}\right) B_j + \left(\vec{B} \bullet \vec{\nabla}\right) B_j - \frac{1}{2}\nabla_j B^2 \right)$$

Thus, we see that the total electromagnetic force per unit volume (*aka* force "density") can be written (much) more compactly (and elegantly) as:

$$\vec{f}_{Tot}^{\,EM} = \vec{\nabla} \bullet \overleftrightarrow{T} - \varepsilon_o \mu_o \frac{\partial \vec{S}}{\partial t} \quad \left(N/m^3\right),$$

Maxwell's stress tensor: $$T_{ij} \equiv \varepsilon_o \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_o}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

Poynting's vector: $$\vec{S} \equiv \frac{1}{\mu_o}\left(\vec{E} \times \vec{B}\right)$$

The ***total*** *EM* force acting on the charges contained within the (source) volume v is given by:

$$\vec{F}_{Tot}^{EM} = \int_v \vec{f}_{Tot}^{\,EM} d\tau' = \int_v \left(\vec{\nabla} \bullet \overleftrightarrow{T} - \varepsilon_o \mu_o \frac{\partial \vec{S}}{\partial t} \right) d\tau'$$

Explicit reminder of the (suppressed) arguments:

$$\vec{F}_{Tot}^{EM}(t) = \int_v \vec{f}_{Tot}^{\,EM}(\vec{r}',t)\, d\tau' = \int_v \left(\vec{\nabla} \bullet \overleftrightarrow{T}(\vec{r}',t) - \varepsilon_o \mu_o \frac{\partial \vec{S}(\vec{r}',t)}{\partial t} \right) d\tau'$$

Using the divergence theorem on the LHS term of the integrand:

$$\vec{F}_{Tot}^{EM} = \oint_S \overleftrightarrow{T} \bullet d\vec{a}' - \varepsilon_o \mu_o \int_v \frac{\partial \vec{S}}{\partial t} d\tau' \quad i.e. \quad \vec{F}_{Tot}^{EM}(t) = \oint_S \overleftrightarrow{T}(\vec{r}',t) \bullet d\vec{a}' - \varepsilon_o \mu_o \int_v \frac{\partial \vec{S}(\vec{r}',t)}{\partial t} d\tau'$$

Finally: $$\vec{F}_{Tot}^{EM} = \oint_S \overleftrightarrow{T} \bullet d\vec{a}' - \varepsilon_o \mu_o \frac{d}{dt}\int_v \vec{S} d\tau' \quad i.e. \quad \vec{F}_{Tot}^{EM}(t) = \oint_S \overleftrightarrow{T}(\vec{r}',t) \bullet d\vec{a}' - \varepsilon_o \mu_o \frac{d}{dt}\int_v \vec{S}(\vec{r}',t)\, d\tau'$$

$$\boxed{\vec{F}_{Tot}^{EM} = \oint_S \overleftrightarrow{T} \cdot d\vec{a}' - \varepsilon_o \mu_o \frac{d}{dt} \int_v \vec{S} d\tau'} \quad i.e. \quad \boxed{\vec{F}_{Tot}^{EM}(t) = \oint_S \overleftrightarrow{T}(\vec{r}',t) \cdot d\vec{a}' - \varepsilon_o \mu_o \frac{d}{dt} \int_v \vec{S}(\vec{r}',t) d\tau'}$$

Note the following important aspects/points about the physical nature of this result:

1.) In the ***static*** case (*i.e.* whenever $\overleftrightarrow{T}(\vec{r},t), \vec{S}(\vec{r},t) \neq fcns(t)$), the second term on the RHS in the above equation vanishes – then the ***total*** *EM* force acting on the charge configuration contained within the (source) volume v can be expressed entirely in terms of Maxwell's stress tensor at the ***boundary*** of the volume v, *i.e.* on the enclosing surface S:

$$\boxed{\vec{F}_{Tot}^{EM} = \oint_S \overleftrightarrow{T}(\vec{r}') \cdot d\vec{a}' \neq fcn(t)}$$

2.) Physically, $\oint_S \overleftrightarrow{T}(\vec{r}',t) \cdot d\vec{a}'$ = net force (SI units: Newtons) acting on the enclosing surface S.

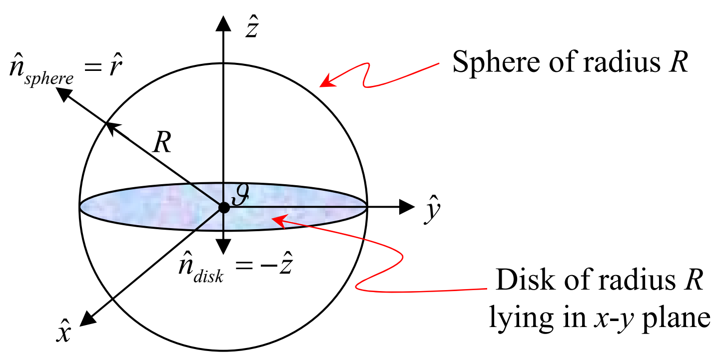
Then {***here***} $\overleftrightarrow{T}$ is the net ***force per unit area*** (SI units: Newtons/m^2) acting on the surface S – *i.e.* $\overleftrightarrow{T}$ corresponds to an {electromagnetically-induced} ***pressure*** (!!!) or a ***stress*** acting on the enclosing surface S.

3.) More precisely: physically, T_{ij} represents the ***force per unit area*** (*Newtons/m*^{2}) in the i^{th} direction ***acting*** on an ***element of the enclosing surface*** S that is ***oriented*** in the j^{th} direction.

Thus, the ***diagonal*** elements of $\overleftrightarrow{T}$ $(i = j)$: $T_{ii} = T_{xx}, T_{yy}, T_{zz}$ physically represent ***pressures***.
The ***off-diagonal*** elements of $\overleftrightarrow{T}$ $(i \neq j)$: $T_{ij} = T_{xy}, T_{yz}, T_{zx}$ physically represent ***shears***.

Griffiths Example 8.2: Use / Application of Maxwell's Stress Tensor $\overleftrightarrow{T}$

Determine the net / total *EM* force acting on the upper ("northern") hemisphere of a uniformly electrically-charged solid non-conducting sphere (*i.e.* uniform/constant electric charge volume density $\rho = Q/\frac{4}{3}\pi R^3$) of radius R and net electric charge Q using Maxwell's Stress Tensor $\overleftrightarrow{T}$ (*c.f.* with Griffiths Problem 2.43, *p.* 107).



⇒ ***n.b. Please work through the details of this problem on your own, as an exercise!!!*** ⇐

First, note that this problem is ***static/time-independent***, thus (***here***): $\boxed{F_{Tot}^{EM} = \oint_S \overleftrightarrow{T}(\vec{r}') \cdot d\vec{a}'}$

Note also that (***here***) $\vec{B}(\vec{r}) = 0$ and thus (***here***): $\boxed{T_{ij} = \varepsilon_o \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right)}$ {***only***}.

The boundary surface S of the "northern" hemisphere consists of two parts – a hemispherical bowl of radius R and a circular disk lying in the x-y plane (*i.e.* $\theta = \pi/2$), also of radius R.

a.) For the hemispherical bowl portion of S, note that: $d\vec{a} = R^2 d\cos\theta d\varphi\hat{r} = R^2 \sin\theta d\theta d\varphi\hat{r}$

The electric field at/on the surface of the charged sphere is: $\vec{E}(r)\Big|_{r=R} = \dfrac{1}{4\pi\varepsilon_o}\dfrac{Q}{R^2}\hat{r}$

In Cartesian coordinates: $\hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}$

Thus:

$$\vec{E}(r)\Big|_{r=R} = E_x(r)\Big|_{r=R}\hat{x} + E_y(r)\Big|_{r=R}\hat{y} + E_z(r)\Big|_{r=R}\hat{z} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R^2}\hat{r}$$
$$= \frac{1}{4\pi\varepsilon_o}\frac{Q}{R^2}\left[\sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}\right]$$

Thus:

$$T_{zx}\big|_{r=R} = T_{xz}\big|_{r=R} = \varepsilon_o E_z E_x = \varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R^2}\right)^2 \sin\theta\cos\theta\cos\varphi$$

$$T_{zy}\big|_{r=R} = T_{yz}\big|_{r=R} = \varepsilon_o E_z E_y = \varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R^2}\right)^2 \sin\theta\cos\theta\sin\varphi$$

$$T_{zz}\big|_{r=R} = \frac{1}{2}\varepsilon_o\left(E_z^2 - E_x^2 - E_y^2\right) = \frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R^2}\right)^2\left(\cos^2\theta - \sin^2\theta\right)$$

The net / total force (due to the symmetry associated with this problem) is obviously only in the $\hat{z}$-direction, thus we "only" need to calculate:

$$\left(\vec{T}\bullet d\vec{a}\right)_z\Big|_{r=R} = \left[T_{zx}da_x + T_{zy}da_y + T_{zz}da_z\right]\Big|_{r=R}$$

Now: $d\vec{a} = R^2\sin\theta d\theta d\varphi\hat{r}$ and $\hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z}$

Then:

$$da_x = \left(R^2\sin\theta d\theta d\varphi\right)\sin\theta\cos\varphi\hat{x}$$
$$da_y = \left(R^2\sin\theta d\theta d\varphi\right)\sin\theta\sin\varphi\hat{y}$$
$$da_z = \left(R^2\sin\theta d\theta d\varphi\right)\cos\theta\hat{z}$$

since $d\vec{a} = da_x\hat{x} + da_y\hat{y} + da_z\hat{z}$

Then:

$$\left(\vec{T}\bullet d\vec{a}\right)_z\Big|_{r=R} = \left[\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o \cancel{R^2}}\right)^2\sin\theta\cos\theta\cos\varphi\right]*\left[\left(\cancel{R^2}\sin\theta d\theta d\varphi\right)\sin\theta\cos\varphi\right]$$
$$+\left[\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o \cancel{R^2}}\right)^2\sin\theta\cos\theta\sin\varphi\right]*\left[\left(\cancel{R^2}\sin\theta d\theta d\varphi\right)\sin\theta\sin\varphi\right]$$
$$+\left[\frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o \cancel{R^2}}\right)^2\left(\cos^2\theta - \sin^2\theta\right)\right]*\left[\left(\cancel{R^2}\sin\theta d\theta d\varphi\right)\cos\theta\right]$$

$$\left(\vec{T}\bullet d\vec{a}\right)_z\Big|_{r=R}$$

$= 1$

$$= \varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R}\right)^2\left\{\left(\sin^2\theta\cos\theta\right)\cos^2\varphi+\left(\sin^2\theta\cos\theta\right)\sin^2\varphi+\frac{1}{2}\left(\cos^2\theta-\sin^2\theta\right)\cos\theta\right\}*\left(\sin\theta d\theta d\varphi\right)$$

$$= \varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R}\right)^2\left\{\left(\cancel{\sin^2\theta}\right)\cos\theta+\frac{1}{2}\left(\cos^2\theta-\cancel{\sin^2\theta}\right)\cos\theta\right\}*\left(\sin\theta d\theta d\varphi\right)$$

$$= \varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R}\right)^2\left\{\frac{1}{2}\left(\sin^2\theta\right)\cos\theta+\frac{1}{2}\left(\cos^2\theta\right)\cos\theta\right\}*\left(\sin\theta d\theta d\varphi\right)$$

$= 1$

$$= \varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R}\right)^2\left\{\frac{1}{2}\cos\theta\right\}*\left(\sin\theta d\theta d\varphi\right)$$

$$= \frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R}\right)^2\cos\theta\sin\theta d\theta d\varphi$$

$$\therefore \boxed{\vec{F}^{EM}_{bowl}\Big|_{r=R} = \oint_{bowl}\left(\vec{T}\bullet d\vec{a}\right)_z\Big|_{r=R} = \frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R}\right)^2 2\pi\underbrace{\int_{\theta=0}^{\theta=\frac{\pi}{2}}\cos\theta\sin\theta d\theta}_{=\int_0^1 udu=\frac{1}{2}u^2\big|_{u=0}^{u=1}=\frac{1}{2}}\hat{z} = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{8R^2}\right)\hat{z}}$$

$$\boxed{\vec{F}^{EM}_{bowl}\Big|_{r=R} = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{8R^2}\right)\hat{z}}$$

b.) For the <u>equatorial disk (*i.e.* the underside) portion</u> of the "northern" hemisphere:

$$d\vec{a}\big|_{disk} = rdrd\varphi\left(-\hat{z}\right) \Leftarrow$$
$$= -rdrd\varphi\hat{z}$$
$$= -da'_z\hat{z}$$

n.b. outward unit normal for equatorial disk (lying in *x-y* plane) points in $-\hat{z}$ direction on this portion of the bounding surface S

And since we are now inside the charged sphere (on the *x-y* plane/at $\theta = \pi/2$):

$$\vec{E}_{Disk}\Big|_{\theta=\pi/2} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R^3}\vec{r}\Big|_{\theta=\pi/2} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R^2}\left(\frac{\vec{r}}{R}\right)\Big|_{\theta=\pi/2}$$

$$= \frac{1}{4\pi\varepsilon_o}\frac{Q}{R^3}\left[r\underbrace{\sin\theta}_{=1}\cos\varphi\hat{x}+r\underbrace{\sin\theta}_{=1}\sin\varphi\hat{y}+r\underbrace{\cos\theta}_{=0}\hat{z}\right]\Big|_{\theta=\pi/2}$$

$$= \frac{1}{4\pi\varepsilon_o}\frac{Q}{R^3}r\left[\cos\varphi\hat{x}+\sin\varphi\hat{y}\right]$$

Then for the equatorial, flat disk lying in the *x-y* plane:

$$\boxed{\left(\vec{T}\bullet d\vec{a}\right)_z = T_{zz}da_z} \quad \text{where:} \quad \boxed{T_{zz} = \frac{1}{2}\varepsilon_o\left(E_z^2-E_x^2-E_y^2\right) = -\frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R^3}\right)^2 r^2}$$

Thus: $\left(\overleftrightarrow{T}\bullet d\vec{a}\right)_z = T_{zz}da_z = \left[-\frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R^3}\right)^2 r^2\hat{z}\hat{z}\right]\bullet\left[-rdrd\varphi\hat{z}\right] = +\frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R^3}\right)^2 r^3drd\varphi\hat{z}$

The *EM* force acting on the disk portion of the "northern" hemisphere is therefore:

$$\vec{F}_{disk}^{EM} = \oint_{disk}\left(\overleftrightarrow{T}\bullet d\vec{a}\right)_z\Big|_{\theta=\pi/2} = \frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o R^3}\right)^2 2\pi\int_0^R r^3dr\,\hat{z} = \frac{1}{4\pi\varepsilon_o}\frac{Q^2}{16R^2}\hat{z}$$

$$\vec{F}_{disk}^{EM} = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{16R^2}\right)\hat{z}$$

The ***total*** *EM* force acting on the upper / "northern" hemisphere is:

$$\vec{F}_{TOT}^{EM} = \vec{F}_{bowl}^{EM} + \vec{F}_{disk}^{EM} = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{8R^2}\right)\hat{z} + \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{16R^2}\right)\hat{z} = \frac{1}{4\pi\varepsilon_o}\left(\frac{3Q^2}{16R^2}\right)\hat{z}$$

Note that in applying $\vec{F}_{TOT}^{EM} = \oint_S \overleftrightarrow{T}(\vec{r},t)\bullet d\vec{a} - \varepsilon_o\mu_o\frac{d}{dt}\int_v \vec{S}(\vec{r},t)d\tau$ that *any* volume *v* that encloses ***all*** of the electric charge will suffice. Thus in above problem, we could ***equally-well*** have instead chosen to use *e.g.* the ***whole*** half-region $z > 0$ – *i.e.* the "disk" consisting of the ***entire*** *x-y* plane ***and*** the upper hemisphere (at $r = \infty$), but note that since $\vec{E} = 0$ at $r = \infty$, this latter surface would contribute nothing to the total *EM* force!!!

Then for the ***outer*** portion of the whole *x-y* plane (*i.e.* $r > R$): $T_{zz} = -\frac{\varepsilon_o}{2}\left(\frac{Q}{4\pi\varepsilon_o}\right)^2\frac{1}{r^4}$

Then for this ***outer*** portion of the *x-y* plane ($r > R$): $\left(\overleftrightarrow{T}\bullet d\vec{a}\right)_z = T_{zz}da_z = +\frac{\varepsilon_o}{2}\left(\frac{Q}{4\pi\varepsilon_o}\right)^2\frac{1}{r^3}drd\varphi$

The corresponding *EM* force on this ***outer*** portion of the *x-y* plane (for $r > R$) is:

$$\vec{F}_{disk}^{EM}(r > R) = \frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o}\right)^2 2\pi\int_R^\infty \frac{1}{r^3}dr\,\hat{z} = \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{8R^2}\right)\hat{z}$$

Thus: $\vec{F}_{TOT}^{EM} = \vec{F}_{disk}^{EM}(r \le R) + \vec{F}_{disk}^{EM}(r > R) == \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{16R^2}\right)\hat{z} + \frac{1}{4\pi\varepsilon_o}\left(\frac{Q^2}{8R^2}\right)\hat{z} = \frac{1}{4\pi\varepsilon_o}\left(\frac{3Q^2}{16R^2}\right)\hat{z}$

n.b. this ***is*** precisely the same result as obtained above via first method!!!

Even though the uniformly charged sphere was a solid object – not a hollow sphere – the use of Maxwell's Stress Tensor allowed us to calculate the net *EM* force acting on the "northern" hemisphere via a ***surface*** integral over the bounding surface *S* enclosing the volume *v* containing the uniform electric charge volume density $\rho = Q/\frac{4}{3}\pi R^3$. That's pretty amazing!!

Further Discussion of the Conservation of Linear Momentum $\vec{p}$

We started out this lecture talking ~ somewhat qualitatively about conservation of linear momentum in electrodynamics; we are now in a position to quantitatively discuss this subject.

Newton's 2nd Law of Motion $\vec{F}(t) = m\vec{a}(t) = m\,d\vec{v}(t)/dt = d\{m\vec{v}(t)\}/dt = d\vec{p}_{mech}(t)/dt$

The total {instantaneous} force $\vec{F}(t)$ acting on an object = {instantaneous} time rate of change of its mechanical linear momentum $d\vec{p}(t)/dt$ *i.e.* $\boxed{\vec{F}(t) = \frac{d\vec{p}_{mech}(t)}{dt}}$. But from above, we know that:

$$\boxed{\vec{F}^{EM}_{Tot}(t) = \frac{d\vec{p}_{mech}(t)}{dt} = \oint_S \vec{T}(\vec{r},t)\cdot d\vec{a} - \varepsilon_o\mu_o \frac{d}{dt}\int_v \vec{S}(\vec{r},t)\,d\tau} \quad **$$

where $\vec{p}_{mech}(t)$ = total {instantaneous} mechanical linear momentum of the particles contained in the (source) volume v. (SI units: kg-m/sec)

We define: $\boxed{\vec{p}_{EM}(t) \equiv \varepsilon_o\mu_o \int_v \vec{S}(\vec{r},t)\,d\tau = \frac{1}{c^2}\int_v \vec{S}(\vec{r},t)\,d\tau}$ {since $\boxed{c^2 = 1/\varepsilon_o\mu_o}$ in free space}

where $\vec{p}_{EM}(t)$ = total {instantaneous} linear momentum carried by / stored in the (macroscopic) electromagnetic fields ($\vec{E}$ and $\vec{B}$) (SI units: kg-m/sec). At the microscopic level – linear momentum is carried by the virtual {and/or real} photons associated with the macroscopic $\vec{E}$ and $\vec{B}$ fields!

We can also define an {instantaneous} *EM* field linear momentum density (SI Units: kg/m^2-sec):

$\boxed{\vec{\wp}_{EM}(\vec{r},t) \equiv \varepsilon_o\mu_o\vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t)}$ = instantaneous *EM* field linear momentum per unit volume

Thus, we see that the total {instantaneous} *EM* field linear momentum $\boxed{\vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\,d\tau}$

Note that the surface integral in ** above, $\oint_S \vec{T}(\vec{r},t)\cdot d\vec{a}$ physically corresponds to the total {instantaneous} *EM* field linear momentum per unit time flowing inwards through the surface S.

Thus, any instantaneous increase in the total linear momentum (mechanical and *EM* field) = the linear momentum brought in by the *EM* fields themselves through the bounding surface S.

Thus: $\boxed{\frac{d\vec{p}_{mech}(t)}{dt} = -\frac{d\vec{p}_{EM}(t)}{dt} + \oint_S \vec{T}(\vec{r},t)\cdot d\vec{a}}$ where: $\boxed{\vec{p}_{EM}(t) \equiv \varepsilon_o\mu_o\int_v \vec{S}(\vec{r},t)\,d\tau = \frac{1}{c^2}\int_v \vec{S}(\vec{r},t)\,d\tau}$

Or: $\boxed{\frac{d\vec{p}_{mech}(t)}{dt} + \frac{d\vec{p}_{EM}(t)}{dt} = \oint_S \vec{T}(\vec{r},t)\cdot d\vec{a}}$

But: $\boxed{\vec{p}_{Tot}(t) \equiv \vec{p}_{mech}(t) + \vec{p}_{EM}(t)}$

$\therefore$ $\boxed{\frac{d\vec{p}_{Tot}(t)}{dt} = \frac{d\vec{p}_{mech}(t)}{dt} + \frac{d\vec{p}_{EM}(t)}{dt} = \oint_S \vec{T}(\vec{r},t)\cdot d\vec{a}}$ $\Leftarrow$ Expresses conservation of linear momentum in electrodynamics

Note that the integral formula expressing conservation of linear momentum in electrodynamics is similar to that of the integral form of Poynting's theorem, expressing conservation of energy in electrodynamics and also to that of the integral form of the Continuity Equation, expressing conservation of electric charge in electrodynamics:

Poynting's Theorem: Energy Conservation

$$P_{Tot}(t) = \frac{dU_{Tot}(t)}{dt} = \frac{dU_{mech}(t)}{dt} + \frac{dU_{EM}(t)}{dt}$$

$$= \frac{d}{dt}\int_v \left(u_{mech}(\vec{r},t) + u_{EM}(\vec{r},t)\right) d\tau = -\oint_S \vec{S}(\vec{r},t) \cdot d\vec{a} = -\int_v \left(\vec{\nabla} \cdot \vec{S}(\vec{r},t)\right) d\tau$$

Continuity Equation: Electric Charge Conservation:

$$\int_v \frac{\partial \rho_{free}(\vec{r},t)}{\partial t} d\tau = -\int_v \vec{\nabla} \cdot \vec{J}_{free}(\vec{r},t)\, d\tau$$

Note further that if the volume v = ***all*** space, then ***no*** linear momentum can flow into / out of v through the bounding surface S. Thus, in this situation:

$$\vec{p}_{Tot}(t) \equiv \vec{p}_{mech}(t) + \vec{p}_{EM}(t) = \text{constant}$$

and:
$$\frac{d\vec{p}_{Tot}(t)}{dt} = \frac{d\vec{p}_{mech}(t)}{dt} + \frac{d\vec{p}_{EM}(t)}{dt} = \oint_S \overleftrightarrow{T}(\vec{r},t) \cdot d\vec{a} = 0 \;\Rightarrow\; \frac{d\vec{p}_{mech}(t)}{dt} = -\frac{d\vec{p}_{EM}(t)}{dt}$$

We can also express conservation of linear momentum via a differential equation, just as we have done in the cases for electric charge and energy conservation. <u>Define</u>:

$\vec{\wp}_{mech}(\vec{r},t) \equiv$ {instantaneous} ***mechanical*** linear momentum density (SI Units: kg/m^2-sec)

$\vec{\wp}_{EM}(\vec{r},t) \equiv$ {instantaneous} ***EM field*** linear momentum density (SI Units: kg/m^2-sec)

$$\vec{\wp}_{EM}(\vec{r},t) \equiv \varepsilon_o \mu_o \vec{S}(\vec{r},t) = \frac{1}{c^2}\vec{S}(\vec{r},t)$$

$\vec{\wp}_{Tot}(\vec{r},t) \equiv$ {instantaneous} <u>total</u> linear momentum density (SI Units: kg/m^2-sec)

$$\vec{\wp}_{Tot}(\vec{r},t) \equiv \vec{\wp}_{mech}(\vec{r},t) + \vec{\wp}_{EM}(\vec{r},t)$$

Then:

$$\vec{p}_{mech}(t) = \int_v \vec{\wp}_{mech}(\vec{r},t)\, d\tau \quad \text{and} \quad \vec{p}_{EM}(t) = \int_v \vec{\wp}_{EM}(\vec{r},t)\, d\tau \text{, thus:}$$

$$\vec{p}_{Tot}(t) = \int_v \vec{\wp}_{Tot}(\vec{r},t)\, d\tau = \int_v \left[\vec{\wp}_{mech}(\vec{r},t) + \vec{\wp}_{EM}(\vec{r},t)\right] d\tau \;\Rightarrow\; \vec{p}_{Tot}(t) = \vec{p}_{mech}(t) + \vec{p}_{EM}(t)$$

Then:
$$\frac{d\vec{p}_{TOT}(t)}{dt} = \frac{d\vec{p}_{mech}(t)}{dt} + \frac{d\vec{p}_{EM}(t)}{dt} = \oint_S \overleftrightarrow{T}(\vec{r},t) \cdot d\vec{a}$$

Using the divergence theorem on the RHS of this relation, this can also be written as:

$$\frac{d}{dt}\int_v \vec{\wp}_{Tot}(\vec{r},t)\, d\tau = \frac{d}{dt}\int_v \vec{\wp}_{mech}(\vec{r},t)\, d\tau + \frac{d}{dt}\int_v \vec{\wp}_{EM}(\vec{r},t)\, d\tau = \int_v \vec{\nabla} \cdot \overleftrightarrow{T}(\vec{r},t)\, d\tau$$

<u>Thus</u>:
$$\int_v \left(\frac{\partial \vec{\wp}_{mech}(\vec{r},t)}{\partial t} + \frac{\partial \vec{\wp}_{EM}(\vec{r},t)}{\partial t} - \vec{\nabla} \cdot \overleftrightarrow{T}(\vec{r},t) \right) d\tau = 0$$

This relation ***must*** hold for ***any*** volume v, and for ***all*** points & times $(\vec{r},t)$ within the volume v:

$$\therefore \boxed{\frac{\partial \vec{\wp}_{mech}(\vec{r},t)}{\partial t}+\frac{\partial \vec{\wp}_{EM}(\vec{r},t)}{\partial t}=\vec{\nabla}\cdot\overleftrightarrow{T}(\vec{r},t)} \Leftarrow \boxed{\text{Differential form of conservation of linear momentum}}$$

$$\boxed{\frac{\partial \vec{\wp}_{mech}(\vec{r},t)}{\partial t}+\frac{\partial \vec{\wp}_{EM}(\vec{r},t)}{\partial t}=\vec{\nabla}\cdot\overleftrightarrow{T}(\vec{r},t)} \quad \text{(Linear momentum conservation)}$$

Continuity Equation: $$\boxed{\frac{\partial}{\partial t}\rho(\vec{r},t)=-\vec{\nabla}\cdot\vec{J}(\vec{r},t)} \quad \text{(Electric charge conservation)}$$

Poynting's Theorem: $$\boxed{\frac{\partial}{\partial t}\left(u_{mech}(\vec{r},t)+u_{EM}(\vec{r},t)\right)=-\vec{\nabla}\cdot\vec{S}(\vec{r},t)} \quad \text{(Energy conservation)}$$

Thus, we see that the ***negative*** of Maxwell's stress tensor, $-\overleftrightarrow{T}(\vec{r},t)$ physically represents linear momentum ***flux density***, analogous to electric current density $\vec{J}(\vec{r},t)$ which physically represents the electric charge ***flux density*** in the continuity equation, and analogous to Poynting's vector $\vec{S}(\vec{r},t)$ which physically represents the energy ***flux density*** in Poynting's theorem.

Note also that relation $\frac{\partial \vec{\wp}_{mech}(\vec{r},t)}{\partial t}+\frac{\partial \vec{\wp}_{EM}(\vec{r},t)}{\partial t}=\vec{\nabla}\cdot\overleftrightarrow{T}(\vec{r},t)$, while mathematically describing the "how" of linear momentum conservation, says nothing about the nature/origin of why linear momentum is conserved, just as in case(s) of the Continuity Equation (electric charge conservation) and Poynting's Theorem (energy conservation).

Since $-\overleftrightarrow{T}(\vec{r},t)$ physically represents the instantaneous linear momentum ***flux density*** (*aka* momentum ***flow*** = momentum ***current density***) at the space-time point $(\vec{r},t)$, the 9 elements of {the negative of} Maxwell's Stress Tensor, $-T_{ij}$ are physically interpreted as the instantaneous *EM* field linear momentum ***flow*** in the i^{th} direction through a surface oriented in the j^{th} direction. Note further that because $-T_{ij}=-T_{ji}$, this is also equal to the instantaneous *EM* field linear momentum *flow* in the j^{th} direction through a surface oriented in the i^{th} direction!

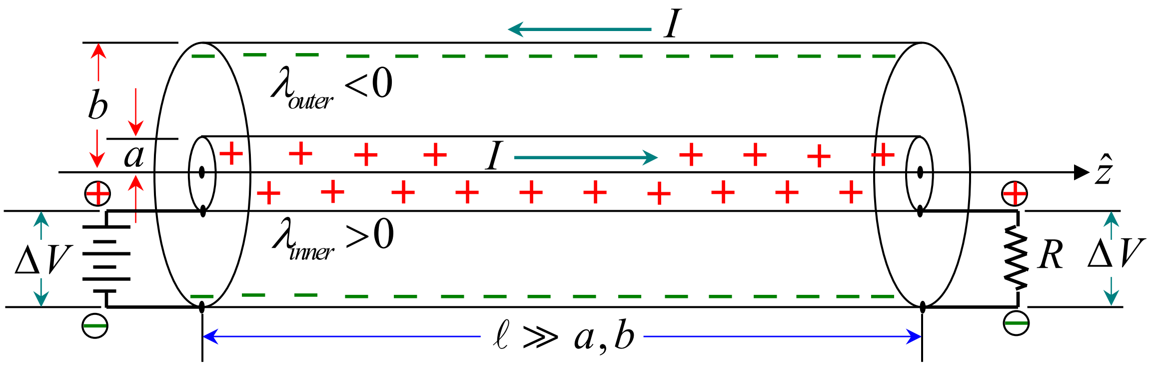
From the equation $\frac{\partial \vec{\wp}_{mech}(\vec{r},t)}{\partial t}+\frac{\partial \vec{\wp}_{EM}(\vec{r},t)}{\partial t}=\vec{\nabla}\cdot\overleftrightarrow{T}(\vec{r},t)$, noting that the del-operator $\vec{\nabla}=\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}$ has SI units of m^{-1}, we see that the 9 elements of $\overleftrightarrow{T}(\vec{r},t)$ {= linear momentum ***flux densities*** = linear momentum ***flows***} have SI units of:

length × linear momentum density/unit time = linear momentum density × length/unit time
= linear momentum density × (length/unit time) = linear momentum density × velocity
= $\{(kg\text{-}m/s)/m^3\} \times (m/s) = kg/m\text{-}s^2$

Earlier (*p*. 9 of these lect. notes), we also said that the 9 elements of $\overleftrightarrow{T}(\vec{r},t)$ have SI units of pressure, p (*Pascals* = $Newtons/m^2 = (kg\text{-}m/s^2)/m^2 = kg/m\text{-}s^2$). Note further that the SI units of pressure are also that of energy ***density***, u ($Joules/m^3$ = (Newton-m)/$m^3 = Newtons/m^2 = kg/m\text{-}s^2$)!

Griffiths Example 8.3 *EM* Field Momentum:

A long coaxial cable of length ℓ consists of a hollow inner conductor (radius a) and hollow outer conductor (radius b). The coax cable is connected to a battery at one end and a resistor at the other end, as shown in figure below:

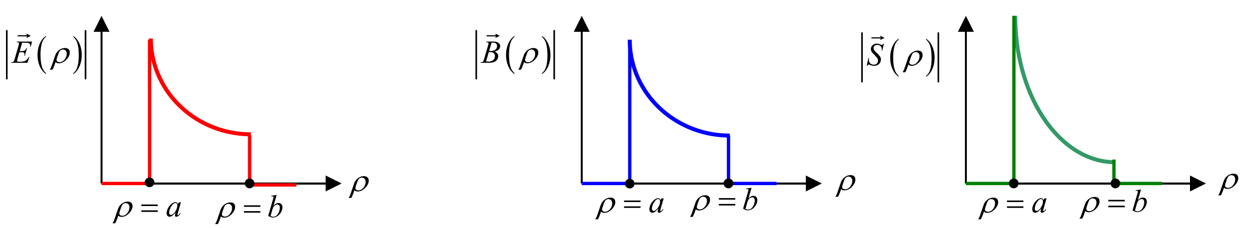


The hollow inner conductor carries uniform charge / unit length $\lambda_{inner} = +Q_{inner}/\ell$ and a steady DC current $\vec{I} = I\hat{z}$ (*i.e.* flowing to the right). The hollow outer conductor has the opposite charge and current. Calculate the *EM* momentum carried by the *EM* fields associated with this system.

Note that this problem has ***no*** time dependence associated with it, *i.e.* it is a ***static*** problem.
∴ The static *EM* fields associated with this long coaxial cable are:

$$\vec{E}(\rho) = \frac{1}{2\pi\varepsilon_o}\frac{\lambda}{\rho}\hat{\rho} \qquad \rho = \sqrt{x^2+y^2} \text{ in cylindrical coordinates}$$

$$\vec{B}(\rho) = \frac{\mu_o}{2\pi}\frac{I}{\rho}\hat{\varphi} \qquad n.b.\ \vec{E}(\rho) \text{ and } \vec{B}(\rho) \text{ only non-zero for } a \le \rho \le b$$



Poynting's vector is: $\vec{S}(\rho) = \frac{1}{\mu_o}\left(\vec{E}(\rho)\times\vec{B}(\rho)\right) = \frac{\lambda I}{4\pi^2\varepsilon_o\rho^2}\overbrace{(\hat{\rho}\times\hat{\varphi})}^{=\hat{z}} = \frac{\lambda I}{4\pi^2\varepsilon_o\rho^2}\hat{z}$ (for $a \le \rho \le b$)

⇒ Even though this is a ***static*** problem (*i.e.* ***no*** explicit time dependence), *EM* energy contained / stored in the *EM* fields {*n.b.* only within the region $a \le \rho \le b$} is flowing down the coax cable in the $+\hat{z}$-direction, from battery to resistor!

The instantaneous *EM* power (= constant $\neq fcn(t)$) transported down the coax cable is obtained by integrating Poynting's vector $\vec{S}(\rho) \| +\hat{z}$ (= energy flux density) over a perpendicular / cross-sectional area of $A_\perp = \pi\left(b^2 - a^2\right)$, with corresponding infinitesimal area element $d\vec{a}_\perp = 2\pi\rho d\rho\hat{z}$:

$$\boxed{P = \int \vec{S}(\rho)\cdot d\vec{a}_\perp = \frac{\lambda I}{4\pi^2\varepsilon_o}\int_{\rho=a}^{\rho=b}\frac{1}{\rho^2}2\pi\rho d\rho = \frac{\lambda I}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)}$$

But: $\boxed{\Delta V = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)}$ $\Rightarrow$ *EM* Power $\boxed{P = \Delta V * I}$ = *EM* Power dissipated in the resistor!

Inside the coax cable (*i.e.* $a \le \rho \le b$), Poynting's vector is: $\boxed{\vec{S}(\rho) = \frac{\lambda I}{4\pi^2\varepsilon_o\rho^2}\hat{z}}$

The linear momentum associated with / carried by / stored in the *EM* field(s) ($a \le \rho \le b$) is:

$$\boxed{\vec{p}_{EM} = \int_v \vec{\wp}_{EM}(\rho)d\tau = \varepsilon_o\mu_o\int_v \vec{S}(\rho)d\tau = \frac{\mu_o\lambda I}{4\pi^2}\hat{z}\int_{\rho=a}^{\rho=b}\frac{1}{\rho^2}\{2\pi\ell\rho d\rho\}}$$

$$\boxed{\vec{p}_{EM} = \frac{\mu_o\lambda I\ell}{2\pi}\ln\left(\frac{b}{a}\right)\hat{z}}$$

Using $\boxed{\Delta V = \frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)}$ and $\boxed{c^2 = \frac{1}{\varepsilon_o\mu_o}}$ we can rewrite this expression as:

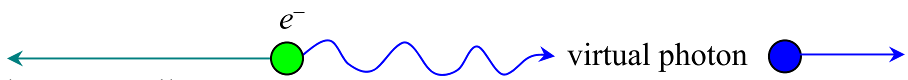
$$\boxed{\vec{p}_{EM} = \mu_o\varepsilon_o\frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)I\ell\hat{z} = \frac{\Delta VI\ell}{c^2}\hat{z}}$$

The *EM* field(s) $\vec{E}$ and $\vec{B}$ (via $\vec{S} = \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)$) in the region $a \le \rho \le b$ are responsible for transporting *EM* power / energy ***as well as*** linear momentum $\vec{p}_{EM}$ down the coaxial cable! Microscopically, *EM* energy ***and*** linear momentum are transported down the coax cable by the {ensemble of} virtual photons associated with the $\vec{E}$ and $\vec{B}$ fields in the region $a \le \rho \le b$.

Transport of non-zero linear momentum down the coax cable might seem bizarre at first encounter, because macroscopically, this is a ***static*** problem – we have a coax cable (at rest in the lab frame), a battery producing a static $\vec{E}$-field and static electric charge distribution, as well as a steady / DC current *I* and static $\vec{B}$-field. How can there be ***any*** net macroscopic linear momentum?

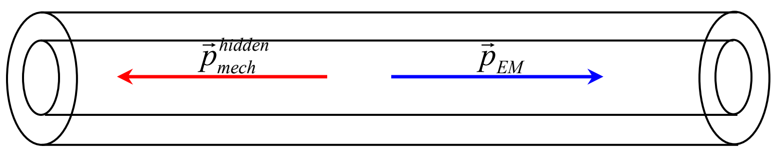
The answer is: There isn't, because there exists a "hidden" mechanical momentum: Microscopically, virtual photons associated with macroscopic $\vec{E}$ and $\vec{B}$ fields are emitted (and absorbed) by electric charges (*e.g.* conduction / "free" electrons flowing as macroscopic current *I* down / back along coax cable and as static charges on conducting surfaces of coax cable (with potential difference ΔV across coax cable). As stated before, virtual photons carry both kinetic energy *K.E.* ***and*** momentum $\vec{p}$.

In the emission (and absorption) process – *e.g.* an electron emitting a virtual photon, again, energy and momentum are conserved (microscopically) – the electron "recoils" emitting a virtual photon, analogous to a rifle firing a bullet:



n.b. emission / absorption of virtual photons *e.g.* by an isolated electron is responsible for a phenomenon known as ***zitterbewegung*** – "trembling motion" of the electron…

The sum total of all electric charges emitting virtual photons gives a net macroscopic "mechanical" linear momentum that is equal / but in the ***opposite*** direction to the net *EM* field momentum. At the microscopic level, individually recoiling electrons (rapidly) interact with the surrounding matter (atoms) making up the coax cable – scattering off of them – thus this (net) recoil momentum (initially associated only with virtual photon-emitting electric charges) very rapidly winds up being transferred (via subsequent scattering interactions with atoms) to the whole/entire macroscopic physical system (here, the coax cable):



$$\vec{p}_{mech}^{\,hidden} = -\vec{p}_{EM}$$

Total linear momentum is conserved in a closed system of volume v (enclosing coax cable):

$$\vec{p}_{tot} = \vec{p}_{EM} + \vec{p}_{mech}^{\,hidden} = \vec{p}_{EM} - \vec{p}_{EM} = 0$$

Now imagine that the resistance R of the load resistor "magically" increases linearly with time {*e.g.* imagine the resistor to be a linear potentiometer (a linear, knob-variable resistor), so we can simply slowly rotate the knob on the linear potentiometer *CW* with time} which causes the current I flowing in the circuit to (slowly) decrease linearly with time.

Then, the slowly linearly-decreasing current will correspondingly have associated with it a slowly linearly-decreasing magnetic field; thus the linearly ***changing*** magnetic field will induce an electric field - via Faraday's Law - using either $\vec{\nabla}\times\vec{E} = -\frac{d\vec{B}}{dt}$ or $\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$:

$$\vec{E}^{ind}(\rho,t) = \left[\frac{\mu_o}{2\pi}\frac{dI(t)}{dt}\ln\rho + K\right]\hat{z}$$ {where K = a constant of integration}

This induced $\vec{E}$-field exerts a net force $\Delta\vec{F}_{ab}^{ind}(t) \equiv \vec{F}_a^{ind}(\rho=a,t) - \vec{F}_b^{ind}(\rho=b,t)$ on the $\pm\lambda$ charges residing on the inner/outer cylinders of the coax cable {where $\vec{F}_i^{ind} = Q_i\vec{E}_i^{ind}$, $i = a,b$} of:

$$\Delta\vec{F}_{ab}^{ind}(t) = \lambda\ell\left[\frac{\mu_o}{2\pi}\frac{dI(t)}{dt}\ln a + K\right]\hat{z} - \lambda\ell\left[\frac{\mu_o}{2\pi}\frac{dI(t)}{dt}\ln b + K\right]\hat{z} = -\frac{\mu_o}{2\pi}\lambda\ell\frac{dI(t)}{dt}\ln\left(\frac{b}{a}\right)\hat{z}$$ {*n.b.* points in the $\hat{z}$-direction}

The ***total***/***net*** mechanical linear momentum ***imparted*** to the coax cable as the current slowly decreases from $I(t=0)=I$ to $I(t=t_{final})=0$, using $\boxed{d\left[\Delta\vec{p}_{mech}(t)\right]=\Delta\vec{F}(t)\,dt}$ is:

$$\boxed{\begin{aligned}\delta\left(\Delta\vec{p}_{mech}\right) &\equiv \int_{\Delta\vec{p}^{mech}_{final}(t=0)=0}^{\Delta\vec{p}^{mech}_{final}(t=t_{final})} d\left[\Delta\vec{p}_{mech}(t)\right] = \Delta\vec{p}^{mech}_{final}\left(t=t_{final}\right) - \underbrace{\cancel{\Delta\vec{p}^{mech}_{final}(t=0)}}_{=0} = \Delta\vec{p}^{mech}_{final}\left(t=t_{final}\right) \\ &= \int_{t=0}^{t=t_{final}} \Delta\vec{F}^{ind}_{ab}(t)\,dt = -\frac{\mu_o}{2\pi}\lambda\ell\left[\int_{t=0}^{t=t_{final}}\frac{dI(t)}{dt}dt\right]\ln\left(\frac{b}{a}\right)\hat{z} \\ &= -\frac{\mu_o}{2\pi}\lambda\ell\left[\int_{I(t=0)=I}^{I(t=t_{final})=0} dI(t)\right]\ln\left(\frac{b}{a}\right)\hat{z} = +\frac{\mu_o\lambda I\ell}{2\pi}\ln\left(\frac{b}{a}\right)\hat{z} = \frac{\Delta VI\ell}{c^2}\hat{z}\end{aligned}}$$

which is ***precisely*** equal to the ***original*** *EM* field momentum$(t\le 0)$: $\boxed{\vec{p}_{EM}=\frac{\mu_o\lambda I\ell}{2\pi}\ln\left(\frac{b}{a}\right)\hat{z}=\frac{\Delta VI\ell}{c^2}\hat{z}}$.

Note that the coax cable will not recoil, because the equal, but opposite impulse is delivered to the coax cable by the "hidden" momentum, microscopically (and macroscopically), in just the same way as described above.

Note further that energy and momentum are able to be transported down the coax cable ***because*** there exists a ***non-zero*** Poynting's vector $\vec{S}=\frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)\ne 0$ and a non-zero linear momentum density $\vec{\wp}_{EM}=\vec{S}(\vec{r},t)/c^2$ due to the {***radial***} electric field $\vec{E}$ in the region $a\le\rho\le b$, arising from the presence of ***static*** electric charges on the surfaces of the inner & outer cylinders, in conjunction with the {***azimuthal***} magnetic field $\vec{B}$ associated with the steady current I flowing down the coax cable. If either $\vec{E}$ or $\vec{B}$ were zero, or their cross-product $\vec{E}\times\vec{B}$ were zero, there would be no transport of *EM* energy & linear momentum down the cable.

Recall that the capacitance C of an electrical device is associated with the ability to store energy in the electric field $\vec{E}$ of that device, and that the {self-} inductance L of an electrical device is associated with the ability to store energy in the magnetic field $\vec{B}$ of that device. We thus realize that:

- The capacitance of the coax cable $\boxed{C=Q/\Delta V=2\pi\varepsilon_o\ell/\ln(b/a)}\Rightarrow\boxed{Q=C\Delta V}$ is responsible for the presence of the surface charges $\sigma_+=+Q/A_{inner}$ and $\sigma_-=-Q/A_{outer}$ on the inner & outer conductors of the coax cable when a potential difference ΔV is imposed between the inner/outer conductors, which also gives rise to the existence of the transverse/radial electric field $\vec{E}=-\vec{\nabla}V$ in the region $a\le\rho\le b$. The energy stored in the electric field $\vec{E}$ of the coax cable is $\boxed{U_E=\tfrac{1}{2}C\Delta V^2=\tfrac{1}{4\pi\varepsilon_o}\lambda^2\ell\ln(b/a)}$ (Joules).

- The {self-} inductance of the coax cable $\boxed{L=\Phi_M/I=\tfrac{\mu_o}{4\pi}\ell\ln(b/a)}\Rightarrow\boxed{\Phi_M=LI=\int_S\vec{B}\cdot d\vec{a}}$ is associated with the azimuthal magnetic field $\vec{B}$ for the in the region $a\le\rho\le b$ resulting from the flow of electrical current I down the inner conductor. The energy stored in the magnetic field $\vec{B}$ of the coax cable is $\boxed{U_M=\tfrac{1}{2}LI^2=\tfrac{\mu_o}{8\pi}I^2\ell\ln(b/a)}$ (Joules).

- The <u>total</u> *EM* energy stored in the coax cable {using the principle of linear superposition} is the sum of these two electric-only and magnetic-only energies:

$$\boxed{U_{Tot} = U_E + U_M = \tfrac{1}{2}C\Delta V^2 + \tfrac{1}{2}LI^2 = \tfrac{1}{4\pi\varepsilon_o}\lambda^2 \ell \ln(b/a) + \tfrac{\mu_o}{8\pi} I^2 \ell \ln(b/a) = \tfrac{1}{4\pi\varepsilon_o}\left(\lambda^2 + \tfrac{1}{2}(I/c)^2\right)\ell \ln(b/a)}$$

- *EM* power transport in a electrical device occurs via the electromagnetic field and <u>necessarily</u> <u>*requires*</u> $\vec{S} = \tfrac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right) \neq 0$ (*i.e.* <u>both</u> $\vec{E}$ <u>*and*</u> $\vec{B}$ <u>*must*</u> be non-zero, <u>*and*</u> must be such that they <u>*also*</u> have non-zero cross-product $\vec{E}\times\vec{B}$).

- *EM* power transport in a electrical device <u>necessarily</u> <u>*requires*</u> the utilization of <u>*both*</u> the capacitance *C* <u>*and*</u> the {self-}inductance *L* of the device in order to do so!

- The *EM* <u>power</u> transported from the battery down the coax cable to the resistor (where it is ultimately dissipated as heat/thermal energy) is: $\boxed{P = \tfrac{1}{2\pi\varepsilon_o}\lambda I \ln(b/a) = \Delta V * I}$ (Watts = J/sec) {*n.b.* the *EM* power is proportional to the <u>product</u> of the charge {per unit length} and the current λI }. But $\boxed{\Delta V = Q/C}$ and $\boxed{I = \Phi_M / L}$; from Gauss' Law $\boxed{\Phi_E \equiv \int_S \vec{E}\bullet d\vec{a} = Q_{encl}/\varepsilon_o}$ we see that $\boxed{P = \tfrac{1}{2\pi\varepsilon_o}\lambda I \ln(b/a) = \Delta V * I = Q\Phi_M / CL = \varepsilon_o\, \Phi_E \Phi_M / CL}$,
  i.e. *EM* power transport in/through an electrical device manifestly involves:
  a.) the <u>product</u> of the electric <u>*and*</u> magnetic fluxes: $\Phi_E\Phi_M$ <u>*and*</u>
  b.) the <u>product</u> of the device's capacitance <u>*and*</u> its inductance: CL !!!

A (Brief) Review of Tensors/Tensor Properties:

Maxwell's stress tensor is a rank-2 3×3 tensor: $\overleftrightarrow{T} = \left(T_{ij}\right) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$

i = ***row*** index
j = ***column*** index

Note that since $\overleftrightarrow{T}$ is a "double vector" the above expression is actually "*short-hand*" notation for:

$$\overleftrightarrow{T} \equiv \left(\overleftrightarrow{T}_{ij}\right) = \begin{pmatrix} \overleftrightarrow{T}_{11} & \overleftrightarrow{T}_{12} & \overleftrightarrow{T}_{13} \\ \overleftrightarrow{T}_{21} & \overleftrightarrow{T}_{22} & \overleftrightarrow{T}_{23} \\ \overleftrightarrow{T}_{31} & \overleftrightarrow{T}_{32} & \overleftrightarrow{T}_{33} \end{pmatrix} = \begin{pmatrix} \overleftrightarrow{T}_{xx} & \overleftrightarrow{T}_{xy} & \overleftrightarrow{T}_{xz} \\ \overleftrightarrow{T}_{yx} & \overleftrightarrow{T}_{yy} & \overleftrightarrow{T}_{yz} \\ \overleftrightarrow{T}_{zx} & \overleftrightarrow{T}_{zy} & \overleftrightarrow{T}_{zz} \end{pmatrix} = \begin{pmatrix} T_{xx}\hat{x}\hat{x} & T_{xy}\hat{x}\hat{y} & T_{xz}\hat{x}\hat{z} \\ T_{yx}\hat{y}\hat{x} & T_{yy}\hat{y}\hat{y} & T_{yz}\hat{y}\hat{z} \\ T_{zx}\hat{z}\hat{x} & T_{zy}\hat{z}\hat{y} & T_{zz}\hat{z}\hat{z} \end{pmatrix}$$

Dot-product multiplication of a tensor with a vector – there exist ***two*** types:

1.) $\boxed{\vec{b} = \vec{a}\cdot\overleftrightarrow{T}}$: $\boxed{b_j = \left(\vec{a}\cdot\overleftrightarrow{T}\right)_j = \sum_{i=1}^{3} a_i T_{ij} = a_i T_{ij}}$ *n.b.* summation is over index i (*i.e.* ***rows*** in $\overleftrightarrow{T}$)!!

The matrix representation of vector $\vec{a} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$ = (1×3) *row* vector, and also vector $\vec{b} = \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$ = (1×3) *row* vector. Thus, $\vec{b} = \vec{a}\cdot\overleftrightarrow{T}$ can thus be written in matrix form as:

$$\underbrace{\begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}}_{(1\times3)} = \underbrace{\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}}_{(1\times3)} \underbrace{\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}}_{(3\times3)}$$

$$= \underbrace{\begin{pmatrix} a_1T_{11} + a_2T_{21} + a_3T_{31} & a_1T_{12} + a_2T_{22} + a_3T_{32} & a_1T_{13} + a_2T_{23} + a_3T_{33} \end{pmatrix}}_{(1\times3)}$$

2.) $\boxed{\vec{c} = \overleftrightarrow{T}\cdot\vec{a}}$: $\boxed{c_i = \left(\overleftrightarrow{T}\cdot\vec{a}\right)_i = \sum_{j=1}^{3} T_{ij} a_j = T_{ij} a_j}$ *n.b.* summation is over index j (*i.e.* ***columns*** in $\overleftrightarrow{T}$)!!

Here, the matrix representation of vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ = (3×1) *column* vector, and also vector

$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ = (3×1) *column* vector. Thus, $\vec{c} = \overleftrightarrow{T}\cdot\vec{a}$ can thus be written in matrix form as:

$$\underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}_{(3\times1)} = \underbrace{\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}}_{(3\times3)} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{(3\times1)} = \underbrace{\begin{pmatrix} a_1T_{11} + a_2T_{12} + a_3T_{13} \\ a_1T_{21} + a_2T_{22} + a_3T_{23} \\ a_1T_{31} + a_2T_{32} + a_3T_{33} \end{pmatrix}}_{(3\times1)}$$

Note that if $\overleftrightarrow{T}$ is a *<u>symmetric</u>* tensor, *i.e.* $T_{ij} = +T_{ji}$, then $\vec{c} = \overleftrightarrow{T}\bullet\vec{a}$ can be equivalently written in matrix form as:

$$\underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}_{(3\times1)} = \underbrace{\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}}_{(3\times3)} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{(3\times1)} = \underbrace{\begin{pmatrix} a_1T_{11} + a_2T_{12} + a_3T_{13} \\ a_1T_{21} + a_2T_{22} + a_3T_{23} \\ a_1T_{31} + a_2T_{32} + a_3T_{33} \end{pmatrix}}_{(3\times1)} = \underbrace{\begin{pmatrix} a_1T_{11} + a_2T_{21} + a_3T_{31} \\ a_1T_{12} + a_2T_{22} + a_3T_{32} \\ a_1T_{13} + a_2T_{23} + a_3T_{33} \end{pmatrix}}_{(3\times1)}$$

But $\vec{c} = \vec{a}\bullet\overleftrightarrow{T}$ written in matrix form is:

$$\underbrace{\begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix}}_{(1\times3)} = \underbrace{\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}}_{(1\times3)} \underbrace{\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}}_{(3\times3)}$$

$$= \underbrace{\begin{pmatrix} a_1T_{11} + a_2T_{21} + a_3T_{31} & a_1T_{12} + a_2T_{22} + a_3T_{32} & a_1T_{13} + a_2T_{23} + a_3T_{33} \end{pmatrix}}_{(1\times3)}$$

Thus, we see/learn that if $\overleftrightarrow{T}$ is a *<u>symmetric</u>* tensor, then: $\vec{c} = \vec{a}\bullet\overleftrightarrow{T} = \overleftrightarrow{T}\bullet\vec{a}$.

If $\overleftrightarrow{T}$ is ***<u>not</u>*** a *<u>symmetric</u>* tensor, then: $\vec{a}\bullet\overleftrightarrow{T} \neq \overleftrightarrow{T}\bullet\vec{a}$.

In general, matrix multiplication $AB \neq BA$ because matrices A and B do not in general ***<u>commute</u>***.

Note further, from the above results, we also see/learn that:

- The vector $\vec{b} = \vec{a}\bullet\overleftrightarrow{T}$ is ***<u>not</u>*** in general parallel to the vector $\vec{a}$.
- Similarly, the vector $\vec{c} = \overleftrightarrow{T}\bullet\vec{a}$ is (also) ***<u>not</u>*** in general parallel to the vector $\vec{a}$.

The divergence of a tensor $\vec{\nabla}\bullet\overleftrightarrow{T}$ is a vector, and has the same mathematical structure as that of $\vec{b} = \vec{a}\bullet\overleftrightarrow{T}$:

$\left(\vec{\nabla}\bullet\overleftrightarrow{T}\right)_j = \sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial r_i}$ *n.b.* summation is over index i (*i.e.* ***<u>rows</u>*** in $\overleftrightarrow{T}$)!!

$$\underbrace{\left(\vec{\nabla}\bullet\overleftrightarrow{T}\right)}_{(1\times3)} = \underbrace{\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}}_{(1\times3)} \underbrace{\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}}_{(3\times3)} = \underbrace{\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}}_{(1\times3)} \underbrace{\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}}_{(3\times3)}$$

$$= \underbrace{\left(\left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \right) \quad \left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \right) \quad \left(\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \right) \right)}_{(1\times3)}$$

$$= \underbrace{\left(\left(\vec{\nabla}\bullet\overleftrightarrow{T}\right)_x \quad \left(\vec{\nabla}\bullet\overleftrightarrow{T}\right)_y \quad \left(\vec{\nabla}\bullet\overleftrightarrow{T}\right)_z \right)}_{(1\times3)}$$

