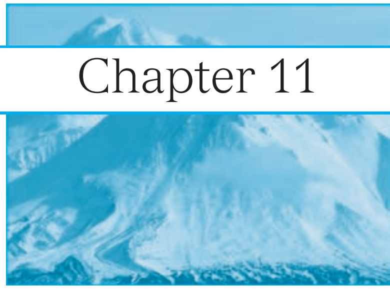


Chapter 11

Legendre Polynomials and Spherical Harmonics

11.1 Introduction

Legendre polynomials appear in many different mathematical and physical situations:

- They originate as solutions of the Legendre ordinary differential equation (ODE), which we have already encountered in the separation of variables (Section 8.9) for Laplace's equation, and similar ODEs in spherical polar coordinates.
- They arise as a consequence of demanding a complete, orthogonal set of functions over the interval $[-1, 1]$ (Gram–Schmidt orthogonalization; Section 9.3).
- In quantum mechanics, they (really the spherical harmonics; Section 11.5) represent angular momentum eigenfunctions. They also appear naturally in problems with azimuthal symmetry, which is the case in the next point.
- They are defined by a generating function: We introduce Legendre polynomials here by way of the electrostatic potential of a point charge, which acts as the generating function.

Physical Basis: Electrostatics

Legendre polynomials appear in an expansion of the electrostatic potential in inverse radial powers. Consider an electric charge q placed on the z-axis at $z = a$. As shown in Fig. 11.1, the electrostatic potential of charge q is

$$\varphi = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r_1} \quad \text{(SI units)}. \tag{11.1}$$

We want to express the electrostatic potential in terms of the spherical polar coordinates r and θ (the coordinate φ is absent because of azimuthal symmetry,

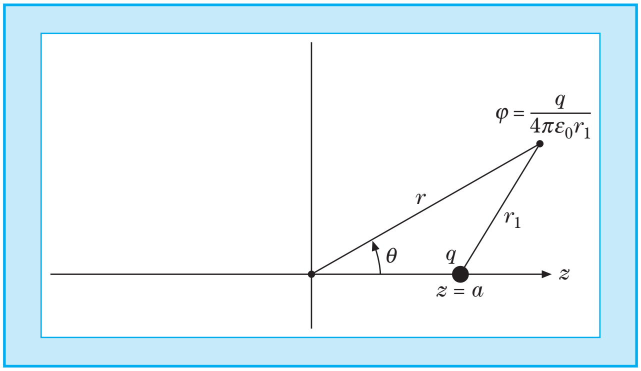


Figure 11.1

Electrostatic Potential. Charge q Displaced from Origin

that is, invariance under rotations about the z-axis). Using the law of cosines in Fig. 11.1, we obtain

$$\varphi = \frac{q}{4\pi\varepsilon_0}(r^2 + a^2 - 2ar\cos\theta)^{-1/2}. \quad (11.2)$$

Generating Function

Consider the case of $r > a$. The radical in Eq. (11.2) may be expanded in a binomial series (see Exercise 5.6.9) for $r^2 > |a^2 - 2ar\cos\theta|$ and then rearranged in powers of (a/r). This yields the coefficient 1 of $(a/r)^0 = 1$, $\cos\theta$ as coefficient of a/r, etc. The **Legendre polynomial** $P_n(\cos\theta)$ (Fig. 11.2) **is defined as the coefficient of** $(a/r)^n$ so that

$$\varphi = \frac{q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{a}{r}\right)^n. \quad (11.3)$$

Dropping the factor $q/4\pi\varepsilon_0 r$ and using $x = \cos\theta$ and $t = a/r$, respectively, we have

$$g(t, x) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |t| < 1, \quad (11.4)$$

defining $g(t, x)$ as the **generating function for the** $P_n(x)$. These polynomials $P_n(x)$, shown in Table 11.1, are the same as those generated in Example 9.3.1 by Gram–Schmidt orthogonalization of powers x^n over the interval $-1 \le x \le 1$. This is no coincidence because $\cos\theta$ varies between the limits ± 1. In the next section, it is shown that $|P_n(\cos\theta)| \le 1$, which means that the series expansion [Eq. (11.4)] is convergent for $|t| < 1$.[1] Indeed, the series is convergent for $|t| = 1$ except for $x = \pm 1$, where $|P_n(\pm 1)| = 1$.

[1]Note that the series in Eq. (11.3) is convergent for $r > a$ even though the binomial expansion involved is valid only for $r > (a^2 + 2ar)^{1/2} \ge |a^2 - 2ar\cos\theta|^{1/2}$ so that $r^2 > a^2 + 2ar$, $(r - a)^2 > 2a^2$, or $r > a(1 + \sqrt{2})$.

Figure 11.2

Legendre Polynomials $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$

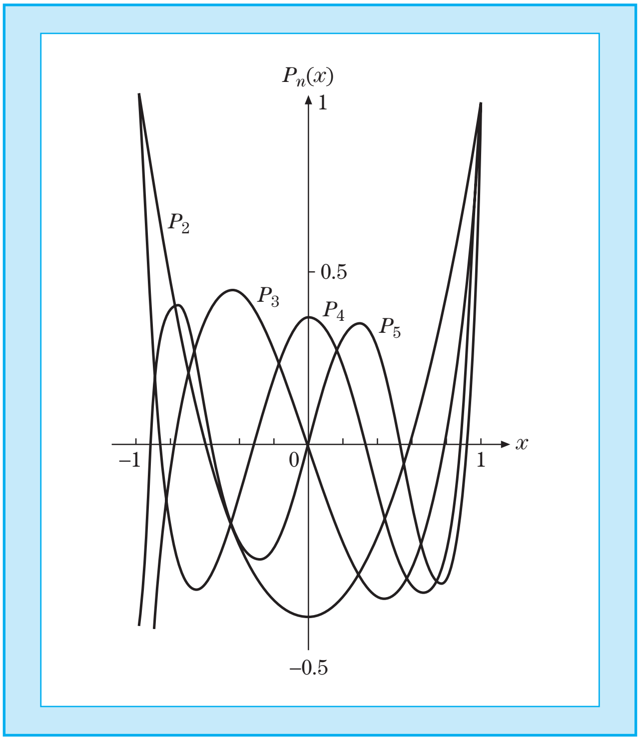


Table 11.1

Legendre Polynomials

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$
$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$
$P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$
$P_8(x) = \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$

In physical applications, such as the Coulomb or gravitational potentials, Eq. (11.4) often appears in the vector form

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \left[r_1^2 + r_2^2 - 2\mathbf{r}_1 \cdot \mathbf{r}_2\right]^{-1/2} = \frac{1}{r_1}\left[1 + \left(\frac{r_2}{r_1}\right)^2 - 2\left(\frac{r_2}{r_1}\right)\cos\theta\right]^{-1/2}.$$

The last equality is obtained by factoring $r_1 = |\mathbf{r}_1|$ from the denominator, which then, for $r_1 > r_2$, we can expand according to Eq. (11.4). In this way, we obtain

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^n P_n(\cos\theta), \tag{11.4a}$$

where

$$\left.\begin{aligned} r_> &= |\mathbf{r}_1| \\ r_< &= |\mathbf{r}_2| \end{aligned}\right\} \quad \text{for} \quad |\mathbf{r}_1| > |\mathbf{r}_2|, \tag{11.4b}$$

and

$$\left.\begin{aligned} r_> &= |\mathbf{r}_2| \\ r_< &= |\mathbf{r}_1| \end{aligned}\right\} \quad \text{for} \quad |\mathbf{r}_2| > |\mathbf{r}_1|. \tag{11.4c}$$

EXAMPLE 11.1.1

Special Values A simple and powerful application of the generating function g is to use it for special values (e.g., $x = \pm 1$) where g can be evaluated explicitly. If we set $x = 1$, that is, point $z = a$ on the positive z-axis, where the potential has a simple form, Eq. (11.4) becomes

$$\frac{1}{(1 - 2t + t^2)^{1/2}} = \frac{1}{1 - t} = \sum_{n=0}^{\infty} t^n, \tag{11.5}$$

using a binomial expansion or the geometric series (Example 5.1.2). However, Eq. (11.4) for $x = 1$ defines

$$\frac{1}{(1 - 2t + t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(1) t^n.$$

Comparing the two series expansions (uniqueness of power series; Section 5.7), we have

$$P_n(1) = 1. \tag{11.6}$$

If we let $x = -1$ in Eq. (11.4), that is, point $z = -a$ on the negative z-axis in Fig. 11.1, where the potential is simple, then we sum similarly

$$\frac{1}{(1 + 2t + t^2)^{1/2}} = \frac{1}{1 + t} = \sum_{n=0}^{\infty} (-t)^n \tag{11.7}$$

so that

$$P_n(-1) = (-1)^n. \tag{11.8}$$

These general results are more difficult to develop from other formulas for Legendre polynomials.

If we take $x = 0$ in Eq. (11.4), using the binomial expansion

$$(1 + t^2)^{-1/2} = 1 - \tfrac{1}{2} t^2 + \tfrac{3}{8} t^4 + \cdots + (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} t^{2n} + \cdots, \tag{11.9}$$

we have[2]

$$P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n n!} = (-1)^n \frac{(2n-1)!!}{(2n)!!} = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \tag{11.10}$$

$$P_{2n+1}(0) = 0, \quad n = 0, 1, 2 \ldots. \tag{11.11}$$

These results can also be verified by inspection of Table 11.1. ■

EXAMPLE 11.1.2

Parity If we replace x by $-x$ and t by $-t$, the generating function is unchanged. Hence,

$$\begin{aligned} g(t, x) = g(-t, -x) &= [1 - 2(-t)(-x) + (-t)^2]^{-1/2} \\ &= \sum_{n=0}^{\infty} P_n(-x)(-t)^n = \sum_{n=0}^{\infty} P_n(x) t^n. \end{aligned} \tag{11.12}$$

Comparing these two series, we have

$$P_n(-x) = (-1)^n P_n(x); \tag{11.13}$$

that is, the polynomial functions are odd or even (with respect to $x = 0$) according to whether the index n is odd or even. This is the parity[3] or reflection property that plays such an important role in quantum mechanics. Parity is conserved when the Hamiltonian is invariant under the reflection of the coordinates $\mathbf{r} \to -\mathbf{r}$. For central forces the index n is the orbital angular momentum [and $n(n+1)$ is the eigenvalue of $\mathbf{L}^2$], thus linking parity and orbital angular momentum. This parity property will be confirmed by the series solution and for the special cases tabulated in Table 11.1. ■

Power Series

Using the binomial theorem (Section 5.6) and Exercise 10.1.15, we expand the generating function as

$$\begin{aligned} (1 - 2xt + t^2)^{-1/2} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} (2xt - t^2)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} (2xt - t^2)^n. \end{aligned} \tag{11.14}$$

Before we expand $(2xt - t^2)^n$ further, let us inspect the lowest powers of t.

---

[2] The double factorial notation is defined in Section 10.1:

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n), \quad (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

[3] In spherical polar coordinates the inversion of the point (r, θ, φ) through the origin is accomplished by the transformation $[r \to r, \theta \to \pi - \theta$, and $\varphi \to \varphi \pm \pi]$. Then, $\cos\theta \to \cos(\pi - \theta) = -\cos\theta$, corresponding to $x \to -x$.

EXAMPLE 11.1.3

Lowest Legendre Polynomials For the first few Legendre polynomials (e.g., P_0, P_1, and P_2), we need the coefficients of t^0, t^1, and t^2 in Eq. (11.14). These powers of t appear only in the terms $n = 0$, 1, and 2; hence, we may limit our attention to the first three terms of the infinite series:

$$\frac{0!}{2^0(0!)^2}(2xt - t^2)^0 + \frac{2!}{2^2(1!)^2}(2xt - t^2)^1 + \frac{4!}{2^4(2!)^2}(2xt - t^2)^2$$
$$= 1t^0 + xt^1 + \left(\frac{3}{2}x^2 - \frac{1}{2}\right)t^2 + \mathcal{O}(t^3).$$

Then, from Eq. (11.4) (and uniqueness of power series) we obtain

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \tag{11.15}$$

confirming the entries of Table 11.1. We repeat this limited development in a vector framework later in this section. ■

In employing a general treatment, we find that the binomial expansion of the $(2xt - t^2)^n$ factor yields the double series

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} t^n \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}(2x)^{n-k}t^k$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{(2n)!}{2^{2n}n!k!(n-k)!} \cdot (2x)^{n-k} t^{n+k}. \tag{11.16}$$

By rearranging the order of summation (valid by absolute convergence), Eq. (11.16) becomes

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^{2n-2k}k!(n-k)!(n-2k)!} \cdot (2x)^{n-2k} t^n, \tag{11.17}$$

with the t^n independent of the index k.[4] Now, equating our two power series [Eqs. (11.4) and (11.17)] term by term, we have[5]

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}. \tag{11.18}$$

We read off this formula, from $k = 0$, that the highest power of $P_n(x)$ is x^n, and the lowest power is $x^0 = 1$ for even n and x for odd n. This is consistent with Example 11.1.3 and Table 11.1. Also, for n even, P_n has only even powers of x and thus even parity [see Eq. (11.13)] and odd powers and odd parity for odd n.

[4] $[n/2] = n/2$ for n even, $(n-1)/2$ for n odd.

[5] Equation (11.18) starts with x^n. By changing the index, we can transform it into a series that starts with x^0 for n even and x^1 for n odd.

Biographical Data

Legendre, Adrien Marie. Legendre, a French mathematician who was born in Paris in 1752 and died there in 1833, made major contributions to number theory, elliptic integrals before Abel and Jacobi, and analysis. He tried in vain to prove the parallel axiom of Euclidean geometry. His taste in selecting research problems was remarkably similar to that of his contemporary Gauss, but nobody could match Gauss's depth and perfection. His great textbooks had enormous impact.

Linear Electric Multipoles

Returning to the electric charge on the z-axis, we demonstrate the usefulness and power of the generating function by adding a charge $-q$ at $z = -a$, as shown in Fig. 11.3, using the superposition principle of electric fields. The potential becomes

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \tag{11.19}$$

and by using the law of cosines, we have for $r > a$

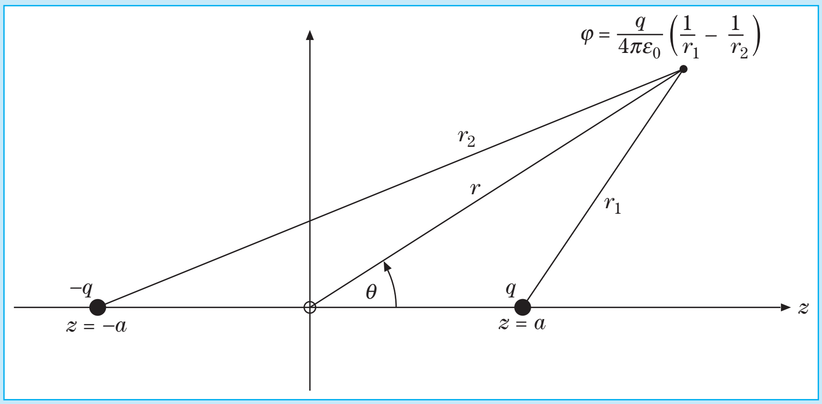
$$\varphi = \frac{q}{4\pi\varepsilon_0 r}\left\{\left[1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right]^{-1/2} - \left[1 + 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right]^{-1/2}\right\},$$

where the second radical is like the first, except that a has been replaced by $-a$. Then, using Eq. (11.4), we obtain

$$\begin{aligned}\varphi &= \frac{q}{4\pi\varepsilon_0 r}\left[\sum_{n=0}^{\infty} P_n(\cos)\left(\frac{a}{r}\right)^n - \sum_{n=0}^{\infty} P_n(\cos\theta)(-1)^n\left(\frac{a}{r}\right)^n\right] \\ &= \frac{2q}{4\pi\varepsilon_0 r}\left[P_1(\cos\theta)\frac{a}{r} + P_3(\cos\theta)\left(\frac{a}{r}\right)^3 + \cdots\right]. \end{aligned} \tag{11.20}$$

Figure 11.3

Electric Dipole



The first term (and dominant term for $r \gg a$) is the **electric dipole** potential

$$\varphi = \frac{2aq}{4\pi\varepsilon_0} \cdot \frac{P_1(\cos\theta)}{r^2}, \tag{11.21}$$

with $2aq$ the **electric dipole moment** (Fig. 11.3). If the potential in Eq. (11.19) is taken to be the dipole potential, then Eq. (11.21) gives its asymptotic behavior for large r. This analysis may be extended by placing additional charges on the z-axis so that the P_1 term, as well as the P_0 (monopole) term, is canceled. For instance, charges of q at $z = a$ and $z = -a$, $-2q$ at $z = 0$ give rise to a potential whose series expansion starts with $P_2(\cos\theta)$. This is a linear electric quadrupole. Two linear quadrupoles may be placed so that the quadrupole term is canceled, but the P_3, the electric octupole term, survives, etc. These expansions are special cases of the general multipole expansion of the electric potential.

Vector Expansion

We consider the electrostatic potential produced by a distributed charge $\rho(\mathbf{r}_2)$:

$$\varphi(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\tau_2. \tag{11.22a}$$

Taking the denominator of the integrand, using first the law of cosines and then a binomial expansion, yields

$$\begin{aligned} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} &= \left(r_1^2 - 2\mathbf{r}_1 \cdot \mathbf{r}_2 + r_2^2\right)^{-1/2} \qquad (11.22\text{b}) \\ &= \frac{1}{r_1}\left[1 + \left(-\frac{2\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1^2} + \frac{r_2^2}{r_1^2}\right)\right]^{-1/2}, \quad \text{for } r_1 > r_2 \\ &= \frac{1}{r_1}\left[1 + \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1^2} - \frac{1}{2}\frac{r_2^2}{r_1^2} + \frac{3}{2}\frac{(\mathbf{r}_1 \cdot \mathbf{r}_2)^2}{r_1^4} + \mathcal{O}\left(\frac{r_2}{r_1}\right)^3\right]. \end{aligned}$$

For $r_1 = 1$, $r_2 = t$, and $\mathbf{r}_1 \cdot \mathbf{r}_2 = xt$, Eq. (11.22b) reduces to the generating function, Eq. (11.4).

The first term in the square bracket, 1, yields a potential

$$\varphi_0(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r_1} \int \rho(\mathbf{r}_2) d\tau_2. \tag{11.22c}$$

The integral contains the total charge. This part of the total potential is an electric **monopole**.

The second term yields

$$\varphi_1(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{r}_1 \cdot}{r_1^3} \int \mathbf{r}_2 \rho(\mathbf{r}_2) d\tau_2, \tag{11.22d}$$

where the integral is the dipole moment whose charge density $\rho(\mathbf{r}_2)$ is weighted by a moment arm $\mathbf{r}_2$. We have an electric dipole potential. For atomic or nuclear states of definite parity, $\rho(\mathbf{r}_2)$ is an even function and the dipole integral is identically zero. However, in the presence of an applied electric field a superposition of odd/even parity states may develop so that the resulting induced dipole moment is no longer zero. The last two terms, both of order $(r_2/r_1)^2$, may be handled by using Cartesian coordinates

$$(\mathbf{r}_1 \cdot \mathbf{r}_2)^2 = \sum_{i=1}^{3} x_{1i}x_{2i} \sum_{j=1}^{3} x_{1j}x_{2j}.$$

Rearranging variables to keep the x_2 inside the integral yields

$$\varphi_2(\mathbf{r}_1) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r_1^5}\sum_{i,j=1}^{3} x_{1i}x_{1j}\int \left[3x_{2i}x_{2j} - \delta_{ij}r_2^2\right]\rho(\mathbf{r}_2)d\tau_2. \tag{11.22e}$$

This is the electric **quadrupole** term. Note that the square bracket in the integrand forms a symmetric tensor of zero trace.

A general electrostatic **multipole expansion** can also be developed by using Eq. (11.22a) for the potential $\varphi(r_1)$ and replacing $1/(4\pi|\mathbf{r}_1 - \mathbf{r}_2|)$ by a (double) series of the angular solutions of the Poisson equation (which are the same as those of the Laplace equation of Section 8.9).

Before leaving multipole fields, we emphasize three points:

- First, an electric (or magnetic) multipole has a value independent of the origin (reference point) only if all lower order terms vanish. For instance, the potential of one charge q at $z = a$ was expanded in a series of Legendre polynomials. Although we refer to the $P_1(\cos\theta)$ term in this expansion as a dipole term, it should be remembered that this term exists only because of our choice of coordinates. We actually have a monopole, $P_0(\cos\theta)$, the term of leading magnitude.
- Second, in physical systems we rarely encounter pure multipoles. For example, the potential of the finite dipole (q at $z = a, -q$ at $z = -a$) contained a $P_3(\cos\theta)$ term. These higher order terms may be eliminated by shrinking the multipole to a point multipole, in this case keeping the product qa constant ($a \to 0, q \to \infty$) to maintain the same dipole moment.
- Third, the multipole expansion is not restricted to electrical phenomena. Planetary configurations are described in terms of mass multipoles.

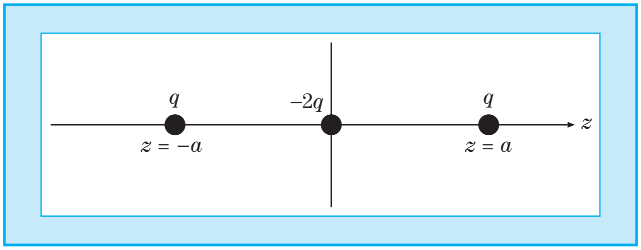
It might also be noted that a multipole expansion is actually a decomposition into the irreducible representations of the rotation group (Section 4.2). The lth multipole involves the eigenfunctions of orbital angular momentum, $|lm\rangle$, one for each component m of the multipole l (see Chapter 4). These $2l + 1$ components of the multipole form an irreducible representation because the lowering operator L_- applied repeatedly to the eigenfunction $|ll\rangle$ generates all other eigenfunctions $|lm\rangle$, down to $m = -l$. The raising and lowering operators $L_\pm$ are generators of the rotation group along with L_z, whose eigenvalue is m.

EXERCISES

11.1.1 Develop the electrostatic potential for the array of charges shown. This is a linear electric quadrupole (Fig. 11.4).

Figure 11.4

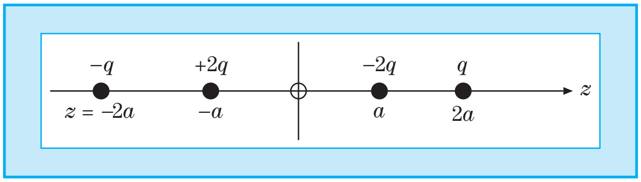
Linear Electric Quadrupole



11.1.2 Calculate the electrostatic potential of the array of charges shown in Fig. 11.5. This is an example of two equal but oppositely directed dipoles. The dipole contributions cancel, but the octupole terms do not cancel.

Figure 11.5

Linear Electric Octupole



11.1.3 Show that the electrostatic potential produced by a charge q at $z = a$ for $r < a$ is

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta).$$

11.1.4 Using $\mathbf{E} = -\boldsymbol{\nabla}\varphi$, determine the components of the electric field corresponding to the (pure) electric dipole potential

$$\varphi(\mathbf{r}) = \frac{2aq\,P_1(\cos\theta)}{4\pi\varepsilon_0 r^2}.$$

Here, it is assumed that $r \gg a$.

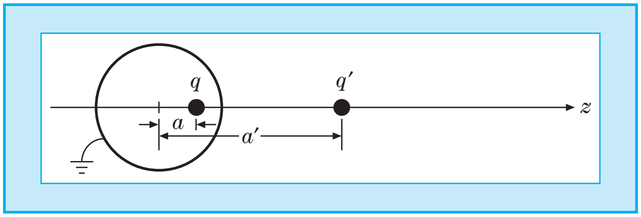
$$\textit{ANS.}\quad E_r = +\frac{4aq\cos\theta}{4\pi\varepsilon_0 r^3},\qquad E_\theta = +\frac{2aq\sin\theta}{4\pi\varepsilon_0 r^3},\quad E_\varphi = 0.$$

11.1.5 A point electric dipole of strength $p^{(1)}$ is placed at $z = a$; a second point electric dipole of equal but opposite strength is at the origin. Keeping the product $p^{(1)}a$ constant, let $a \to 0$. Show that this results in a point electric quadrupole.

11.1.6 A point charge q is in the interior of a hollow conducting sphere of radius r_0. The charge q is displaced a distance a from the center of the sphere. If the conducting sphere is grounded, show that the potential in the interior produced by q and the distributed induced charge is the same as that produced by q and its image charge q'. The image charge is at a distance $a' = r_0^2/a$ from the center, collinear with q and the origin (Fig. 11.6).

Figure 11.6

Image Charge q'



Hint. Calculate the electrostatic potential for $a < r_0 < a'$. Show that the potential vanishes for $r = r_0$ if we take $q' = -qr_0/a$.

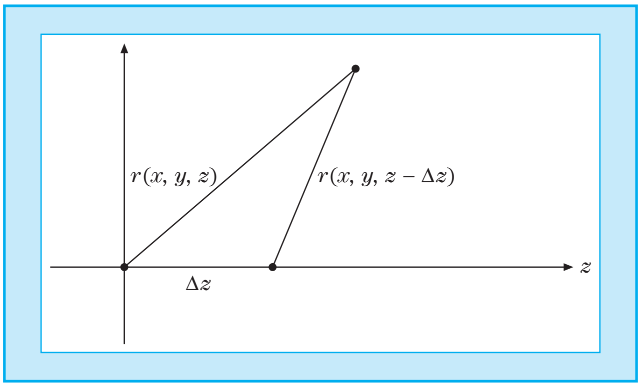
11.1.7 Prove that

$$P_n(\cos\theta) = (-1)^n \frac{r^{n+1}}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

Hint. Compare the Legendre polynomial expansion of the generating function ($a \to \Delta z$; Fig. 11.1) with a Taylor series expansion of $1/r$, where z dependence of r changes from z to $z - \Delta z$ (Fig. 11.7).

Figure 11.7

Geometry for $z \longrightarrow z - \Delta z$



11.1.8 By differentiation and direct substitution of the series form, Eq. (11.18), show that $P_n(x)$ satisfies Legendre's ODE. Note that we may have any x, $-\infty < x < \infty$ and indeed any z in the entire finite complex plane.

11.2 Recurrence Relations and Special Properties

Recurrence Relations

The Legendre polynomial generating function provides a convenient way of deriving the recurrence relations[6] and some special properties. If our generating function [Eq. (11.4)] is differentiated with respect to t, we obtain

$$\frac{\partial g(t,x)}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}. \tag{11.23}$$

By substituting Eq. (11.4) into this and rearranging terms, we have

$$(1-2xt+t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x)t^n = 0. \tag{11.24}$$

The left-hand side is a power series in t. Since this power series vanishes for all values of t, the coefficient of each power of t is equal to zero; that is, our power series is unique (Section 5.7). These coefficients are found by separating the individual summations and using appropriate summation indices as follows:

$$\sum_{m=0}^{\infty} mP_m(x)t^{m-1} - \sum_{n=0}^{\infty} 2nxP_n(x)t^n + \sum_{s=0}^{\infty} sP_s(x)t^{s+1} + \sum_{s=0}^{\infty} P_s(x)t^{s+1} - \sum_{n=0}^{\infty} xP_n(x)t^n = 0. \tag{11.25}$$

Now letting $m = n+1$, $s = n-1$, we find

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \quad n = 1, 2, 3, \ldots. \tag{11.26}$$

With this three-term recurrence relation we may easily construct the higher Legendre polynomials. If we take $n = 1$ and insert the values of $P_0(x)$ and $P_1(x)$ [Exercise 11.1.7 or Eq. (11.18)], we obtain

$$3xP_1(x) = 2P_2(x) + P_0(x), \quad \text{or } P_2(x) = \frac{1}{2}(3x^2-1).$$

This process may be continued indefinitely; the first few Legendre polynomials are listed in Table 11.1.

Cumbersome as it may appear at first, this technique is actually more efficient for a computer than is direct evaluation of the series [Eq. (11.18)]. For greater stability (to avoid undue accumulation and magnification of round off error), Eq. (11.26) is rewritten as

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) - [xP_n(x) - P_{n-1}(x)]/(n+1). \tag{11.26a}$$

[6] We can also apply the explicit series form [Eq. (11.18)] directly.

One starts with $P_0(x) = 1$, $P_1(x) = x$, and computes the **numerical** values of all the $P_n(x)$ for a given value of x, up to the desired $P_N(x)$. The values of $P_n(x)$, $0 \le n < N$ are available as a fringe benefit.

To practice, let us derive another recursion relation from the generating function.

EXAMPLE 11.2.1

Recursion Formula Consider the product

$$g(t, x)g(t, -x) = (1 - 2xt + t^2)^{-1/2}(1 + 2xt + t^2)^{-1/2}$$
$$= [(1 + t^2)^2 - 4x^2t^2]^{-1/2} = [t^4 + 2t^2(1 - 2x^2) + 1]^{-1/2}$$

and recognize the generating function, upon replacing $t^2 \to t$, $2x^2 - 1 \to x$. Using Eq. (11.4) and comparing coefficients of the power series in t we therefore have derived

$$g(t, x)g(t, -x) = \sum_{m,n} P_m(x)P_n(-x)t^{m+n} = \sum_N P_N(2x^2 - 1)t^{2N},$$

or, for $m + n = 2N$ and $m + n = 2N - 1$, respectively,

$$P_N(2x^2 - 1) = \sum_{n=0}^{2N} P_{2N-n}(x)P_n(-x), \tag{11.27a}$$

$$\sum_{n=0}^{2N-1} P_{2N-n-1}(x)P_n(-x) = 0. \tag{11.27b}$$

For $N = 1$ we check first that

$$\sum_{n=0}^{1} P_{1-n}(x)P_n(-x) = x \cdot 1 - 1 \cdot x = 0$$

and second that

$$P_1(2x^2 - 1) = \sum_{n=0}^{2} P_{2-n}(x)P_n(-x) = x - x - x^2 + 2\left(\frac{3}{2}x^2 - \frac{1}{2}\right) = 2x^2 - 1. \quad \blacksquare$$

Differential Equations

More information about the behavior of the Legendre polynomials can be obtained if we now differentiate Eq. (11.4) with respect to x. This gives

$$\frac{\partial g(t, x)}{\partial x} = \frac{t}{(1 - 2xt + t^2)^{3/2}} = \sum_{n=0}^{\infty} P'_n(x)t^n \tag{11.28}$$

or

$$(1 - 2xt + t^2)\sum_{n=0}^{\infty} P'_n(x)t^n - t\sum_{n=0}^{\infty} P_n(x)t^n = 0. \tag{11.29}$$

As before, the coefficient of each power of t is set equal to zero and we obtain

$$P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \tag{11.30}$$

A more useful relation may be found by differentiating Eq. (11.26) with respect to x and multiplying by 2. To this we add $(2n+1)$ times Eq. (11.30), canceling the P'_n term. The result is

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{11.31}$$

From Eqs. (11.30) and (11.31) numerous additional equations may be developed,[7] including

$$P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x), \tag{11.32}$$

$$P'_{n-1}(x) = -nP_n(x) + xP'_n(x), \tag{11.33}$$

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x), \tag{11.34}$$

$$(1-x^2)P'_n(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x). \tag{11.35}$$

By differentiating Eq. (11.34) and using Eq. (11.33) to eliminate $P'_{n-1}(x)$, we find that $P_n(x)$ satisfies the linear, second-order ODE

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$$

or

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0. \tag{11.36}$$

In the second form the ODE is self-adjoint. The previous equations, Eqs. (11.30)–(11.35), are all first-order ODEs but with polynomials of two different indices. The price for having all indices alike is a second-order differential equation. Equation (11.36) is **Legendre's** ODE. We now see that the polynomials $P_n(x)$ generated by the power series for $(1-2xt+t^2)^{-1/2}$ satisfy Legendre's equation, which, of course, is why they are called Legendre polynomials.

In Eq. (11.36) differentiation is with respect to x ($= \cos\theta$). Frequently, we encounter Legendre's equation expressed in terms of differentiation with respect to θ:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_n(\cos\theta)}{d\theta}\right) + n(n+1)P_n(\cos\theta) = 0. \tag{11.37}$$

[7] Using the equation number in parentheses to denote the left-hand side of the equation, we may write the derivatives as

$$\begin{aligned}
2\cdot\tfrac{d}{dx}(11.26) + (2n+1)\cdot(11.30) &\Rightarrow (11.31)\\
\tfrac{1}{2}\{(11.30)+(11.31)\} &\Rightarrow (11.32)\\
\tfrac{1}{2}\{(11.30)-(11.31)\} &\Rightarrow (11.33)\\
(11.32)_{n\to n-1} + x\cdot(11.33) &\Rightarrow (11.34)\\
\tfrac{d}{dx}(11.34) + n\cdot(11.33) &\Rightarrow (11.36).
\end{aligned}$$

Upper and Lower Bounds for $P_n(\cos\theta)$

Finally, in addition to these results, our generating function enables us to set an upper limit on $|P_n(\cos\theta)|$. We have

$$\begin{aligned}(1-2t\cos\theta+t^2)^{-1/2} &= (1-te^{i\theta})^{-1/2}(1-te^{-i\theta})^{-1/2}\\ &= \left(1+\tfrac{1}{2}te^{i\theta}+\tfrac{3}{8}t^2e^{2i\theta}+\cdots\right)\\ &\quad\cdot\left(1+\tfrac{1}{2}te^{-i\theta}+\tfrac{3}{8}t^2e^{-2i\theta}+\cdots\right),\end{aligned} \tag{11.38}$$

with all coefficients **positive**. Our Legendre polynomial, $P_n(\cos\theta)$, still the coefficient of t^n, may now be written as a sum of terms of the form

$$\tfrac{1}{2}a_m(e^{im\theta}+e^{-im\theta}) = a_m\cos m\theta \tag{11.39a}$$

with all the a_m **positive**. Then

$$P_n(\cos\theta) = \sum_{m=0 \text{ or } 1}^{n} a_m \cos m\theta. \tag{11.39b}$$

This series, Eq. (11.39b), is clearly a maximum when $\theta = 0$ and all $\cos m\theta = 1$ are maximal. However, for $x = \cos\theta = 1$, Eq. (11.6) shows that $P_n(1) = 1$. Therefore,

$$|P_n(\cos\theta)| \le P_n(1) = 1. \tag{11.39c}$$

A fringe benefit of Eq. (11.39b) is that it shows that **our Legendre polynomial is a linear combination of** $\cos m\theta$. This means that the **Legendre polynomials form a complete set for any functions that may be expanded in series of** $\cos m\theta$ **over the interval** $[0, \pi]$.

SUMMARY

In this section, various useful properties of the Legendre polynomials are derived from the generating function, Eq. (11.4). The explicit series representation, Eq. (11.18), offers an alternate and sometimes superior approach.

EXERCISES

11.2.1 Given the series

$$\alpha_0+\alpha_2\cos^2\theta+\alpha_4\cos^4\theta+\alpha_6\cos^6\theta = a_0P_0+a_2P_2+a_4P_4+a_6P_6,$$

express the coefficients α_i as a column vector $\boldsymbol{\alpha}$ and the coefficients a_i as a column vector $\mathbf{a}$ and determine the matrices A and B such that

$$\mathsf{A}\boldsymbol{\alpha} = \mathbf{a} \qquad \text{and} \qquad \mathsf{B}\mathbf{a} = \boldsymbol{\alpha}.$$

Check your computation by showing that $\mathsf{AB} = 1$ (unit matrix). Repeat for the odd case

$$\alpha_1\cos\theta+\alpha_3\cos^3\theta+\alpha_5\cos^5\theta+\alpha_7\cos^7\theta = a_1P_1+a_3P_3+a_5P_5+a_7P_7.$$

Note. $P_n(\cos\theta)$ and $\cos^n\theta$ are tabulated in terms of each other in AMS-55.

11.2.2 By differentiating the generating function, $g(t, x)$, with respect to t, multiplying by $2t$, and then adding $g(t, x)$, show that

$$\frac{1-t^2}{(1-2tx+t^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n.$$

This result is useful in calculating the charge induced on a grounded metal sphere by a point charge q.

11.2.3 (a) Derive Eq. (11.35)

$$(1-x^2)P_n'(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x).$$

(b) Write out the relation of Eq. (11.35) to preceding equations in symbolic form analogous to the symbolic forms for Eqs. (11.31)–(11.34).

11.2.4 A point electric octupole may be constructed by placing a point electric quadrupole (pole strength $p^{(2)}$ in the z-direction) at $z = a$ and an equal but opposite point electric quadrupole at $z = 0$ and then letting $a \to 0$, subject to $p^{(2)}a = \text{constant}$. Find the electrostatic potential corresponding to a point electric octupole. Show from the construction of the point electric octupole that the corresponding potential may be obtained by differentiating the point quadrupole potential.

11.2.5 Operating in **spherical polar coordinates**, show that

$$\frac{\partial}{\partial z}\left[\frac{P_n(\cos\theta)}{r^{n+1}}\right] = -(n+1)\frac{P_{n+1}(\cos\theta)}{r^{n+2}}.$$

This is the key step in the mathematical argument that the derivative of one multipole leads to the next higher multipole.
Hint. Compare Exercise 2.5.12.

11.2.6 From

$$P_L(\cos\theta) = \frac{1}{L!}\frac{\partial^L}{\partial t^L}(1-2t\cos\theta+t^2)^{-1/2}\Big|_{t=0}$$

show that

$$P_L(1) = 1, \quad P_L(-1) = (-1)^L.$$

11.2.7 Prove that

$$P_n'(1) = \frac{d}{dx}P_n(x)\Big|_{x=1} = \frac{1}{2}n(n+1).$$

11.2.8 Show that $P_n(\cos\theta) = (-1)^n P_n(-\cos\theta)$ by use of the recurrence relation relating P_n, P_{n+1}, and P_{n-1} and your knowledge of P_0 and P_1.

11.2.9 From Eq. (11.38) write out the coefficient of t^2 in terms of $\cos n\theta$, $n \le 2$. This coefficient is $P_2(\cos\theta)$.

11.3 Orthogonality

Legendre's ODE [Eq. (11.36)] may be written in the form (Section 9.1)

$$\frac{d}{dx}[(1-x^2)P_n'(x)] + n(n+1)P_n(x) = 0, \tag{11.40}$$

showing clearly that it is self-adjoint. Subject to satisfying certain boundary conditions, then, we know that the eigenfunction solutions $P_n(x)$ are orthogonal. Upon comparing Eq. (11.40) with Eqs. (9.6) and (9.8) we see that the weight function $w(x) = 1$, $\mathcal{L} = (d/dx)(1-x^2)(d/dx)$, $p(x) = 1 - x^2$ and the eigenvalue $\lambda = n(n+1)$. The integration limits on x are ± 1, where $p(\pm 1) = 0$. Then for $m \neq n$, Eq. (9.34) becomes

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0,^{8} \tag{11.41}$$

$$\int_{0}^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta\, d\theta = 0, \tag{11.42}$$

showing that $P_n(x)$ and $P_m(x)$ are orthogonal for the interval $[-1, 1]$.

We need to evaluate the integral [Eq. (11.41)] when $n = m$. Certainly, it is no longer zero. From our generating function

$$(1 - 2tx + t^2)^{-1} = \left[\sum_{n=0}^{\infty} P_n(x)t^n\right]^2. \tag{11.43}$$

Integrating from $x = -1$ to $x = +1$, we have

$$\int_{-1}^{1} \frac{dx}{1-2tx+t^2} = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} [P_n(x)]^2\, dx. \tag{11.44}$$

The cross terms in the series vanish by means of Eq. (11.41). Using $y = 1 - 2tx + t^2$, we obtain

$$\int_{-1}^{1} \frac{dx}{1-2tx+t^2} = \frac{1}{2t}\int_{(1-t)^2}^{(1+t)^2} \frac{dy}{y} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right). \tag{11.45}$$

Expanding this in a power series (Exercise 5.4.1) gives us

$$\frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}. \tag{11.46}$$

[8] In Section 9.4 such integrals are intepreted as inner products in a linear vector (function) space. Alternate notations are

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx \equiv \langle P_n(x)|P_m(x)\rangle \equiv (P_n(x), P_m(x)).$$

The $\langle\,\rangle$ form, popularized by Dirac, is common in physics literature. The form $(\,,\,)$ is more common in mathematics literature.

Comparing power series coefficients of Eqs. (11.44) and (11.46), we must have

$$\int_{-1}^{1} [P_n(x)]^2 \, dx = \frac{2}{2n+1}. \tag{11.47}$$

Combining Eq. (11.41) with Eq. (11.47) we have the orthonormality condition

$$\int_{-1}^{1} P_m(x) P_n(x) \, dx = \frac{2\delta_{mn}}{2n+1}. \tag{11.48}$$

Therefore, P_n **are not normalized to unity**. We return to this normalization in Section 11.5, when we construct the orthonormal spherical harmonics.

Expansion of Functions, Legendre Series

In addition to orthogonality, the Sturm–Liouville theory shows that the Legendre polynomials form a complete set. Let us assume, then, that the series

$$\sum_{n=0}^{\infty} a_n P_n(x) = f(x), \qquad \text{or} \quad |f\rangle = \sum_n a_n |P_n\rangle, \tag{11.49}$$

defines $f(x)$ in the sense of convergence in the mean (Section 9.4) in the interval $[-1, 1]$. This demands that $f(x)$ and $f'(x)$ be at least sectionally continuous in this interval. The coefficients a_n are found by multiplying the series by $P_m(x)$ and integrating term by term. Using the orthogonality property expressed in Eqs. (11.42) and (11.48), we obtain

$$\frac{2}{2m+1} a_m = \int_{-1}^{1} P_m(x) f(x) \, dx = \langle P_m | f \rangle = \sum_n a_n \langle P_m | P_n \rangle. \tag{11.50}$$

We replace the variable of integration x by t and the index m by n. Then, substituting into Eq. (11.49), we have

$$f(x) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \left(\int_{-1}^{1} f(t) P_n(t) \, dt \right) P_n(x). \tag{11.51}$$

This expansion in a series of Legendre polynomials is usually referred to as a Legendre series.[9] Its properties are quite similar to the more familiar Fourier series (Chapter 14). In particular, we can use the orthogonality property [Eq. (11.48)] to show that the series is unique.

On a more abstract (and more powerful) level, Eq. (11.51) gives the representation of $f(x)$ in the linear vector space of Legendre polynomials (a Hilbert space; Section 9.4).

Equation (11.51) may also be interpreted in terms of the **projection operators** of quantum theory. We may define a projection operator

$$\mathcal{P}_m \equiv P_m(x) \frac{2m+1}{2} \int_{-1}^{1} P_m(t) [\;] dt$$

[9] Note that Eq. (11.50) gives a_m as a **definite** integral, that is, a number for a given $f(x)$.

as an (integral) operator, ready to operate on $f(t)$. [The $f(t)$ would go in the square bracket as a factor in the integrand.] Then, from Eq. (11.50)

$$\mathcal{P}_m f = a_m P_m(x).^{10}$$

The operator $\mathcal{P}_m$ projects out the mth component of the function f.

EXAMPLE 11.3.1

Legendre Expansion Expand $f(x) = x(x+1)(x-1)$ in the interval $-1 \le x \le 1$.

Because $f(x)$ is odd under parity and is a third-order polynomial, we expect only P_1, P_3. However, we check all coefficients:

$$2a_0 = \int_{-1}^{1} (x^3 - x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]\Big|_{-1}^{1} = 0, \quad \text{also by parity,}$$

$$\frac{2}{3}a_1 = \int_{-1}^{1} (x^4 - x^2)\,dx = \left[\frac{1}{5}x^5 - \frac{1}{3}x^3\right]\Big|_{-1}^{1} = \frac{2}{5} - \frac{2}{3} = -\frac{4}{15},$$

$$\frac{2}{5}a_2 = \frac{1}{2}\int_{-1}^{1} (x^3 - x)(3x^2 - 1)\,dx = 0, \quad \text{by parity;}$$

$$\frac{2}{7}a_3 = \frac{1}{2}\int_{-1}^{1} (x^3 - x)(5x^3 - 3x)\,dx = \frac{1}{2}\int_{-1}^{1} (5x^6 - 8x^4 + 3x^2)\,dx$$

$$= \frac{1}{2}\left[\frac{5}{7}x^7 - \frac{8}{5}x^5 + x^3\right]\Big|_{-1}^{1} = \frac{5}{7} - \frac{8}{5} + 1 = \frac{4}{35}.$$

Finally, using a_1, a_3, we verify that $-\frac{2}{5}x + \frac{1}{5}(5x^3 - 3x) = x(x^2 - 1)$. ■

Equation (11.3), which leads directly to the generating function definition of Legendre polynomials, is a Legendre expansion of $1/r_1$. Going beyond a simple Coulomb field, the $1/r_{12}$ is often replaced by a potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ and the solution of the problem is again effected by a Legendre expansion.

The Legendre series, Eq. (11.49), has been treated as a **known** function $f(x)$ that we arbitrarily chose to expand in a series of Legendre polynomials. Sometimes the origin and nature of the Legendre series are different. In the next examples we consider **unknown** functions we know can be represented by a Legendre series because of the differential equation the unknown functions satisfy. As before, the problem is to determine the unknown coefficients in the series expansion. Here, however, the coefficients are not found by Eq. (11.50). Rather, they are determined by demanding that the Legendre series match a known solution at a boundary. These are boundary value problems.

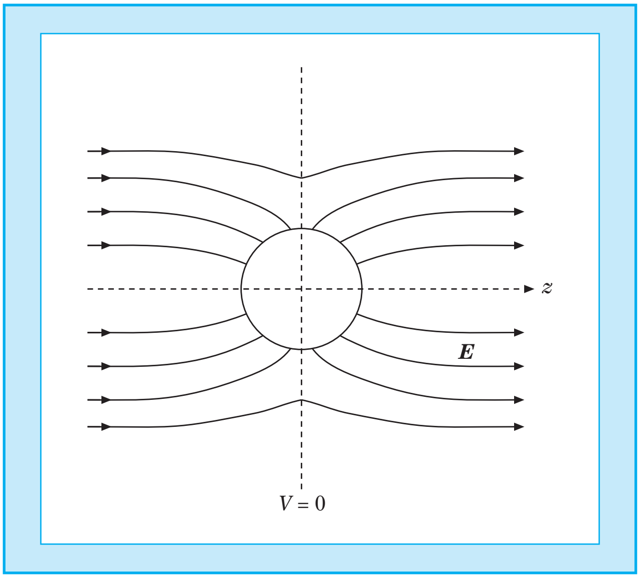
EXAMPLE 11.3.2

Sphere in a Uniform Field Another illustration of the use of Legendre polynomials is provided by the problem of a neutral conducting sphere (radius r_0) placed in a (previously) uniform electric field (Fig. 11.8). The problem is to

[10] The dependent variables are arbitrary. Here, x came from the x in $\mathcal{P}_m$.

Figure 11.8

Conducting Sphere in a Uniform Field



find the new, perturbed, electrostatic potential. The electrostatic potential[11] V satisfies

$$\nabla^2 V = 0, \tag{11.52}$$

Laplace's equation. We select spherical polar coordinates because of the spherical shape of the conductor. (This will simplify the application of the boundary condition at the surface of the conductor.) We can write the unknown potential $V(r, \theta)$ in the region outside the sphere as a linear combination of solutions of the Laplace equation, called harmonic polynomials (check by applying the Laplacian in spherical polar coordinates from Chapter 2):

$$V(r, \theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} b_n \frac{P_n(\cos\theta)}{r^{n+1}}. \tag{11.53}$$

No φ dependence appears because of the **axial (azimuthal) symmetry** of our problem. (The center of the conducting sphere is taken as the origin and the z-axis is oriented parallel to the original uniform field.)

It might be noted here that n is an integer because only for integral n is the θ dependence well behaved at $\cos\theta = \pm 1$. For nonintegral n, the solutions of Legendre's equation diverge at the ends of the interval $[-1, 1]$, the poles $\theta = 0, \pi$ of the sphere (compare Exercises 5.2.11 and 8.5.5). It is for this same reason that the irregular solution of Legendre's equation is also excluded.

[11]It should be emphasized that this is not a presentation of a Legendre series expansion of a known $V(\cos\theta)$. Here, we deal with a **boundary value** problem of a partial differential equation (see Section 8.9).

Now we turn to our (Dirichlet) boundary conditions to determine the unknown a_n and b_n of our series solution, Eq. (11.53). If the original unperturbed electrostatic field is $E_0 = |\mathbf{E}_0|$, we require, as one boundary condition,

$$V(r \to \infty) = -E_0 z = -E_0 r \cos\theta = -E_0 r P_1(\cos\theta). \tag{11.54}$$

Since our Legendre series is unique, we may equate coefficients of $P_n(\cos\theta)$ in Eq. (11.53) ($r \to \infty$) and Eq. (11.54) to obtain

$$a_n = 0, \quad n > 1 \quad \text{and} \quad n = 0, \qquad a_1 = -E_0. \tag{11.55}$$

If $a_n \neq 0$ for $n > 1$, these terms would dominate at large r and the boundary condition [Eq. (11.54)] could not be satisfied.

As a second boundary condition, we may choose the conducting sphere and the plane $\theta = \pi/2$ to be at zero potential, which means that Eq. (11.53) now becomes

$$V(r = r_0) = \frac{b_0}{r_0} + \left(\frac{b_1}{r_0^2} - E_0 r_0\right) P_1(\cos\theta) + \sum_{n=2}^{\infty} b_n \frac{P_n(\cos\theta)}{r_0^{n+1}} = 0. \tag{11.56}$$

In order that this may hold for all values of θ, each coefficient of $P_n(\cos\theta)$ must vanish.[12] Hence,

$$b_0 = 0,^{13} \qquad b_n = 0, \quad n \geq 2, \tag{11.57}$$

whereas

$$b_1 = E_0 r_0^3. \tag{11.58}$$

The electrostatic potential (outside the sphere) is then

$$V = -E_0 r P_1(\cos\theta) + \frac{E_0 r_0^3}{r^2} P_1(\cos\theta) = -E_0 r P_1(\cos\theta)\left(1 - \frac{r_0^3}{r^3}\right). \tag{11.59}$$

It can be shown that a solution of Laplace's equation that satisfies the boundary conditions over the entire boundary is unique. The electrostatic potential V, as given by Eq. (11.59), is a solution of Laplace's equation. It satisfies our boundary conditions and therefore is the solution of Laplace's equation for this problem.

It may further be shown (Exercise 11.3.13) that there is an induced surface charge density

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial r}\bigg|_{r=r_0} = 3\varepsilon_0 E_0 \cos\theta \tag{11.60}$$

[12] Again, this is equivalent to saying that a series expansion in Legendre polynomials (or any complete orthogonal set) is unique.

[13] The coefficient of P_0 is b_0/r_0. We set $b_0 = 0$ since there is no net charge on the sphere. If there is a net charge q, then $b_0 \neq 0$.

on the surface of the sphere and an induced electric dipole moment

$$P = 4\pi r_0^3 \varepsilon_0 E_0. \tag{11.61}$$

■

EXAMPLE 11.3.3

Electrostatic Potential of a Ring of Charge As a further example, consider the electrostatic potential produced by a conducting ring carrying a total electric charge q (Fig. 11.9). From electrostatics (and Section 1.14) the potential ψ satisfies Laplace's equation. Separating variables in spherical polar coordinates, we obtain

$$\psi(r, \theta) = \sum_{n=0}^{\infty} c_n \frac{a^n}{r^{n+1}} P_n(\cos\theta), \quad r > a, \tag{11.62a}$$

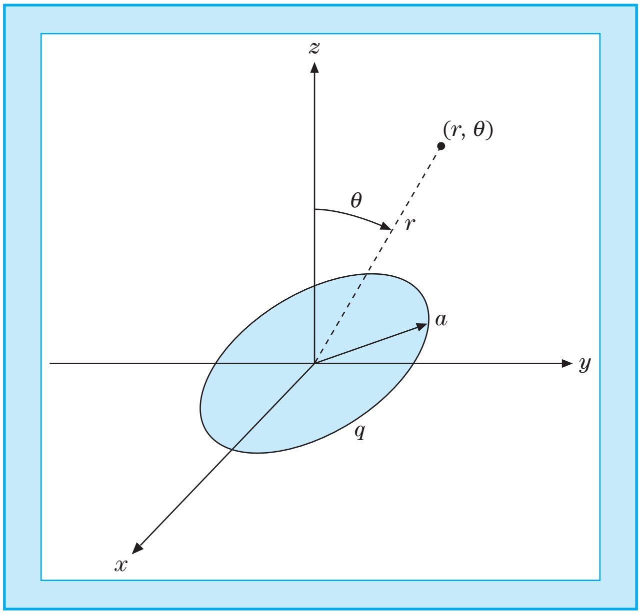
where a is the radius of the ring that is assumed to be in the $\theta = \pi/2$ plane. There is no φ (azimuthal) dependence because of the cylindrical symmetry of the system. The terms with positive exponent in the radial dependence have been rejected since the potential must have an asymptotic behavior

$$\psi \sim \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r}, \quad r \gg a. \tag{11.62b}$$

The problem is to determine the coefficients c_n in Eq. (11.62a). This may be done by evaluating $\psi(r, \theta)$ at $\theta = 0$, $r = z$, and comparing with an independent

Figure 11.9

Charged, Conducting Ring



calculation of the potential from Coulomb's law. In effect, we are using a boundary condition along the z-axis. From Coulomb's law (with all charge equidistant),

$$\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{(z^2+a^2)^{1/2}}, \quad \begin{cases}\theta = 0\\ r = z,\end{cases}$$

$$= \frac{q}{4\pi\varepsilon_0 z}\sum_{s=0}^{\infty}(-1)^s\frac{(2s)!}{2^{2s}(s!)^2}\left(\frac{a}{z}\right)^{2s}, \quad z > a. \tag{11.62c}$$

The last step uses the result of Exercise 10.1.15. Now, Eq. (11.62a) evaluated at $\theta = 0$, $r = z$ [with $P_n(1) = 1$], yields

$$\psi(r,\theta) = \sum_{n=0}^{\infty} c_n \frac{a^n}{z^{n+1}}, \quad r = z. \tag{11.62d}$$

Comparing Eqs. (11.62c) and (11.62d), we get $c_n = 0$ for n odd. Setting $n = 2s$, we have

$$c_{2s} = \frac{q}{4\pi\varepsilon_0}(-1)^s\frac{(2s)!}{2^{2s}(s!)^2}, \tag{11.62e}$$

and our electrostatic potential $\psi(r,\theta)$ is given by

$$\psi(r,\theta) = \frac{q}{4\pi\varepsilon_0 r}\sum_{s=0}^{\infty}(-1)^s\frac{(2s)!}{2^{2s}(s!)^2}\left(\frac{a}{r}\right)^{2s}P_{2s}(\cos\theta), \quad r > a. \tag{11.62f}$$

■

EXERCISES

11.3.1 You have constructed a set of orthogonal functions by the Gram–Schmidt process (Section 9.3), taking $u_m(x) = x^n$, $n = 0, 1, 2, \ldots$, in increasing order with $w(x) = 1$ and an interval $-1 \le x \le 1$. Prove that the nth such function constructed is proportional to $P_n(x)$.
Hint. Use mathematical induction.

11.3.2 Expand the Dirac delta function in a series of Legendre polynomials using the interval $-1 \le x \le 1$.

11.3.3 Verify the Dirac delta function expansions

$$\delta(1-x) = \sum_{n=0}^{\infty}\frac{2n+1}{2}P_n(x),$$

$$\delta(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{2n+1}{2}P_n(x).$$

These expressions appear in a resolution of the Rayleigh plane wave expansion (Exercise 11.4.7) into incoming and outgoing spherical waves.

Note. Assume that the **entire** Dirac delta function is covered when integrating over $[-1, 1]$.

11.3.4 Neutrons (mass 1) are being scattered by a nucleus of mass $A (A > 1)$. In the center of the mass system the scattering is isotropic. Then, in the lab system the average of the cosine of the angle of deflection of the neutron is

$$\langle \cos\psi \rangle = \frac{1}{2}\int_0^\pi \frac{A\cos\theta + 1}{(A^2 + 2A\cos\theta + 1)^{1/2}} \sin\theta \, d\theta.$$

Show, by expansion of the denominator, that $\langle \cos\psi \rangle = 2/3A$.

11.3.5 A particular function $f(x)$ defined over the interval $[-1, 1]$ is expanded in a Legendre series over this same interval. Show that the expansion is unique.

11.3.6 A function $f(x)$ is expanded in a Legendre series $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$. Show that

$$\int_{-1}^{1} [f(x)]^2 \, dx = \sum_{n=0}^{\infty} \frac{2a_n^2}{2n+1}.$$

This is the Legendre form of the Fourier series Parseval identity (Exercise 14.4.2). It also illustrates Bessel's inequality [Eq. (9.73)] becoming an equality for a complete set.

11.3.7 Derive the recurrence relation

$$(1 - x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$$

from the Legendre polynomial generating function.

11.3.8 Evaluate $\int_0^1 P_n(x)\, dx$.

ANS. $n = 2s$; 1 for $s = 0$, 0 for $s > 0$,
$n = 2s + 1$; $P_{2s}(0)/(2s + 2) = (-1)^s(2s - 1)!!/1(2s + 2)!!$

Hint. Use a recurrence relation to replace $P_n(x)$ by derivatives and then integrate by inspection. Alternatively, you can integrate the generating function.

11.3.9 (a) For

$$f(x) = \begin{cases} +1, & 0 < x < 1 \\ -1, & -1 < x < 0, \end{cases}$$

show that

$$\int_{-1}^{1} [f(x)]^2 \, dx = 2\sum_{n=0}^{\infty} (4n + 3)\left[\frac{(2n-1)!!}{(2n+2)!!}\right]^2.$$

(b) By testing the series, prove that the series is convergent.

11.3.10 Prove that

$$\begin{aligned}\int_{-1}^{1} x(1-x^2)P_n' P_m'\,dx &= 0, \text{ unless } m = n \pm 1, \\ &= \frac{2n(n^2-1)}{4n^2-1}\delta_{m,n-1}, \qquad \text{if } m < n. \\ &= \frac{2n(n+2)(n+1)}{(2n+1)(2n+3)}\delta_{m,n+1}, \quad \text{if } m > n.\end{aligned}$$

11.3.11 The amplitude of a scattered wave is given by

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\exp[i\delta_l]\sin\delta_l P_l(\cos\theta),$$

where θ is the angle of scattering, l is the angular momentum, $\hbar k$ is the incident momentum, and δ_l is the phase shift produced by the central potential that is doing the scattering. The total cross section is $\sigma_{\text{tot}} = \int |f(\theta)|^2 d\Omega$. Show that

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l.$$

11.3.12 The coincidence counting rate, $W(\theta)$, in a gamma–gamma angular correlation experiment has the form

$$W(\theta) = \sum_{n=0}^{\infty} a_{2n}P_{2n}(\cos\theta).$$

Show that data in the range $\pi/2 \le \theta \le \pi$ can, in principle, define the function $W(\theta)$ (and permit a determination of the coefficients a_{2n}). This means that although data in the range $0 \le \theta < \pi/2$ may be useful as a check, they are not essential.

11.3.13 A conducting sphere of radius r_0 is placed in an initially uniform electric field, $\mathbf{E}_0$. Show the following:

(a) The induced surface charge density is

$$\sigma = 3\varepsilon_0 E_0\cos\theta.$$

(b) The induced electric dipole moment is

$$P = 4\pi r_0^3\varepsilon_0 E_0.$$

The induced electric dipole moment can be calculated either from the surface charge [part (a)] or by noting that the final electric field $\mathbf{E}$ is the result of superimposing a dipole field on the original uniform field.

11.3.14 A charge q is displaced a distance a along the z-axis from the center of a spherical cavity of radius R.

(a) Show that the electric field averaged over the volume $a \le r \le R$ is zero.

(b) Show that the electric field averaged over the volume $0 \le r \le a$ is

$$\mathbf{E} = \hat{\mathbf{z}} E_z = -\hat{\mathbf{z}} \frac{q}{4\pi\varepsilon_0 a^2} \quad \text{(SI units)} = -\hat{\mathbf{z}} \frac{nqa}{3\varepsilon_0},$$

where n is the number of such displaced charges per unit volume. This is a basic calculation in the polarization of a dielectric.
Hint. $\mathbf{E} = -\nabla\varphi$.

11.3.15 Determine the electrostatic potential (Legendre expansion) of a circular ring of electric charge for $r < a$.

11.3.16 Calculate the electric field produced by the charged conducting ring of Exercise 11.3.15 for
(a) $r > a$, (b) $r < a$.

11.3.17 Find the potential $\psi(r, \theta)$ produced by a charged conducting disk (Fig. 11.10) for $r > a$, the radius of the disk. The charge density σ (on each side of the disk) is

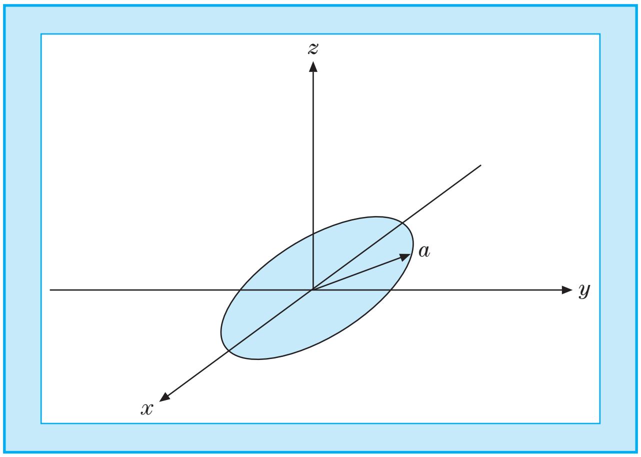
$$\sigma(\rho) = \frac{q}{4\pi a(a^2 - \rho^2)^{1/2}}, \quad \rho^2 = x^2 + y^2.$$

Hint. The definite integral you get can be evaluated as a beta function. For more details, see Section 5.03 of Smythe in Additional Reading.

$$\textit{ANS.} \quad \psi(r, \theta) = \frac{q}{4\pi\varepsilon_0 r} \sum_{l=0}^{\infty} (-1)^l \frac{1}{2l+1} \left(\frac{a}{r}\right)^{2l} P_{2l}(\cos\theta).$$

Figure 11.10

Charged, Conducting Disk



11.3.18 From the result of Exercise 11.3.17 calculate the potential of the disk. Since you are violating the condition $r > a$, justify your calculation carefully.
Hint. You may run into the hypogeometric series given in Exercise 5.2.9.

11.3.19 The hemisphere defined by $r = a$, $0 \le \theta < \pi/2$ has an electrostatic potential $+V_0$. The hemisphere $r = a$, $\pi/2 < \theta \le \pi$ has an electrostatic potential $-V_0$. Show that the potential at interior points is

$$V = V_0 \sum_{n=0}^{\infty} \frac{4n+3}{2n+2}\left(\frac{r}{a}\right)^{2n+1} P_{2n}(0)P_{2n+1}(\cos\theta)$$
$$= V_0 \sum_{n=0}^{\infty} (-1)^n \frac{(4n+3)(2n-1)!!}{(2n+2)!!}\left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

Hint. You need Exercise 11.3.8.

11.3.20 A conducting sphere of radius a is divided into two electrically separate hemispheres by a thin insulating barrier at its equator. The top hemisphere is maintained at a potential V_0 and the bottom hemisphere at $-V_0$.

(a) Show that the electrostatic potential **exterior** to the two hemispheres is

$$V(r,\theta) = V_0 \sum_{s=0}^{\infty} (-1)^s (4s+3)\frac{(2s-1)!!}{(2s+2)!!}\left(\frac{a}{r}\right)^{2s+2} P_{2s+1}(\cos\theta).$$

(b) Calculate the electric charge density σ on the outside surface. Note that your series diverges at $\cos\theta = \pm 1$ as you expect from the infinite capacitance of this system (zero thickness for the insulating barrier).

$$ANS. \quad \sigma = \varepsilon_0 E_n = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a}$$
$$= \varepsilon_0 V_0 \sum_{s=0}^{\infty} (-1)^s (4s+3)\frac{(2s-1)!!}{(2s)!!} P_{2s+1}(\cos\theta).$$

11.3.21 In the notation of Section 9.4 $\langle x|\varphi_s\rangle = \sqrt{(2s+1)/2}P_s(x)$, a Legendre polynomial is renormalized to unity. Explain how $|\varphi_s\rangle\langle\varphi_s|$ acts as a projection operator. In particular, show that if $|f\rangle = \sum_n a'_n|\varphi_n\rangle$, then

$$|\varphi_s\rangle\langle\varphi_s|f\rangle = a'_s|\varphi_s\rangle.$$

11.3.22 Expand x^8 as a Legendre series. Determine the Legendre coefficients from Eq. (11.50),

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} x^8 P_m(x)\,dx.$$

Check your values against AMS-55, Table 22.9. This illustrates the expansion of a simple function. Actually, if $f(x)$ is expressed as a power series, the recursion Eq. (11.26) is both faster and more accurate.
Hint. Gaussian quadrature can be used to evaluate the integral.

11.3.23 Expand $\arcsin x$ in Legendre polynomials.

11.3.24 Expand the polynomials $2+5x$, $1+x+x^3$ in a Legendre series and plot your results and the polynomials for the larger interval $-2 \le x \le 2$.

11.4 Alternate Definitions of Legendre Polynomials

Rodrigues's Formula

The series form of the Legendre polynomials [Eq. (11.18)] of Section 11.1 may be transformed as follows. From Eq. (11.18)

$$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n r!(n-2r)!} x^{n-2r}. \tag{11.63}$$

For n an integer

$$\begin{aligned} P_n(x) &= \sum_{r=0}^{[n/2]} (-1)^r \frac{1}{2^n r!(n-r)!} \left(\frac{d}{dx}\right)^n x^{2n-2r} \\ &= \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n \sum_{r=0}^{n} \frac{(-1)^r n!}{r!(n-r)!} x^{2n-2r}. \end{aligned} \tag{11.64}$$

Note the extension of the upper limit. The reader is asked to show in Exercise 11.4.1 that the additional terms $[n/2]+1$ to n in the summation contribute nothing. However, the effect of these extra terms is to permit the replacement of the new summation by $(x^2-1)^n$ (binomial theorem once again) to obtain

$$P_n(x) = \frac{1}{2^n n!} \left(\frac{d}{dx}\right)^n (x^2-1)^n. \tag{11.65}$$

This is Rodrigues's formula. It is useful in proving many of the properties of the Legendre polynomials, such as orthogonality. A related application is seen in Exercise 11.4.3. The Rodrigues definition can be extended to define the associated Legendre functions.

EXAMPLE 11.4.1

Lowest Legendre Polynomials For $n=0$, $P_0 = 1$ follows right away from Eq. (11.65), as well as $P_1(x) = \frac{2x}{2} = x$. For $n=2$ we obtain

$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = \frac{1}{8}(12x^2 - 4) = \frac{3}{2}x^2 - \frac{1}{2},$$

and for $n=3$

$$P_3(x) = \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) = \frac{1}{48}(120x^3 - 72x) = \frac{5}{2}x^3 - \frac{3}{2}x,$$

in agreement with Table 11.1. ■

EXERCISES

11.4.1 Show that **each** term in the summation

$$\sum_{r=[n/2]+1}^{n} \left(\frac{d}{dx}\right)^n \frac{(-1)^r n!}{r!(n-r)!} x^{2n-2r}$$

vanishes (r and n integral).

11.4.2 Using Rodrigues's formula, show that the $P_n(x)$ are orthogonal and that

$$\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}.$$

Hint. Use Rodrigues's formula and integrate by parts.

11.4.3 Show that $\int_{-1}^{1} x^m P_n(x) dx = 0$ when $m < n$.
Hint. Use Rodrigues's formula or expand x^m in Legendre polynomials.

11.4.4 Show that

$$\int_{-1}^{1} x^n P_n(x) dx = \frac{2^{n+1} n! n!}{(2n+1)!}.$$

Note. You are expected to use Rodrigues's formula and integrate by parts, but also see if you can get the result from Eq. (11.18) by inspection.

11.4.5 Show that

$$\int_{-1}^{1} x^{2r} P_{2n}(x) dx = \frac{2^{2n+1}(2r)!(r+n!)}{(2r+2n+1)!(r-n)!}, \qquad r \geq n.$$

11.4.6 As a generalization of Exercises 11.4.4 and 11.4.5, show that the Legendre expansions of x^s are

(a) $x^{2r} = \sum_{n=0}^{r} \frac{2^{2n}(4n+1)(2r)!(r+n)!}{(2r+2n+1)!(r-n)!} P_{2n}(x), \qquad s = 2r,$

(b) $x^{2r+1} = \sum_{n=0}^{r} \frac{2^{2n+1}(4n+3)(2r+1)!(r+n+1)!}{(2r+2n+3)!(r-n)!} P_{2n+1}(x),$
$s = 2r+1.$

11.4.7 A plane wave may be expanded in a series of spherical waves by the Rayleigh equation

$$e^{ikr\cos\gamma} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\gamma).$$

Show that $a_n = i^n(2n+1)$.
Hint.
1. Use the orthogonality of the P_n to solve for $a_n j_n(kr)$.
2. Differentiate n times with respect to (kr) and set $r = 0$ to eliminate the r dependence.
3. Evaluate the remaining integral by Exercise 11.4.4.

Note. This problem may also be treated by noting that both sides of the equation satisfy the Helmholtz equation. The equality can be established by showing that the solutions have the same behavior at the origin and also behave alike at large distances.

11.4.8 Verify the Rayleigh equation of Exercise 11.4.7 by starting with the following steps:

(a) Differentiate with respect to (kr) to establish

$$\sum_n a_n j_n'(kr) P_n(\cos\gamma) = i \sum_n a_n j_n(kr) \cos\gamma\, P_n(\cos\gamma).$$

(b) Use a recurrence relation to replace $\cos\gamma\, P_n(\cos\gamma)$ by a linear combination of P_{n-1} and P_{n+1}.

(c) Use a recurrence relation to replace j_n' by a linear combination of j_{n-1} and j_{n+1}. See Chapter 12 for Bessel functions.

11.4.9 In numerical work (Gauss–Legendre quadrature) it is useful to establish that $P_n(x)$ has n real zeros in the interior of $[-1, 1]$. Show that this is so.

Hint. Rolle's theorem shows that the first derivative of $(x^2-1)^{2n}$ has one zero in the interior of $[-1, 1]$. Extend this argument to the second, third, and ultimately to the nth derivative.

11.5 Associated Legendre Functions

When Laplace's equation is separated in spherical polar coordinates (Section 8.9), one of the separated ODEs is the associated Legendre equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP_n^m(\cos\theta)}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right] P_n^m(\cos\theta) = 0. \quad (11.66)$$

With $x = \cos\theta$, this becomes

$$(1-x^2)\frac{d^2}{dx^2}P_n^m(x) - 2x\frac{d}{dx}P_n^m(x) + \left[n(n+1) - \frac{m^2}{1-x^2}\right] P_n^m(x) = 0. \quad (11.67)$$

If the azimuthal separation constant $m^2 = 0$ we have Legendre's equation, Eq. (11.36). The regular solutions (with m not necessarily zero), relabeled $P_n^m(x)$, are

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \quad (11.68)$$

These are the associated Legendre functions.[14] Since the highest power of x in $P_n(x)$ is x^n, we must have $m \le n$ (or the m-fold differentiation will drive our function to zero). In quantum mechanics the requirement that $m \le n$ has

[14] One finds (as in AMS-55) the associated Legendre functions defined with an additional factor of $(-1)^m$. This phase $(-1)^m$ seems an unnecessary complication at this point. It will be included in the definition of the spherical harmonics $Y_n^m(\theta, \varphi)$. Note also that **the upper index** m **is not an exponent**.

the physical interpretation that the expectation value of the square of the z-component of the angular momentum is less than or equal to the expectation value of the square of the angular momentum vector **L** (Section 4.3),

$$\langle L_z^2 \rangle \leq \langle L^2 \rangle \equiv \int \psi_{nm}^* \mathbf{L}^2 \psi_{nm} d^3 r,$$

where m is the eigenvalue of L_z, and $n(n+1)$ is the eigenvalue of $\mathbf{L}^2$. From the form of Eq. (11.68), we might expect m to be nonnegative. However, if $P_n(x)$ is expressed by Rodrigues's formula, this limitation on m is relaxed and we may have $-n \leq m \leq n$, with negative as well as positive values of m being permitted. Using Leibniz's differentiation formula once again, the reader may show that $P_n^m(x)$ and $P_n^{-m}(x)$ are related by

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x). \tag{11.69}$$

From our definition of the associated Legendre functions, $P_n^m(x)$,

$$P_n^0(x) = P_n(x). \tag{11.70}$$

As with the Legendre polynomials, a generating function for the associated Legendre functions is obtained via Eq. (11.67) from that of ordinary Legendre polynomials:

$$\frac{(2m)!(1-x^2)^{m/2}}{2^m m!(1-2tx+t^2)^{m+1/2}} = \sum_{s=0}^{\infty} P_{s+m}^m(x) t^s. \tag{11.71}$$

If we drop the factor $(1-x^2)^{m/2} = \sin^m \theta$ from this formula and define the **polynomials** $\mathcal{P}_{s+m}^m(x) = P_{s+m}^m(x)(1-x^2)^{-m/2}$, then we obtain a practical form of the generating function

$$g_m(x,t) \equiv \frac{(2m)!}{2^m m!(1-2tx+t^2)^{m+1/2}} = \sum_{s=0}^{\infty} \mathcal{P}_{s+m}^m(x) t^s. \tag{11.72}$$

We can derive a recursion relation for associated Legendre polynomials that is analogous to Eqs. (11.23) and (11.26) by differentiation as follows:

$$(1-2tx+t^2)\frac{\partial g_m}{\partial t} = (2m+1)(x-t) g_m(x,t).$$

Substituting the defining expansions for associated Legendre polynomials we get

$$(1-2tx+t^2)\sum_s s\mathcal{P}_{s+m}^m(x) t^{s-1} = (2m+1)\sum_s \left[x\mathcal{P}_{s+m}^m t^s - \mathcal{P}_{s+m}^m t^{s+1}\right].$$

Comparing coefficients of powers of t in these power series, we obtain the recurrence relation

$$(s+1)\mathcal{P}_{s+m+1}^m - (2m+1+2s)x\mathcal{P}_{s+m}^m + (s+2m)\mathcal{P}_{s+m-1}^m = 0. \tag{11.73}$$

For $m = 0$ and $s = n$ this relation is Eq. (11.26).

Before we can use this relation we need to initialize it, that is, relate the associated Legendre polynomials with $m = 1$ to ordinary Legendre polynomials. We observe that

$$(1 - 2xt + t^2)g_1(x, t) = (1 - 2xt + t^2)^{-1/2} = \sum_s P_s(x)t^s \qquad (11.74)$$

so that upon inserting Eq. (11.72) we get the recursion

$$\mathcal{P}^1_{s+1} - 2x\mathcal{P}^1_s + \mathcal{P}^1_{s-1} = P_s(x). \qquad (11.75)$$

More generally, we also have the identity

$$(1 - 2xt + t^2)g_{m+1}(x, t) = (2m + 1)g_m(x, t), \qquad (11.76)$$

from which we extract the recursion

$$\mathcal{P}^{m+1}_{s+m+1} - 2x\mathcal{P}^{m+1}_{s+m} + \mathcal{P}^{m+1}_{s+m-1} = (2m + 1)\mathcal{P}^m_{s+m}(x), \qquad (11.77)$$

which relates the associated Legendre polynomials with superindex $m + 1$ to those with m. For $m = 0$ we recover the initial recursion Eq. (11.75).

EXAMPLE 11.5.1

Lowest Associated Legendre Polynomials Now we are ready to derive the entries of Table 11.2. For $m = 1$ and $s = 0$ Eq. (11.75) yields $\mathcal{P}^1_1 = 1$ because $\mathcal{P}^1_0 = 0 = \mathcal{P}^1_{-1}$ do not occur in the definition, Eq. (11.72), of the associated Legendre polynomials. Multiplying by $(1 - x^2)^{1/2} = \sin\theta$ we get the first line of Table 11.2. For $s = 1$ we find from Eq. (11.75),

$$\mathcal{P}^1_2(x) = P_1 + 2x\mathcal{P}^1_1 = x + 2x = 3x,$$

from which the second line of Table 11.2, $3\cos\theta\sin\theta$, follows upon multiplying by $\sin\theta$. For $s = 2$ we get

$$\mathcal{P}^1_3(x) = P_2 + 2x\mathcal{P}^1_2 - \mathcal{P}^1_1 = \frac{1}{2}(3x^2 - 1) + 6x^2 - 1 = \frac{15}{2}x^2 - \frac{3}{2},$$

Table 11.2

Associated Legendre Functions

$P^1_1(x) = (1 - x^2)^{1/2} = \sin\theta$
$P^1_2(x) = 3x(1 - x^2)^{1/2} = 3\cos\theta\sin\theta$
$P^2_2(x) = 3(1 - x^2) = 3\sin^2\theta$
$P^1_3(x) = \frac{3}{2}(5x^2 - 1)(1 - x^2)^{1/2} = \frac{3}{2}(5\cos^2\theta - 1)\sin\theta$
$P^2_3(x) = 15x(1 - x^2) = 15\cos\theta\sin^2\theta$
$P^3_3(x) = 15(1 - x^2)^{3/2} = 15\sin^3\theta$
$P^1_4(x) = \frac{5}{2}(7x^3 - 3x)(1 - x^2)^{1/2} = \frac{5}{2}(7\cos^3\theta - 3\cos\theta)\sin\theta$
$P^2_4(x) = \frac{15}{2}(7x^2 - 1)(1 - x^2) = \frac{15}{2}(7\cos^2\theta - 1)\sin^2\theta$
$P^3_4(x) = 105x(1 - x^2)^{3/2} = 105\cos\theta\sin^3\theta$
$P^4_4(x) = 105(1 - x^2)^2 = 105\sin^4\theta$

in agreement with line 4 of Table 11.2. To get line 3 we use Eq. (11.76). For $m = 1,\ s = 0$, this gives $\mathcal{P}_2^2(x) = 3\mathcal{P}_1^1(x) = 3$, and multiplying by $1-x^2 = \sin^2\theta$ reproduces line 3 of Table 11.2. For lines 5, 8, and 9, Eq. (11.72) may be used, which we leave as an exercise. ■

EXAMPLE 11.5.2

Special Values For $x = 1$ we use

$$(1-2t+t^2)^{-m-1/2} = (1-t)^{-2m-1} = \sum_{s=0}^{\infty} \binom{-2m-1}{s} t^s$$

in Eq. (11.72) and find

$$\mathcal{P}_{s+m}^m(1) = \frac{(2m)!}{2^m m!} \binom{-2m-1}{s}. \tag{11.78}$$

For $m = 1,\ s = 0$ we have $\mathcal{P}_1^1(1) = \binom{-3}{0} = 1$; for $s = 1,\ \mathcal{P}_2^1(1) = -\binom{-3}{1} = 3$; and for $s = 2$, $\mathcal{P}_3^1(1) = \binom{-3}{2} = \frac{(-3)(-4)}{2} = 6 = \frac{3}{2}(5-1)$. These all agree with Table 11.2.

For $x = 0$ we can also use the binomial expansion, which we leave as an exercise. ■

EXAMPLE 11.5.3

Parity From the identity $g_m(-x, -t) = g_m(x, t)$ we obtain the parity relation

$$\mathcal{P}_{s+m}^m(-x) = (-1)^s \mathcal{P}_{s+m}^m(x). \tag{11.79}$$

■

We have the **orthogonality integral**

$$\int_{-1}^{1} P_p^m(x) P_q^m(x)\, dx = \frac{2}{2q+1} \cdot \frac{(q+m)!}{(q-m)!} \delta_{pq} \tag{11.80}$$

or, in spherical polar coordinates,

$$\int_0^{\pi} P_p^m(\cos\theta) P_q^m(\cos\theta) \sin\theta\, d\theta = \frac{2}{2q+1} \cdot \frac{(q+m)!}{(q-m)!} \delta_{pq}. \tag{11.81}$$

The orthogonality of the Legendre polynomials is a special case of this result, obtained by setting m equal to zero; that is, for $m = 0$, Eq. (11.80) reduces to Eqs. (11.47) and (11.48). In both Eqs. (11.80) and (11.81) our Sturm–Liouville theory of Chapter 9 could provide the Kronecker delta. A special calculation is required for the normalization constant.

Spherical Harmonics

The functions $\Phi_m(\varphi) = e^{im\varphi}$ are orthogonal when integrated over the azimuthal angle φ, whereas the functions $P_n^m(\cos\theta)$ are orthogonal upon integrating over the polar angle θ. We take the product of the two and define

$$Y_n^m(\theta, \varphi) \equiv (-1)^m \sqrt{\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta) e^{im\varphi} \tag{11.82}$$

Table 11.3

Spherical Harmonics (Condon-Shortley Phase)

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$
$$Y_1^1(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$$
$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_1^{-1}(\theta,\varphi) = +\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}$$
$$Y_2^2(\theta,\varphi) = \sqrt{\frac{5}{96\pi}}3\sin^2\theta e^{2i\varphi}$$
$$Y_2^1(\theta,\varphi) = -\sqrt{\frac{5}{24\pi}}3\sin\theta\cos\theta e^{i\varphi}$$
$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$
$$Y_2^{-1}(\theta,\varphi) = +\sqrt{\frac{5}{24\pi}}3\sin\theta\cos\theta e^{-i\varphi}$$
$$Y_2^{-2}(\theta,\varphi) = \sqrt{\frac{5}{96\pi}}3\sin^2\theta e^{-2i\varphi}$$

to obtain functions of two angles (and two indices) that are orthonormal over the spherical surface. These $Y_n^m(\theta,\varphi)$ are spherical harmonics. The complete orthogonality integral becomes

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} Y_{n_1}^{m_1 *}(\theta,\varphi)Y_{n_2}^{m_2}(\theta,\varphi)\sin\theta\, d\theta\, d\varphi = \delta_{n_1 n_2}\delta_{m_1 m_2} \tag{11.83}$$

and explains the presence of the complicated normalization constant in Eq. (11.82).

The extra $(-1)^m$ included in the defining equation of $Y_n^m(\theta,\varphi)$ with $-n \le m \le n$ deserves some comment. It is clearly legitimate since Eq. (11.68) is linear and homogeneous. It is not necessary, but in moving on to certain quantum mechanical calculations, particularly in the quantum theory of angular momentum, it is most convenient. The factor $(-1)^m$ is a phase factor often called the Condon–Shortley phase after the authors of a classic text on atomic spectroscopy. The effect of this $(-1)^m$ [Eq. (11.82)] and the $(-1)^m$ of Eq. (11.69) for $P_n^{-m}(\cos\theta)$ is to introduce an alternation of sign among the positive m spherical harmonics. This is shown in Table 11.3.

The functions $Y_n^m(\theta,\varphi)$ acquired the name spherical harmonics because they are defined over the surface of a sphere with θ the polar angle and φ the azimuth. The "harmonic" was included because solutions of Laplace's equation were called harmonic functions and $Y_n^m(\cos,\varphi)$ is the angular part of such a solution.

EXAMPLE 11.5.4

Lowest Spherical Harmonics For $n = 0$ we have $m = 0$ and $Y_0^0 = \frac{1}{\sqrt{4\pi}}$ from Eq. (11.82). For $n = 1$ we have $m = \pm 1, 0$ and $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, whereas for $m = \pm 1$ we see from Table 11.2 that $\cos\theta$ is replaced by $\sin\theta$ and we have the additional factor $(\mp 1)\frac{e^{\pm i\varphi}}{\sqrt{2}}$, which checks with Table 11.3. ■

In the framework of quantum mechanics Eq. (11.67) becomes an orbital angular momentum equation and the solution $Y_L^M(\theta, \varphi)$ (n replaced by L and m by M) is an angular momentum eigenfunction, with L being the angular momentum quantum number and M the z-axis projection of L. These relationships are developed in more detail in Section 4.3.

EXAMPLE 11.5.5

Spherical Symmetry of Probability Density of Atomic States What is the angular dependence of the probability density of the degenerate atomic states with principal quantum number $n = 2$?

Here we have to sum the absolute square of the wave functions for $n = 2$ and orbital angular momentum $l = 0,\ m = 0;\ l = 1,\ m = -1, 0, +1$; that is, s and three p states. We ignore the radial dependence. The s state has orbital angular momentum $l = 0$ and $m = 0$ and is independent of angles. For the p states with $l = 1$ and $m = \pm 1, 0$

$$\psi_{200} \sim Y_0^0 = \frac{1}{\sqrt{4\pi}}, \qquad \psi_{21m} \sim Y_1^m,$$

we have to evaluate the sum

$$\sum_{m=-1}^{1} |Y_1^m|^2 = \frac{3}{4\pi}\left[2\left(\frac{1}{\sqrt{2}}\sin\theta\right)^2 + \cos^2\theta\right] = 1$$

upon substituting the spherical harmonics from Table 11.3. This result is spherically symmetric, as is the density for the s state alone or the sum of the three p states. These results can be generalized to higher orbital angular momentum l. ■

SUMMARY

Legendre polynomials are naturally defined by their generating function in a multipole expansion of the Coulomb potential. They appear in physical systems with azimuthal symmetry. They also arise in the separation of partial differential equations with spherical or cylindrical symmetry or as orthogonal eigenfunctions of the Sturm–Liouville theory of their second-order differential equation. Associated Legendre polynomials appear as ingredients of the spherical harmonics in situations that lack in azimuthal symmetry.

EXERCISES

11.5.1 Show that the parity of $Y_L^M(\theta, \varphi)$ is $(-1)^L$. Note the disappearance of any M dependence.

Hint. For the parity operation in spherical polar coordinates, see Section 2.5 and Section 11.2.

11.5.2 Prove that

$$Y_L^M(0, \varphi) = \left(\frac{2L+1}{4\pi}\right)^{1/2} \delta_{M0}.$$

11.5.3 In the theory of Coulomb excitation of nuclei we encounter $Y_L^M(\pi/2, 0)$. Show that

$$Y_L^M\left(\frac{\pi}{2}, 0\right) = \left(\frac{2L+1}{4\pi}\right)^{1/2} \frac{[(L-M)!(L+M)!]^{1/2}}{(L-M)!!(L+M)!!}(-1)^{(L+M)/2}$$
$$\text{for } L+M \text{ even},$$
$$= 0 \quad \text{for } L+M \text{ odd}.$$

Here,

$$(2n)!! = 2n(2n-2)\cdots 6\cdot 4\cdot 2,$$
$$(2n+1)!! = (2n+1)(2n-1)\cdots 5\cdot 3\cdot 1.$$

11.5.4 (a) Express the elements of the quadrupole moment tensor $x_i x_j$ as a linear combination of the spherical harmonics Y_2^m (and Y_0^0).
Note. The tensor $x_i x_j$ is reducible. The Y_0^0 indicates the presence of a scalar component.
(b) The quadrupole moment tensor is usually defined as

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\mathbf{r})\, d\tau,$$

with $\rho(\mathbf{r})$ the charge density. Express the components of $(3x_i x_j - r^2\delta_{ij})$ in terms of $r^2 Y_2^M$.
(c) What is the significance of the $-r^2\delta_{ij}$ term?
Hint. Contract the indices i, j.

11.5.5 The orthogonal azimuthal functions yield a useful representation of the Dirac delta function. Show that

$$\delta(\varphi_1 - \varphi_2) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \exp[im(\varphi_1 - \varphi_2)].$$

11.5.6 Derive the spherical harmonic closure relation

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y_l^m(\theta_1, \varphi_1) Y_l^{m*}(\theta_2, \varphi_2) = \frac{1}{\sin\theta_1}\delta(\theta_1 - \theta_2)\delta(\varphi_1 - \varphi_2)$$
$$= \delta(\cos\theta_1 - \cos\theta_2)\delta(\varphi_1 - \varphi_2).$$

11.5.7 The quantum mechanical angular momentum operators $L_x \pm iL_y$ are given by

$$L_x + iL_y = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right),$$
$$L_x - iL_y = -e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right).$$

Show that
(a) $(L_x + iL_y)Y_L^M(\theta, \varphi), = \sqrt{(L-M)(L+M+1)}\,Y_L^{M+1}(\theta, \varphi)$,
(b) $(L_x - iL_y)Y_L^M(\theta, \varphi), = \sqrt{(L-M)(L-M+1)}\,Y_L^{M-1}(\theta, \varphi)$.

11.5.8 With $L_\pm$ given by

$$L_\pm = L_x \pm iL_y = \pm e^{\pm i\varphi}\left[\frac{\partial}{\partial\theta} \pm i\cot\theta\,\frac{\partial}{\partial\varphi}\right],$$

show that

(a) $Y_l^m = \sqrt{\dfrac{(l+m)!}{(2l)!(l-m)!}}(L_-)^{l-m}Y_l^l$,

(b) $Y_l^m = \sqrt{\dfrac{(l-m)!}{(2l)!(l+m)!}}(L_+)^{l+m}Y_l^{-l}$.

11.5.9 In some circumstances it is desirable to replace the imaginary exponential of our spherical harmonic by sine or cosine. Morse and Feshbach define

$$Y_{mn}^e = P_n^m(\cos\theta)\cos m\varphi,$$
$$Y_{mn}^o = P_n^m(\cos\theta)\sin m\varphi,$$

where

$$\int_0^{2\pi}\int_0^{\pi}[Y_{mn}^{e \text{ or } o}(\theta,\varphi)]^2 \sin\theta\,d\theta\,d\varphi = \frac{4\pi}{2(2n+1)}\frac{(n+m)!}{(n-m)!},\quad n = 1, 2, \ldots$$
$$= 4\pi \quad \text{for} \quad n = 0 \ (Y_{00}^o \text{ is undefined}).$$

These spherical harmonics are often named according to the patterns of their positive and negative regions on the surface of a sphere zonal harmonics for $m = 0$, sectoral harmonics for $m = n$, and tesseral harmonics for $0 < m < n$. For Y_{mn}^e, $n = 4$, $m = 0, 2, 4$, indicate on a diagram of a hemisphere (one diagram for each spherical harmonic) the regions in which the spherical harmonic is positive.

11.5.10 A function $f(r,\theta,\varphi)$ may be expressed as a Laplace series

$$f(r,\theta,\varphi) = \sum_{l,m} a_{lm} r^l Y_l^m(\theta,\varphi).$$

With $\langle\rangle_{\text{sphere}}$ used to mean the average over a sphere (centered on the origin), show that

$$\langle f(r,\theta,\varphi)\rangle_{\text{sphere}} = f(0,0,0).$$

Additional Reading

Hobson, E. W. (1955). *The Theory of Spherical and Ellipsoidal Harmonics.* Chelsea, New York. This is a very complete reference, which is the classic text on Legendre polynomials and all related functions.

Smythe, W. R. (1989). *Static and Dynamic Electricity*, 3rd ed. McGraw-Hill, New York.

See also the references listed in Section 4.4 and at the end of Chapter 13.