Lecture 5: The Governing Equations

Williams Ch. 6.1 – 6.6
Weather Analysis and Forecasting
Ch. 4.1 – 4.5, 4.8 – 4.9
Outline

• The Hydrostatic Equation
• The Thickness Equation
• Dines Compensation Principle
• Geostrophic Balance
• Thermal Wind Balance
Introduction

• Study of the physical world tends to be focused on the quantities such as mass, momentum, and energy. The behavior of the atmosphere is no exception to this rule.
• In these lecture notes, we will investigate the manner in which these quantities and their various interactions serve to describe the building blocks of a dynamical understanding of the atmosphere at middle latitudes.
Distribution of Mass in the Atmosphere

• Recall that mass can be described as the measure of the substance of an object.

• The pressure exerted by Earth’s atmosphere decreases with increasing distance away from the surface as the depth of the fluid decreases.

• This implies that the mass of the atmosphere also decreases with height.
Hydrostatic Equation

\[ \frac{dP}{dz} = -\rho g \]

- This expression is known as the **hydrostatic equation** and it represents the fundamental vertical balance condition in the Earth’s atmosphere.
- Hydrostatic balance is obeyed to great accuracy under nearly all conditions in the Earth’s atmosphere.
Thickenes

• Meteorologists often think of the atmosphere in layers, that is, from one height to another, or from one pressure level to another.
• The vertical distance between two pressure levels is called that layer's **thickness**.
**Thickness**

- Less (more) dense air will correspond to a greater (smaller) thickness.
- Less (more) dense air will correspond to a higher (lower) average temperature.
- Therefore, the temperature should have a bearing on the thickness between two isobaric levels.
Thickness

- Average height of 500-mb surface (5600 m)
- 500-mb surface
- High height
- Low height
- Warm air
- Cold air
- North
- South
- 5600 m
Thickness Equation

• Using the ideal gas law, the hydrostatic equation can be rewritten as

$$\frac{dP}{dz} = -\frac{g}{R_d T} P$$

• Integrating this expression between pressure levels $P_1$ and $P_2$ at which the heights are $z_1$ and $z_2$, we obtain

$$\frac{R_d \bar{T}}{g} \ln \left( \frac{P_1}{P_2} \right) = z_2 - z_1 = \Delta z$$

• This expression is known as the thickness equation, or the hypsometric equation
Thickness Equation

- We can express the thickness equation in terms of a quantity called the **geopotential** $\Phi$.
- The geopotential is defined as the work required to raise a unit mass a distance $dz$ above sea level.
- Therefore, geopotential is given as $d\Phi = gdz$.
- Using this expression, we can rewrite the thickness equation as

$$R_d \bar{T} \ln \left( \frac{P_1}{P_2} \right) = \Phi_2 - \Phi_1 = \Delta \Phi$$
Thickness and Temperature Advection

• We will often refer to geopotential height \((Z)\) in subsequent discussions and map analysis. The geopotential height is simply given by

\[
Z = \frac{\Phi}{g}
\]

• The thickness equation allows us to make a general relationship between temperature and geopotential heights on isobaric maps.

• As a rule, warmer temperatures in the lower troposphere imply higher geopotential heights at upper levels.

• Since warm air leads to a greater thickness than cold air, thickness is another easy way to diagnose where warm and cold air masses are present in the atmosphere.
Thickness and Temperature Advection
Thickness and Precipitation Type

• Thickness is often used to determine the 50% probability of snow, given that precipitation is occurring.
• This is referred to as the rain-snow line.
• The table shows thickness values for the 1000-500 mb layer associated with a 50% chance of snow given that precipitation is occurring.
One of the most common analysis products used to characterize and understand the weather is a sea level pressure map. In geographical regions characterized by high terrain, such as the Rocky Mountains of North America, the elevation is so far above sea level that use of the station pressure does not effectively contribute to this goal. In such regions the thickness equation can be used to calculate a reduced sea-level pressure, which is an estimate of what the sea-level pressure would be were the surface elevation 0 m.
Altimeter Equation

• We begin with the thickness equation
\[
\frac{R_d \bar{T}}{g} \ln \left( \frac{P_1}{P_2} \right) = z_2 - z_1 = \Delta z
\]

• Let \( P_1 = P_{RSLP} \) and \( z_1 = 0 \text{ m} \) be the desired values at sea level and let \( P_2 = P_{STA} \) and \( z_2 = z_{STA} \) be the observed station values.

• Rearranging the thickness equation gives us
\[
\frac{g z_{STA}}{R_d \bar{T}} = \ln \left( \frac{P_{RSLP}}{P_{STA}} \right)
\]

• Solving for the \( p_{RSLP} \) gives the \textbf{altimeter equation}
\[
p_{RSLP} = p_{STA} \exp \left( \frac{g z_{STA}}{R_d \bar{T}} \right)
\]
Conservation of Mass

- Considering an infinitesimal cube, fixed in space, through which air flows.
- The rate at which mass flows at the center of the cube (known as the mass flux) is given as
  \[ \text{Flux} = \rho \vec{V} \]
- If the inflow rate exceeds the outflow rate, we would say that mass is accumulating towards the center of the cube and thus there is net convergence into the infinitesimal cube.
- Conversely, if the outflow rate exceeds the inflow rate, we would say that mass is evacuating away from the center of the cube and thus there is net divergence out of the infinitesimal cube.
Conservation of Mass

- Hence, the divergence of the wind field is a measure of the rate at which mass is removed from a given volume of air.
- Likewise, the convergence of the wind field as a measure of the rate at which mass is accumulates into a given volume of air.
- This physically suggests that the net rate of mass accumulation in the cube is represented by the divergence/convergence of the wind field.
- Therefore,
\[ \frac{d\rho}{dt} \propto \text{Net Divergence} \]
Conservation of Mass

- Physically, mass conservation states that regions of local convergence (divergence) leads to an increase (decrease) in mass.
- The mass continuity equation also demonstrates that horizontal convergence (divergence) leads to rising (sinking) motion in the atmosphere.
In synoptic meteorology, this statement can be explained in terms of **Dines compensation principle**.

The Dines compensation principle states that there must be at least one level of nondivergence in the troposphere [typically called the **level of non-divergence (LND)**].

This level is usually around 550 mb, but can be highly variable depending on atmospheric stability.

The compensation principle states that the convergence (divergence) that occurs above the LND tends to be offset by divergence (convergence) that occurs below the LND.
Dines Compensation Principle

• Thus, if upper-level divergence occurs, the troposphere will attempt to compensate by initiating rising motion to “fill the void”.

• Increased convergence in the lower troposphere will usually result. This process is sometimes referred to as the chimney effect.

• When air is forced to rise vertically from the surface, it rises into regions where pressure is lower.

• If the divergence aloft is stronger than the convergence in the lower levels, surface pressure falls will occur since mass is being removed from the column by divergence.
Dines Compensation Principle

• If upper-level convergence occurs, the troposphere will attempt to compensate by initiating sinking motion (also called subsidence).

• Increased divergence in the lower troposphere will usually result. This process is sometimes referred to as the **damper effect**.

• If the convergence aloft is stronger than the divergence in the lower-levels, surface pressure rises will occur since mass is being added to the column by the convergence.
Convergence and Vertical Motion
Convergence and Vertical Motion
Mass Continuity Equation

• The rate of mass inflow through the left-hand face (per unit area) is
  \[ \rho u - \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \]

• The rate of mass outflow through the right face (per unit area) is
  \[ \rho u + \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \]
Mass Continuity Equation

• The net flow rate into the volume due to the x-velocity component is

\[
\left[ \rho u - \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right] - \left[ \rho u + \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right] = - \frac{\partial}{\partial x} (\rho u) \delta x \delta y \delta z
\]

• Generalizing to three-dimensions, the net flow rate is given by gives

\[
- \left[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] \delta x \delta y \delta z
\]

• Since the increase of mass (per unit) volume is just the local density change, we have

\[
\frac{\partial \rho}{\partial t} = - \left[ \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right] = - \nabla \cdot (\rho \vec{V})
\]

• This is known as the mass divergence form of the continuity equation.
Conservation of Momentum

- Newton’s 2nd law is a statement of the conservation of momentum
  \[ \frac{d}{dt}(m\vec{V}) = \vec{F}_{\text{net}} \]
- As discussed previously, the five major forces that impact atmospheric motion are the
  - Pressure Gradient Force
  - Gravitational Force
  - Viscous Force
  - Centrifugal Force
  - Coriolis Force
- Observations show that the atmosphere is largely in hydrostatic balance and geostrophic balance.
Geostrophic Wind

- In geostrophic balance, there is a balance between the pressure gradient force (PGF) and the Coriolis force (CF).
- PGF is always directed from high to low pressure.
- For balance, CF must be equal and opposite to the PGF as depicted.
- The CF displaces parcels to the right of its motion, causing the air parcels to flow parallel to isobars.
Geostrophic Wind

• For upper air maps, geopotential heights are plotted on isobaric surfaces.
• Here, the geostrophic wind is parallel to the geopotential height contours with a magnitude dependent on the spacing of the isopleths.
• For synoptic-scale motions, the actual wind within 10-15% of the observed wind.
The Momentum Equations

• The momentum equations, neglecting frictional effects and including the effects of latitude, can be written in Cartesian coordinates as

\[
\begin{align*}
\frac{du}{dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + f v \\
\frac{dv}{dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} - f u \\
\frac{dw}{dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial z} - g
\end{align*}
\]

• Here, \( f = 2\Omega \sin \phi \) is the Coriolis parameter
Hydrostatic Balance

\[ \frac{dw}{dt} = - \frac{1}{\rho} \frac{\partial P}{\partial z} - g \]

- For synoptic-scale motions, \( w \) and \( \frac{dw}{dt} \) is very small. Therefore, we have
  \[ - \frac{1}{\rho} \frac{dP}{dz} \approx g \]
- Therefore, the dominant balance will be between the vertical pressure gradient force and the gravitational force.
- To first order, the atmosphere is in hydrostatic balance vertically.
Geostrophic Balance

\[
\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + fv, \quad \frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial P}{\partial y} - fu
\]

- For synoptic-scale motions, \(du/dt\) and \(dv/dt\) are small. Therefore, we have

\[
\frac{1}{\rho} \frac{\partial P}{\partial x} \approx fv_g, \quad \frac{1}{\rho} \frac{\partial P}{\partial y} \approx -fu_g
\]

- This can be written in vector notation as

\[
\vec{V}_g = \frac{\hat{k}}{\rho f} \times \nabla z P
\]

- Therefore, the dominant balance will be between the horizontal pressure gradient force and the Coriolis force. To first order, the atmosphere is in geostrophic balance horizontally.
Horizontal Momentum Equations

• In order to recast the momentum equations in isobaric coordinates, we must convert the pressure gradient force term into an equivalent expression in isobaric coordinates.

• This is done by considering the differential $dP$ on a constant pressure surface:

\[
   dP = \left( \frac{\partial P}{\partial x} \right)_{y,z} dx_P + \left( \frac{\partial P}{\partial y} \right)_{x,z} dy_P + \left( \frac{\partial P}{\partial z} \right)_{x,y} dz_P
\]

• Since there’s no change in pressure on an isobaric surface, then $dP = 0$ so that

\[
   0 = \left( \frac{\partial P}{\partial x} \right)_{y,z} dx_P + \left( \frac{\partial P}{\partial y} \right)_{x,z} dy_P + \left( \frac{\partial P}{\partial z} \right)_{x,y} dz_P
\]
Horizontal Momentum Equations

Next, we expand $dz_p$ as a function of $x$ and $y$ to yield

$$0 = \left( \frac{\partial P}{\partial x} \right)_{y,z} dx_p + \left( \frac{\partial P}{\partial y} \right)_{x,z} dy_p + \left( \frac{\partial P}{\partial z} \right)_{x,y} \left[ \left( \frac{\partial z}{\partial x} \right)_{y,p} dx_p + \left( \frac{\partial z}{\partial y} \right)_{x,p} dy_p \right]$$

This can be rearranged into

$$0 = \left[ \left( \frac{\partial P}{\partial x} \right)_{y,z} + \left( \frac{\partial P}{\partial z} \right)_{x,y} \left( \frac{\partial z}{\partial x} \right)_{y,p} \right] dx_p + \left[ \left( \frac{\partial P}{\partial y} \right)_{x,z} + \left( \frac{\partial P}{\partial z} \right)_{x,y} \left( \frac{\partial z}{\partial y} \right)_{x,p} \right] dy_p$$

Since this statement is true for all $dx$ and $dy$, this implies that

$$\left( \frac{\partial P}{\partial x} \right)_{y,z} = -\left( \frac{\partial P}{\partial z} \right)_{x,y} \left( \frac{\partial z}{\partial x} \right)_{y,p} \quad \left( \frac{\partial P}{\partial y} \right)_{x,z} = -\left( \frac{\partial P}{\partial z} \right)_{x,y} \left( \frac{\partial z}{\partial y} \right)_{x,p}$$
Horizontal Momentum Equations

• Using the hydrostatic equation and dividing by $\rho$ gives

$$-rac{1}{\rho} \left( \frac{\partial P}{\partial x} \right)_{y,z} = -g \left( \frac{\partial z}{\partial x} \right)_{y,p} = - \left( \frac{\partial \Phi}{\partial x} \right)_{y,p}$$

$$-rac{1}{\rho} \left( \frac{\partial P}{\partial y} \right)_{x,z} = -g \left( \frac{\partial z}{\partial y} \right)_{x,p} = - \left( \frac{\partial \Phi}{\partial y} \right)_{y,p}$$

• The frictionless momentum equations in isobaric coordinates can be written as

$$\frac{du}{dt} = - \frac{\partial \Phi}{\partial x} + fv, \quad \frac{dv}{dt} = - \frac{\partial \Phi}{\partial y} - fu$$

• For synoptic-scale motions, $du/dt$ and $dv/dt$ are small. Therefore, we have

$$u_g = - \frac{1}{f} \frac{\partial \Phi}{\partial y}, \quad v_g = \frac{1}{f} \frac{\partial \Phi}{\partial x}$$

• This can be written in vector notation as

$$\vec{V}_g = \frac{\hat{k}}{f} \times \nabla_P \Phi$$
Momentum Equations
Thermal Wind Balance

- To first order, the synoptic-scale environment is in geostrophic and hydrostatic balance.
- These conditions can be combined to produce a single balance condition known as **thermal wind balance**.
- Thermal wind balance indicates that there is a relationship between the vertical shear of the geostrophic wind and the horizontal temperature gradient.
- Thermal wind balance provides us with a powerful diagnostic tool for understanding the structure, dynamics, and evolution of mid-latitude weather systems.
Thermal Wind Balance

- Consider a hypothetical example in which a cold column and a warm column are horizontally juxtaposed.
- The thickness must be larger in the warm air than the cold such that the 800 and 500 hPa surfaces slope downward toward the cold air.
- Note that the slope of the isobaric surfaces increases with increasing height in the presence of horizontal gradient in column average temperature.
Thermal Wind Balance

• The increased slope of the isobaric surfaces mean that the magnitude of the horizontal pressure gradient force increases with height.

• Therefore, there is a relationship between the vertical shear of the geostrophic wind and the horizontal temperature gradient.
Thermal Wind Balance

• From the discussion of geostrophic wind, we saw that the magnitude of the geostrophic wind was

\[ V_g \approx \frac{1}{f} \frac{\Delta \Phi}{L} \]

• If we have geostrophic wind at two different isobaric levels, then the thermal wind is given by

\[ V_T = V_{g,2} - V_{g,1} = \frac{1}{f} \frac{\Delta (\Phi_2 - \Phi_1)}{L} \]

• Using the thickness equation, the thermal wind equation gives

\[ V_T = \frac{1}{f} \frac{\Delta (\Phi_2 - \Phi_1)}{L} = \frac{R_d}{f} \ln \left( \frac{P_1}{P_2} \right) \frac{\Delta (\bar{T}_2 - \bar{T}_1)}{L} \]
Thermal Wind Balance

\[ V_T = \frac{R_d}{f} \ln \left( \frac{P_1}{P_2} \right) \frac{\Delta(\bar{T}_2 - \bar{T}_1)}{L} \]

- Therefore, the thermal wind is proportional to the temperature gradient in the layer.
- Just as the geostrophic wind is proportional to the geopotential height gradient on an isobaric surface, the thermal wind is proportional to the thickness gradient between two isobaric surfaces, which is determined by the temperature gradient.
- The thermal wind relationship suggests that wind, geopotential height, and temperature are all locked together such that a change in one results in a change in the others.
Thermal Wind Balance

\[ V_T = \frac{R_d}{f} \ln \left( \frac{P_1}{P_2} \right) \frac{\Delta(\bar{T}_2 - \bar{T}_1)}{L} \]

- In graphical form, the thermal wind vector is simply the vector difference between the geostrophic wind at some upper level in the atmosphere and the geostrophic wind at some lower level.
- Using vector addition and subtraction, we see that when thickness contours are parallel to geopotential height contours, the geostrophic winds at different levels will be parallel. This implies that the thermal wind is parallel to the geostrophic wind.
Thermal Wind Balance

• Writing the hydrostatic equation in terms of the geopotential gives
  \[ \frac{\partial \Phi}{\partial P} = -\frac{RT}{P} \]

• The vertical derivative of the geostrophic wind relationship is
  \[ \frac{\partial \vec{V}_g}{\partial P} = \frac{\hat{k}}{f} \times \nabla \frac{\partial \Phi}{\partial P} \]

• Using the isobaric form of the hydrostatic equation yields
  \[ \frac{\partial \vec{V}_g}{\partial P} = -\frac{R\hat{k}}{fP} \times \nabla T \]

• The vertical shear of the geostrophic wind is known as the **thermal wind**.
Thermal Wind Balance

• The component form of the thermal wind equation yields

\[
\frac{\partial u_g}{\partial P} = \frac{R}{fP} \frac{\partial y}{\partial y} \quad \frac{\partial v_g}{\partial P} = -\frac{R}{fP} \frac{\partial T}{\partial x}
\]

• In the figure, we see that

\[
\frac{\partial T}{\partial x} > 0 \quad \text{and} \quad \frac{\partial v_g}{\partial p} < 0
\]
Thermal Wind Balance

• We can also express the thermal wind in terms of the geopotential difference using the thickness equation

\[
\frac{\partial u_g}{\partial P} = \frac{R}{fP} \frac{\partial T}{\partial y} \quad \frac{\partial v_g}{\partial P} = -\frac{R}{fP} \frac{\partial T}{\partial x}
\]

• Solving for the thermal wind vector gives

\[
\begin{align*}
    u_T &= -\frac{R}{fP} \frac{\partial \langle T \rangle}{\partial y} \ln \left( \frac{P_0}{P_1} \right) = \frac{1}{f} \frac{\partial}{\partial y} (\Phi_1 - \Phi_0) \\
    v_T &= \frac{R}{fP} \frac{\partial \langle T \rangle}{\partial x} \ln \left( \frac{P_0}{P_1} \right) = \frac{1}{f} \frac{\partial}{\partial x} (\Phi_1 - \Phi_0)
\end{align*}
\]

• The vector form of the thermal wind relation is

\[
\vec{V}_T = \frac{R}{f} \ln \left( \frac{P_0}{P_1} \right) \hat{k} \times \nabla T = \frac{\hat{k}}{f} \times \nabla (\Phi_1 - \Phi_0)
\]
Thermal Wind Vector

- When height and thickness contours are parallel, the geostrophic winds at different levels are parallel, too.
- Thus, the thermal wind is parallel to the geostrophic wind.
Thermal Wind Vector

- When thickness contours are not parallel to height contours, the wind changes direction with height.
- In the figure below, we have a Northern Hemisphere case with 1000 mb geopotential heights oriented perpendicular to thickness contours for the mean 1000-500 mb layer.
- When cold air is to the north, the top of the 1000-500 mb thickness layer tilts down to the north. This makes the 1000-500 mb contours perpendicular to those at 1000 mb.
Thermal Wind Vector

• Adding the 1000-500 mb thickness (that slopes down to the north) to the 1000 mb heights (that slope down to the west) results in 500 mb heights that slope down to the northwest and a little more steeply than 1000 mb heights. This requires that the wind is somewhat stronger at 500 mb.
• The thermal wind is oriented parallel to the thickness contours of the 1000-500 mb layer, which are parallel to the isotherms for the average temperature in the layer.
• This case clearly demonstrates the simple rule that (in the Northern Hemisphere) ``with the thermal wind at your back, cold air is to your left!"
Thermal Wind and Temperature Advection

- The thermal wind relation indicates that the thermal wind blows parallel to isopleths with the warm air to the right facing downstream in the Northern Hemisphere.
- Therefore, geostrophic wind that turns counterclockwise (clockwise) with height is associated with cold advection (warm advection).
Thermal Wind and Temperature Advection

- Because of the thermal wind relationship, meteorologists are often interested in how the wind direction changes with respect to time, or across a given space.
- The vertical variation of the wind field at a fixed location is sometimes plotted as a hodograph.
- Thus, hodographs are useful not only because they depict the wind as a function of height, but they also because they show the vertical wind shear as a function of height.
- If the meteorological wind direction increases with time or height, then the wind field is veering with time or height.
- If the wind direction decreases with time or height, then the wind field is backing with time or height.
- Thus, hodographs can be used to anticipate warm air advection and cold air advection into a region.