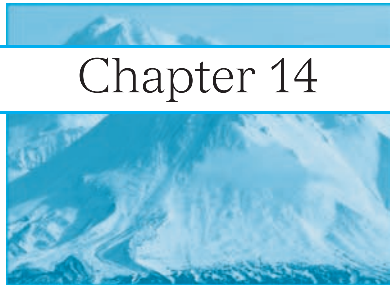


Chapter 14

Fourier Series

14.1 General Properties

Periodic phenomena involving waves [$\sim \sin(2\pi x/\lambda)$ as a crude approximation to water waves, for example], motors, rotating machines (harmonic motion), or some repetitive pattern of a driving force are described by periodic functions. Fourier series are a basic tool for solving ordinary differential equations (ODEs) and partial differential equations (PDEs) with periodic boundary conditions. Fourier integrals for nonperiodic phenomena are developed in Chapter 15. The common name for the whole field is **Fourier analysis**.

A Fourier series is defined as an expansion of a function or representation of a function in a series of sines and cosines, such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \tag{14.1}$$

The coefficients a_0, a_n, and b_n are related to the periodic function $f(x)$ by definite integrals:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx, \tag{14.2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx, \qquad n = 0, 1, 2, \ldots. \tag{14.3}$$

This result, of course, is subject to the requirement that the integrals exist. They do if $f(x)$ is piecewise continuous (or square integrable). Notice that a_0 is singled out for special treatment by the inclusion of the factor $\frac{1}{2}$. This is done so that Eq. (14.2) will apply to all a_n, $n = 0$ as well as $n > 0$.

The Sturm–Liouville theory of the harmonic oscillator in Example 9.1.4 guarantees the validity of Eqs. (14.2) and (14.3) and, by use of the orthogonality relations (Example 9.2.1), allows us to compute the expansion coefficients.

Another way of describing what we are doing here is to say that $f(x)$ is part of an infinite-dimensional Hilbert space, with the orthogonal $\cos nx$ and $\sin nx$ as the basis. The statement that $\cos nx$ and $\sin nx\,(n = 0, 1, 2, \ldots)$ span this Hilbert space is equivalent to saying that they form a complete set. Finally, the **expansion coefficients** a_n and b_n **correspond to the projections** of $f(x)$, with the integral inner products [Eqs. (14.2) and (14.3)] playing the role of the dot product of Section 1.2. These points are outlined in Section 9.4.

The conditions imposed on $f(x)$ to make Eq. (14.1) valid, and the series convergent, are that $f(x)$ has only a finite number of finite discontinuities and only a finite number of extreme values, maxima, and minima in the interval $[0, 2\pi]$.[1] Functions satisfying these conditions are called **piecewise regular**. The conditions are known as the **Dirichlet conditions**. Although there are some functions that do not obey these Dirichlet conditions, they may well be labeled pathological for purposes of Fourier expansions. In the vast majority of physical problems involving a Fourier series, these conditions will be satisfied. In most physical problems we shall be interested in functions that are square integrable, for which the sines and cosines form a complete orthogonal set. This in turn means that Eq. (14.1) is valid in the sense of convergence in the mean (see Eq. 9.63).

Completeness

The Fourier expansion and the completeness property may be expected because the functions $\sin nx$, $\cos nx$, e^{inx} are all eigenfunctions of a self-adjoint linear ODE,

$$y'' + n^2 y = 0. \tag{14.4}$$

We obtain orthogonal eigenfunctions for different values of the eigenvalue n for the interval $[0, 2\pi]$ to satisfy the boundary conditions in the Sturm–Liouville theory (Chapter 9). The different eigenfunctions for the same eigenvalue n are orthogonal. We have

$$\int_0^{2\pi} \sin mx \sin nx\, dx = \begin{cases} \pi\,\delta_{mn}, & m \neq 0, \\ 0, & m = 0, \end{cases} \tag{14.5}$$

$$\int_0^{2\pi} \cos mx \cos nx\, dx = \begin{cases} \pi\,\delta_{mn}, & m \neq 0, \\ 2\pi, & m = n = 0, \end{cases} \tag{14.6}$$

$$\int_0^{2\pi} \sin mx \cos nx\, dx = 0, \quad \text{for all integral } m \text{ and } n. \tag{14.7}$$

Note that any interval $x_0 \leq x \leq x_0 + 2\pi$ will be equally satisfactory. Frequently, we use $x_0 = -\pi$ to obtain the interval $-\pi \leq x \leq \pi$. For the complex eigenfunctions $e^{\pm inx}$, orthogonality is usually **defined** in terms of the complex

[1]These conditions are **sufficient** but not **necessary**.

conjugate of one of the two factors,

$$\int_0^{2\pi} (e^{imx})^* e^{inx} dx = 2\pi\,\delta_{mn}. \tag{14.8}$$

This agrees with the treatment of the spherical harmonics (Section 11.5).

EXAMPLE 14.1.1

Sawtooth Wave Let us apply Eqs. (14.2) and (14.3) to the sawtooth shape shown in Fig. 14.1 to derive its Fourier series. Our sawtooth function can also be expressed as

$$f(x) = \begin{cases} x, & 0 \le x < \pi, \\ x - 2\pi, & \pi \le x \le 2\pi, \end{cases}$$

which is an odd function of the variable x. Hence, we expect a pure sine expansion. Integrating by parts, we indeed find

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{x}{n\pi}\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin nx\,dx \\ &= \frac{1}{n^2\pi}\cos nx\Big|_{-\pi}^{\pi} = \frac{1}{n^2\pi}[(-1)^n - (-1)^n] = 0, \end{aligned}$$

while

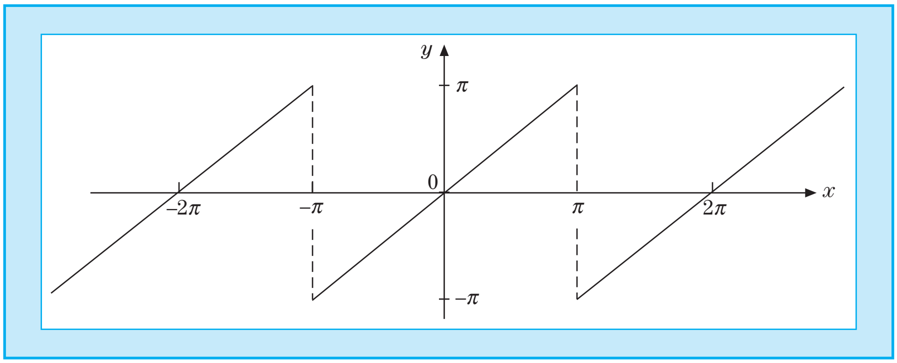
$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = -\frac{x}{n\pi}\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\sin nx\,dx \\ &= -\frac{2}{n}(-1)^n - \frac{1}{n^2\pi}\cos nx\Big|_{-\pi}^{\pi} = -\frac{2}{n}(-1)^n. \end{aligned}$$

This establishes the Fourier expansion

$$\begin{aligned} f(x) &= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin nx}{n} + \cdots\right] \\ &= 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} = x, \quad -\pi < x < \pi, \end{aligned} \tag{14.9}$$

Figure 14.1

Sawtooth Wave



which converges only conditionally, not absolutely, because of the discontinuity of $f(x)$ at $x = \pm\pi$. It makes no difference whether a discontinuity is in the interior of the expansion interval or at its ends: It will give rise to conditional convergence of the Fourier series. In terms of physical applications with $x = \omega$ a frequency, conditional convergence means that our square wave is dominated by **high-frequency components**. ■

Behavior of Discontinuities

The behavior at $x = n\pi$ is an example of a general rule that at a finite discontinuity the series converges to the arithmetic mean. For a discontinuity at $x = x_0$ the series yields

$$f(x_0) = \frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)], \tag{14.10}$$

where the arithmetic mean of the right and left approaches to $x = x_0$. A general proof using partial sums is given by Jeffreys and by Carslaw (see Additional Reading).

An idea of the convergence of a Fourier series and the error in using only a finite number of terms in the series may be obtained by considering the expansion of the sawtooth shape of Fig. 14.1. Figure 14.2 shows $f(x)$ for $0 \leq x < \pi$ for the sum of 4, 6, and 10 terms of the series. Three features deserve comment:

1. There is a steady increase in the accuracy of the representation as the number of terms included is increased.
2. All the curves pass through the midpoint $f(x) = 0$ at $x = \pi$.
3. In the vicinity of $x = \pi$ there is an overshoot that persists and shows no sign of diminishing. This **overshoot (and undershoot)** is called the **Gibbs phenomenon** and is a typical feature of Fourier series. The inclusion of more terms does nothing to remove the overshoot (undershoot) but merely moves it closer to the point of discontinuity. The Gibbs phenomenon is not

Figure 14.2

Fourier Representation of Sawtooth Wave

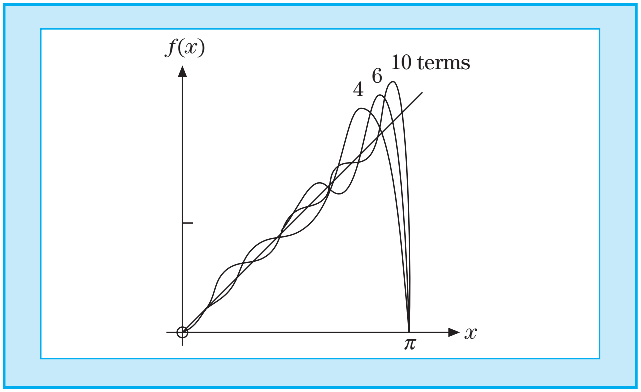
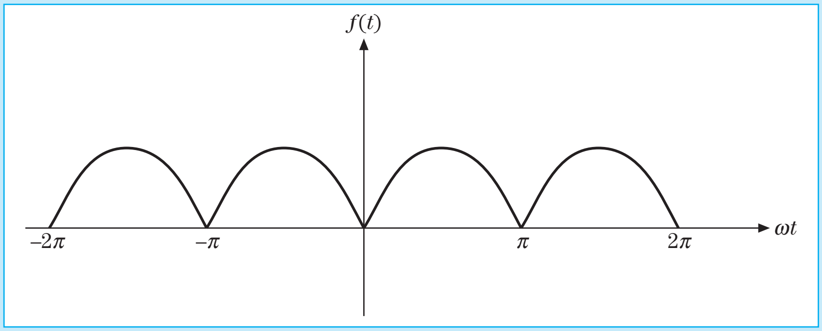


Figure 14.3

Full Wave Rectifier



limited to Fourier series. It occurs with other eigenfunction expansions. For more details, see W. J. Thomson, Fourier series and the Gibbs phenomenon, *Am. J. Phys.* **60**, 425 (1992).

One of the advantages of a Fourier representation over some other representation, such as a Taylor series, is that it can represent a discontinuous function. An example is the sawtooth wave in the preceding section and Example 14.1.3. Other examples are considered in the exercises.

EXAMPLE 14.1.2

Full-Wave Rectifier Consider the case of an absolutely convergent Fourier series representing a continuous periodic function, displayed in Fig. 14.3. Let us ask how well the output of a full-wave rectifier approaches pure direct current. Our rectifier may be thought of as having passed the positive peaks of an incoming sine wave and inverting the negative peaks. This yields

$$\begin{aligned} f(t) &= \sin\omega t, \qquad 0 < \omega t < \pi, \\ f(t) &= -\sin\omega t, \quad -\pi < \omega t < 0. \end{aligned} \tag{14.11}$$

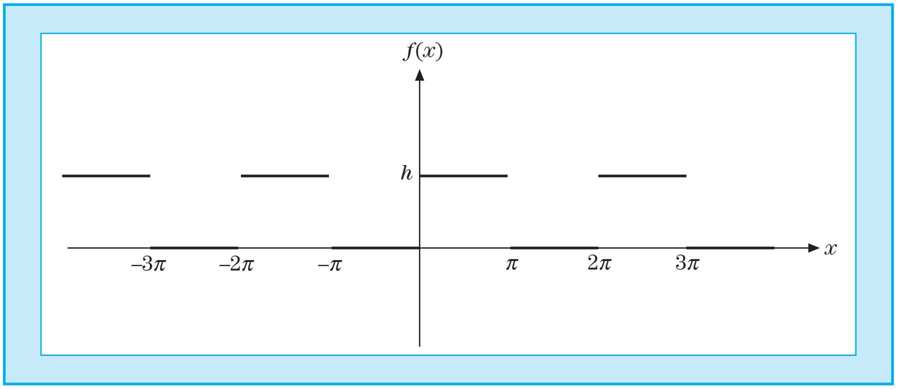
Since $f(t)$ defined here is even, no terms of the form $\sin n\omega t$ will appear. Again, from Eqs. (14.2) and (14.3), we have

$$\begin{aligned} a_0 &= -\frac{1}{\pi}\int_{-\pi}^{0} \sin\omega t\, d(\omega t) + \frac{1}{\pi}\int_{0}^{\pi} \sin\omega t\, d(\omega t) \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin\omega t\, d(\omega t) = \frac{4}{\pi}, \end{aligned} \tag{14.12}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{0}^{\pi} \sin\omega t \cos n\omega t\, d(\omega t) \\ &= -\frac{2}{\pi}\frac{2}{n^2-1}, \quad n \text{ even}, \\ &= 0, \quad n \text{ odd}. \end{aligned} \tag{14.13}$$

Figure 14.4

Square Wave



Note that $(0, \pi)$ is not an orthogonality interval for both sines and cosines together and we do not get zero for even n. The resulting series is

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \frac{\cos n\omega t}{n^2 - 1}. \tag{14.14}$$

The original frequency ω has been eliminated. The lowest frequency oscillation is 2ω. The high-frequency components fall off as n^{-2}, showing that the full-wave rectifier does a fairly good job of approximating direct current. Whether this good approximation is adequate depends on the particular application. If the remaining ac components are objectionable, they may be further suppressed by appropriate filter circuits. These two examples highlight two features characteristic of Fourier expansions:[2]

- If $f(x)$ has discontinuities (as in the sawtooth wave in Example 14.1.1), we can expect the nth coefficient to be decreasing as $\mathcal{O}(1/n)$. Convergence is conditional only.[3]
- If $f(x)$ is continuous (although possibly with discontinuous derivatives as in the full-wave rectifier of Example 14.1.2), we can expect the nth coefficient to be decreasing as $1/n^2$, that is, absolute convergence. ■

EXAMPLE 14.1.3

Square Wave–High Frequencies One application of Fourier series, the analysis of a "square" wave (Fig. 14.4) in terms of its Fourier components, occurs in electronic circuits designed to handle sharply rising pulses. This example explains the physical meaning of conditional convergence. Suppose that our wave is defined by

$$\begin{aligned} f(x) &= 0, \quad -\pi < x < 0, \\ f(x) &= h, \quad 0 < x < \pi. \end{aligned} \tag{14.15}$$

[2] Raisbeek, G. (1955). Order of magnitude of Fourier coefficients. *Am. Math. Mon.* **62**, 149–155.
[3] A technique for improving the rate of convergence is developed in the exercises of Section 5.9.

From Eqs. (14.2) and (14.3) we find

$$a_0 = \frac{1}{\pi}\int_0^\pi h\,dt = h, \tag{14.16}$$

$$a_n = \frac{1}{\pi}\int_0^\pi h\cos nt\,dt = 0, \quad n = 1, 2, 3, \ldots, \tag{14.17}$$

$$b_n = \frac{1}{\pi}\int_0^\pi h\sin nt\,dt = \frac{h}{n\pi}(1-\cos n\pi); \tag{14.18}$$

$$b_n = \frac{2h}{n\pi}, \quad n \text{ odd}, \tag{14.19}$$

$$b_n = 0, \quad n \text{ even}. \tag{14.20}$$

The resulting series is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{14.21}$$

Except for the first term, which represents an average of $f(x)$ over the interval $[-\pi, \pi]$, all the cosine terms have vanished. Since $f(x) - h/2$ is odd, we have a Fourier sine series. Although only the odd terms in the sine series occur, they fall only as n^{-1}. This **conditional convergence** is like that of the harmonic series. Physically, this means that our square wave contains a lot of **high-frequency components**. If the electronic apparatus will not pass these components, our square wave input will emerge more or less rounded off, perhaps as an amorphous blob. ■

Biographical Data

Fourier, Jean Baptiste Joseph, Baron. Fourier, a French mathematician, was born in 1768 in Auxerre, France, and died in Paris in 1830. After his graduation from a military school in Paris, he became a professor at the school in 1795. In 1808, after his great mathematical discoveries involving the series and integrals named after him, he was made a baron by Napoleon. Earlier, he had survived Robespierre and the French Revolution. When Napoleon returned to France in 1815 after his abdication and first exile to Elba, Fourier rejoined him and, after Waterloo, fell out of favor for a while. In 1822, his book on the *Analytic Theory of Heat* appeared and inspired Ohm to new thoughts on the flow of electricity.

SUMMARY

Fourier series are finite or infinite sums of sines and cosines that describe periodic functions that can have discontinuities and thus represent a wider class of functions than we have considered so far. Because $\sin nx$, $\cos nx$ are eigenfunctions of a self-adjoint ODE, the classical harmonic oscillator equation, the Hilbert space properties of Fourier series are consequences of the Sturm–Liouville theory.

EXERCISES

14.1.1 A function $f(x)$ (quadratically integrable) is to be represented by a **finite** Fourier series. A convenient measure of the accuracy of the series is given by the integrated square of the deviation

$$\Delta_p = \int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{p} (a_n \cos nx + b_n \sin nx) \right]^2 dx.$$

Show that the requirement that Δ_p be minimized, that is,

$$\frac{\partial \Delta_p}{\partial a_n} = 0, \quad \frac{\partial \Delta_p}{\partial b_n} = 0,$$

for all n, leads to choosing a_n and b_n, as given in Eqs. (14.2) and (14.3). *Note*. Your coefficients a_n and b_n are independent of p. This independence is a consequence of orthogonality and would not hold for powers of x, fitting a curve with polynomials.

14.1.2 In the analysis of a complex waveform (ocean tides, earthquakes, musical tones, etc.) it might be more convenient to have the Fourier series written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx - \theta_n).$$

Show that this is equivalent to Eq. (14.1) with

$$a_n = \alpha_n \cos\theta, \quad \alpha_n^2 = a_n^2 + b_n^2,$$
$$b_n = \alpha_n \sin\theta, \quad \tan\theta_n = b_n/a_n.$$

Note. The coefficients α_n^2 as a function of n define what is called the **power spectrum**. The importance of α_n^2 lies in its invariance under a shift in the phase θ_n.

14.1.3 Assuming that $\int_{-\pi}^{\pi} [f(x)]^2 dx$ is finite, show that

$$\lim_{m\to\infty} a_m = 0, \quad \lim_{m\to\infty} b_m = 0.$$

Hint. Integrate $[f(x) - s_n(x)]^2$, where $s_n(x)$ is the nth partial sum, and use Bessel's inequality (Section 9.4). For our finite interval the assumption that $f(x)$ is square integrable ($\int_{-\pi}^{\pi} |f(x)|^2\, dx$ is finite) implies that $\int_{-\pi}^{\pi} |f(x)|\, dx$ is also finite. The converse does not hold.

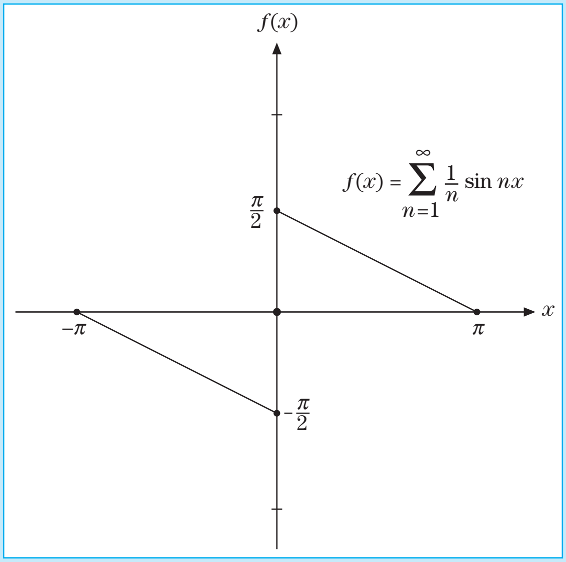
14.1.4 Apply the summation technique of this section to show that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} \frac{1}{2}(\pi - x), & 0 < x \le \pi \\ -\frac{1}{2}(\pi + x), & -\pi \le x < 0 \end{cases}$$

(Fig. 14.5).

Figure 14.5

Reverse Sawtooth Wave



14.1.5 Sum the trigonometric series

$$\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1},$$

and show that it equals

$$\begin{cases} \pi/4, & 0 < x < \pi \\ -\pi/4, & -\pi < x < 0. \end{cases}$$

14.2 Advantages and Uses of Fourier Series

Periodic Functions

Related to the advantage of describing discontinuous functions is the usefulness of a Fourier series in representing a periodic function. If $f(x)$ has a period of 2π, perhaps it is only natural that we expand it in a series of functions with period $2\pi, 2\pi/2, 2\pi/3, \ldots$. This guarantees that if our periodic $f(x)$ is represented over one interval $[0, 2\pi]$ or $[-\pi, \pi]$, the representation holds for all finite x.

At this point, we may conveniently consider the properties of **symmetry**. Using the interval $[-\pi, \pi]$, $\sin x$ is odd and $\cos x$ is an even function of x. Hence, by Eqs. (14.2) and (14.3),[4] if $f(x)$ is odd, all $a_n = 0$, and if $f(x)$ is even,

[4]With the range of integration $-\pi \le x \le \pi$.

all $b_n = 0$. In other words,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad f(x) \text{ even}, \tag{14.22}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad f(x) \text{ odd}. \tag{14.23}$$

Frequently, these properties are helpful in expanding a given function.

We have noted that the Fourier series is periodic. This is important in considering whether Eq. (14.1) holds outside the initial interval. Suppose we are given only that

$$f(x) = x, \quad 0 \leq x < \pi \tag{14.24}$$

and are asked to represent $f(x)$ by a series expansion. Let us take three of the infinite number of possible expansions:

1. If we assume a Taylor expansion, we have

$$f(x) = x, \tag{14.25}$$

   a one-term series. This (one-term) series is defined for all finite x.
2. Using the Fourier cosine series [Eq. (14.22)], we predict that

$$\begin{aligned} f(x) &= -x, & -\pi < x \leq 0, \\ f(x) &= 2\pi - x, & \pi < x \leq 2\pi. \end{aligned} \tag{14.26}$$

3. Finally, from the Fourier sine series [Eq. (14.23)], we have

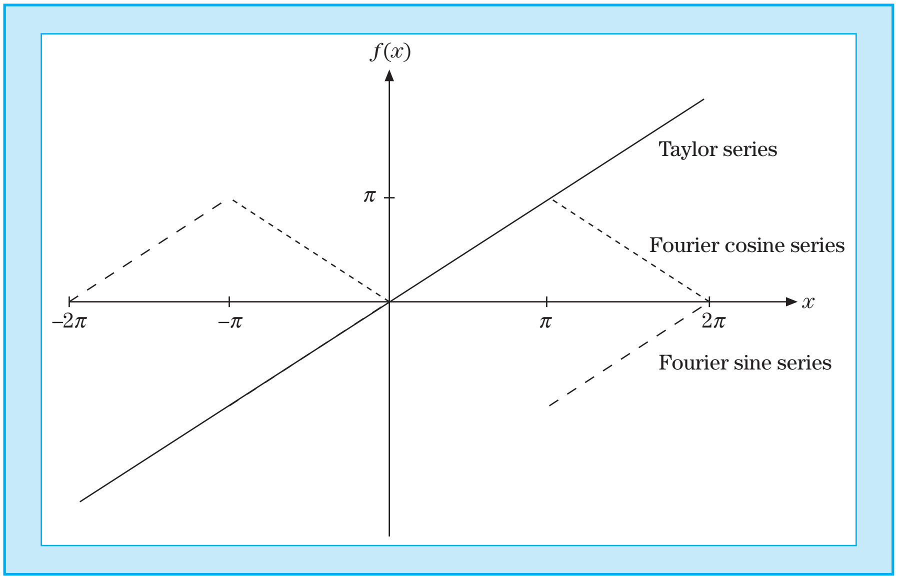
$$\begin{aligned} f(x) &= x, & -\pi < x \leq 0, \\ f(x) &= x - 2\pi, & \pi < x \leq 2\pi. \end{aligned} \tag{14.27}$$

These three possibilities—Taylor series, Fourier cosine series, and Fourier sine series—are each perfectly valid in the original interval $[0, \pi]$. Outside, however, their behavior is strikingly different (compare Fig. 14.6). Which of the three, then, is correct? This question has no answer, unless we are given more information about $f(x)$. It may be any of the three or none of them. Our Fourier expansions are valid over the basic interval. Unless the function $f(x)$ is known to be periodic, with a period equal to our basic interval, or $(1/n)$th of our basic interval, there is no assurance whatever that the representation [Eq. (14.1)] will have any meaning outside the basic interval. Clearly, the interval of length 2π, which defines the expansion, makes a real difference for a nonperiodic function because the Fourier series repeats the pattern of the basic interval in adjacent intervals. This also follows from Fig. 14.6.

In addition to the advantages of representing discontinuous and periodic functions, there is a third very real advantage in using a Fourier series. Suppose that we are solving the equation of motion of an oscillating particle, subject to

Figure 14.6

Comparison of Fourier Cosine Series, Fourier Sine Series, and Taylor Series



a periodic driving force. The Fourier expansion of the driving force then gives us the fundamental term and a series of harmonics. The (linear) ODE may be solved for each of these harmonics individually, a process that may be much easier than dealing with the original driving force. Then, as long as the ODE is linear, all the solutions may be added together to obtain the final solution.[5] This is more than just a clever mathematical trick.

- It corresponds to finding the response of the system to the fundamental frequency and to each of the harmonic frequencies, called Fourier analysis.

The following question is sometimes raised: Were the harmonics there all along, or were they created by our Fourier analysis? One answer compares the functional resolution into harmonics with the resolution of a vector into rectangular components. The components may have been present in the sense that they may be isolated and observed, but the resolution is certainly not unique. Hence, many authorities prefer to say that the harmonics were created by our choice of expansion. Other expansions in other sets of orthogonal functions would give different results. For further discussion, we refer to a series of notes and letters in the *American Journal of Physics*.[6]

[5]One of the nastier features of nonlinear differential equations is that this principle of superposition is not valid.

[6]Robinson, B. L. (1953). Concerning frequencies resulting from distortion. *Am. J. Phys.* **21**, 391; Van Name, F. W., Jr. (1954). Concerning frequencies resulting from distortion. *Am. J. Phys.* **22**, 94.

Change of Interval

So far, attention has been restricted to an interval of length 2π. This restriction may easily be relaxed. If $f(x)$ is periodic with a period $2L$, we may write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right], \tag{14.28}$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L}\, dt, \quad n = 0, 1, 2, 3, \ldots, \tag{14.29}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}\, dt, \quad n = 1, 2, 3, \ldots, \tag{14.30}$$

replacing x in Eq. (14.1) with $\pi x/L$ and t in Eqs. (14.2) and (14.3) with $\pi t/L$. [For convenience, the interval in Eqs. (14.2) and (14.3) is shifted to $-\pi \le t \le \pi$.] The choice of the symmetric interval $[-L, L]$ is **not** essential. For $f(x)$ periodic with a period of $2L$, any interval $(x_0, x_0 + 2L)$ will do. The choice is a matter of convenience or literally personal preference.

If our function $f(-x) = f(x)$ is even in x, then it has a pure cosine series because, by substituting $t \to -t$,

$$\int_{-L}^{0} f(t) \cos \frac{n\pi t}{L} dt = \int_{0}^{L} f(t) \cos \frac{n\pi t}{L} dt,$$
$$\int_{-L}^{0} f(t) \sin \frac{n\pi t}{L} dt = -\int_{0}^{L} f(t) \sin \frac{n\pi t}{L} dt$$

so that all $b_n = 0$ and

$$a_n = \frac{2}{L} \int_{0}^{L} f(t) \cos \frac{n\pi t}{L} dt.$$

Similarly, if $f(-x) = -f(x)$ is odd in x, then $f(x)$ has a pure sine series with coefficients

$$b_n = \frac{2}{L} \int_{0}^{L} f(t) \sin \frac{n\pi t}{L} dt.$$

EXAMPLE 14.2.1

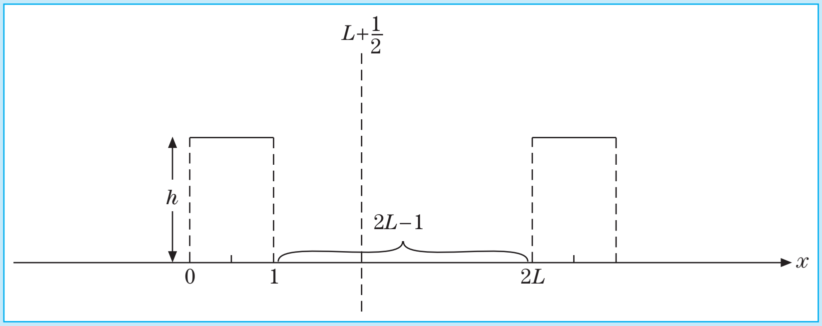
Asymmetric Square Wave Let us derive the Fourier expansion for the square wave in Fig. 14.7:

$$f(x) = h, \quad 0 < x < 1,$$
$$f(x) = 0, \quad 1 < x < 2L$$

for $L \ge 1$ and $h > 0$. The expansion interval is $[-L, L]$.

Figure 14.7

Asymmetric Square Wave



The geometry of the square pulse implies that its Fourier coefficients are given by Eqs. (14.29) and (14.30) as

$$a_n = \frac{h}{L}\int_0^1 \cos\frac{n\pi x}{L}dx = \frac{h}{n\pi}\sin\frac{n\pi x}{L}\bigg|_0^1 = \frac{h}{n\pi}\sin\frac{n\pi}{L}, \quad n = 0, 1, \dots,$$

$$b_n = \frac{h}{L}\int_0^1 \sin\frac{n\pi x}{L}dx = -\frac{h}{n\pi}\cos\frac{n\pi x}{L}\bigg|_0^1 = \frac{h}{n\pi}\left(1-\cos\frac{n\pi}{L}\right), \quad n = 1, 2, \dots .$$

The resulting Fourier series

$$f(x) = \frac{h}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\frac{n\pi}{L}\cos\frac{n\pi x}{L} + \left(1-\cos\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L}\right]$$

for $L = 1$ agrees with the square wave series of Example 14.1.3 because $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

Because f is symmetric about $x = \frac{2L+1}{2}$ (dashed vertical line in Fig. 14.7), we shift the origin of our coordinates from $x = 0$ to this new origin, calling $\xi = x - \frac{2L+1}{2}$ the new variable. We expect a pure cosine series with coefficients

$$A_n = \frac{h}{L}\int_{L-1/2}^{L+1/2}\cos\frac{n\pi\xi}{L}d\xi = \frac{h}{n\pi}\sin\frac{n\pi\xi}{L}\bigg|_{L-1/2}^{L+1/2}$$
$$= \frac{h}{n\pi}\left(\sin\frac{n\pi(2L+1)}{2L} - \sin\frac{n\pi(2L-1)}{2L}\right) = (-1)^n\frac{2h}{n\pi}\sin\frac{n\pi}{2L},$$

and $B_n = 0$. The Fourier series

$$f(x) = \frac{2h}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{2L}\cos\frac{n\pi(x-L-1/2)}{L}$$

can be seen to be equivalent to our first result using the addition formulas for trigonometric functions

$$\cos\frac{n\pi(x-L-1/2)}{L} = (-1)^n\left[\cos\frac{n\pi}{2L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi}{2L}\sin\frac{n\pi x}{L}\right]. \quad \blacksquare$$

EXERCISES

14.2.1 The boundary conditions [such as $\psi(0) = \psi(l) = 0$] may suggest solutions of the form $\sin(n\pi x/l)$ and eliminate the corresponding cosines.

(a) Verify that the boundary conditions used in the Sturm–Liouville theory are satisfied for the interval $(0, l)$. Note that this is only half the usual Fourier interval.

(b) Show that the set of functions $\varphi_n(x) = \sin(n\pi x/l)$, $n = 1, 2, 3, \ldots$ satisfies an orthogonality relation

$$\int_0^l \varphi_m(x)\varphi_n(x)dx = \frac{l}{2}\delta_{mn}, \quad n > 0.$$

14.2.2 (a) Expand $f(x) = x$ in the interval $[0, 2L]$. Sketch the series you have found (right-hand side of Answer) over $[-2L, 2L]$.

ANS. $x = L - \dfrac{2L}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\left(\dfrac{n\pi x}{L}\right).$

(b) Expand $f(x) = x$ as a sine series in the **half** interval $(0, L)$. Sketch the series you have found (right-hand side of Answer) over $[-2L, 2L]$.

ANS. $x = \dfrac{4}{\pi}\sum_{n=0}^{\infty}\dfrac{1}{2n+1}\sin\dfrac{(2n+1)\pi x}{L}.$

14.2.3 In some problems it is convenient to approximate $\sin\pi x$ over the interval $[0, 1]$ by a parabola $ax(1-x)$, where a is a constant. To get a feeling for the accuracy of this approximation, expand $4x(1-x)$ in a Fourier sine series:

$$f(x) = \begin{Bmatrix} 4x(1-x), & 0 \le x \le 1 \\ 4x(1+x), & -1 \le x \le 0 \end{Bmatrix} = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

ANS. $b_n = \dfrac{32}{\pi^3}\cdot\dfrac{1}{n^3}, \quad n$ odd

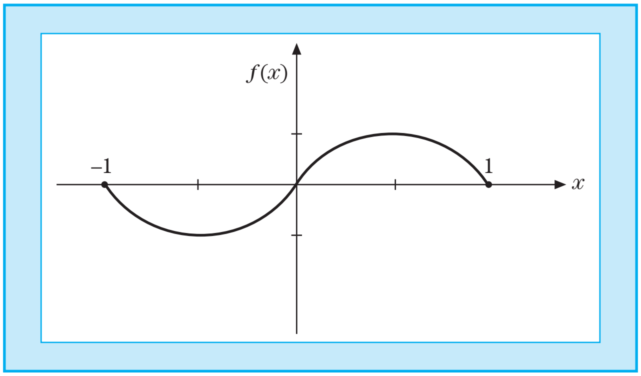
$b_n = 0, \quad n$ even

(Fig. 14.8).

14.2.4 Take $-\pi \le x \le \pi$ as the basic interval for the function $f(x) = x$ and repeat the arguments leading to Eqs. (14.25)–(14.27). Compare your results with those in Fig. 14.6 and plot them.

Figure 14.8

Parabolic Sine Wave



14.3 Complex Fourier Series

One way of summing a trigonometric Fourier series is to transform it into exponential form and compare it with a Laurent series (see Section 6.5). If we expand $f(z)$ in a Laurent series [assuming $f(z)$ is analytic],

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \tag{14.31}$$

then on the unit circle $z = e^{i\theta} = \cos\theta + i\sin\theta$ by Euler's identity so that

$$f(z) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}. \tag{14.32}$$

Expressing $\cos nx$ and $\sin nx$ in exponential form and setting $x = \theta$,

$$\cos nx = \frac{1}{2}(e^{inx} + e^{-inx}), \qquad \sin nx = \frac{1}{2i}(e^{inx} - e^{-inx}),$$

we may rewrite the general Fourier series [Eq. (14.1)] as

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \frac{1}{2}(e^{inx} + e^{-inx}) + b_n \frac{1}{2i}(e^{inx} - e^{-inx})\right] \\ &= \sum_{n=-\infty}^{\infty} c_n e^{inx}, \end{aligned} \tag{14.33}$$

in which

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \quad n > 0, \quad c_0 = \frac{1}{2}a_0. \tag{14.34}$$

The Laurent expansion on the unit circle [Eq. (14.32)] has the same form as the complex Fourier series [Eq. (14.33)], which shows the equivalence between the two expansions.

When the function $f(z)$ is known, the complex Fourier coefficients may be directly derived by projection from it:

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx, \quad -\infty < n < \infty,$$

based on the orthogonality relation [Eq. (14.8)].

Abel's Theorem

Consider a function $f(z)$ represented by a convergent power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n r^n e^{in\theta}. \tag{14.35}$$

This is our Fourier exponential series [Eq. (14.32)]. Separating real and imaginary parts

$$u(r,\theta) = \sum_{n=0}^{\infty} c_n r^n \cos n\theta, \qquad v(r,\theta) = \sum_{n=1}^{\infty} c_n r^n \sin n\theta, \tag{14.36}$$

the Fourier cosine and sine series. Abel's theorem asserts that if $u(1,\theta)$ and $v(1,\theta)$ are convergent for a given θ, then

$$u(1,\theta) + iv(1,\theta) = \lim_{r\to 1} f(re^{i\theta}). \tag{14.37}$$

An application of this theorem appears as Exercise 14.3.11, and it is used in the next example.

EXAMPLE 14.3.1

Summation of a Complex Fourier Series Consider the series $\sum_{n=1}^{\infty}(1/n)\cos nx$, $x \in (0, 2\pi)$. Since this series is only conditionally convergent (see Example 5.3.1) and diverges at $x = 0$ so that Dirichlet's conditions are violated, we take

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \lim_{r\to 1} \sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} \tag{14.38}$$

absolutely convergent for $|r| < 1$. Our procedure is again to try forming a power series by transforming the trigonometric functions into exponential form:

$$\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{r^n e^{inx}}{n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{r^n e^{-inx}}{n}. \tag{14.39}$$

Now these power series may be identified as Maclaurin expansions of $-\ln(1-z)$, $z = re^{ix}, re^{-ix}$ [Eq. (5.65)], and

$$\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = -\frac{1}{2}[\ln(1-re^{ix}) + \ln(1-re^{-ix})]$$

$$= -\ln[(1+r^2) - 2r\cos x]^{1/2}. \tag{14.40}$$

Table 14.1

	Fourier Series	Reference
1.	$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin nx = \begin{cases} -\frac{1}{2}(\pi+x), & -\pi \le x < 0 \\ \frac{1}{2}(\pi - x), & 0 \le x < \pi \end{cases}$	Exercise 14.1.4 Exercise 14.3.3
2.	$\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin nx = \frac{1}{2}x, \quad -\pi < x < \pi$	Example 14.1.1 Exercise 14.3.2
3.	$\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x = \begin{cases} -\pi/4, & -\pi < x < 0 \\ +\pi/4, & 0 < x < \pi \end{cases}$	Exercise 14.1.5 Eq. (14.52)
4.	$\displaystyle\sum_{n=1}^{\infty}\frac{\cos nx}{n} = -\ln\left[2\sin\left(\frac{\lvert x\rvert}{2}\right)\right], \quad -\pi < x < \pi$	Eq. (14.38)
5.	$\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}\cos nx = -\ln\left[2\cos\left(\frac{x}{2}\right)\right], \quad -\pi < x < \pi$	Exercise 14.3.11
6.	$\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos(2n+1)x = \frac{1}{2}\ln\left[\cot\frac{\lvert x\rvert}{2}\right], \quad -\pi < x < \pi$	(Item 5 − Item 4)
7.	$\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{\cos nx}{n^2} = \frac{x^2}{4} - \frac{\pi^2}{12}, \quad -\pi < x < \pi$	Example 14.4.2
8.	$\displaystyle\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} = \frac{\pi}{4}\left(\frac{\pi}{2} - \lvert x\rvert\right), \quad -\pi < x < \pi$	Exercise 14.3.4

Letting $r = 1$ based on Abel's theorem, we obtain item 4 of Table 14.1

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n} = -\ln(2 - 2\cos x)^{1/2}$$
$$= -\ln\left(2\sin\frac{x}{2}\right), \qquad x \in (0, 2\pi).^7 \tag{14.41}$$

Both sides of this expression diverge as $x \to 0$ and 2π. ■

EXERCISES

14.3.1 Develop the Fourier series representation of

$$f(t) = \begin{cases} 0, & -\pi \le \omega t \le 0, \\ \sin\omega t, & 0 \le \omega t \le \pi. \end{cases}$$

This is the output of a simple half-wave rectifier. It is also an approximation of the solar thermal effect that produces "tides" in the atmosphere.

$$ANS. \quad f(t) = \frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos n\omega t}{n^2 - 1}.$$

[7] The limits may be shifted to $(-\pi, \pi)$ (and $x \neq 0$) using $|x|$ on the right-hand side.

14.3.2 A sawtooth wave is given by

$$f(x) = x, \quad -\pi < x < \pi.$$

Show that

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

14.3.3 A different sawtooth wave is described by

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x), & -\pi \le x < 0 \\ +\frac{1}{2}(\pi - x), & 0 < x \le \pi. \end{cases}$$

Show that $f(x) = \sum_{n=1}^{\infty}(\sin nx/n)$.

14.3.4 A triangular wave (Fig. 14.9) is represented by

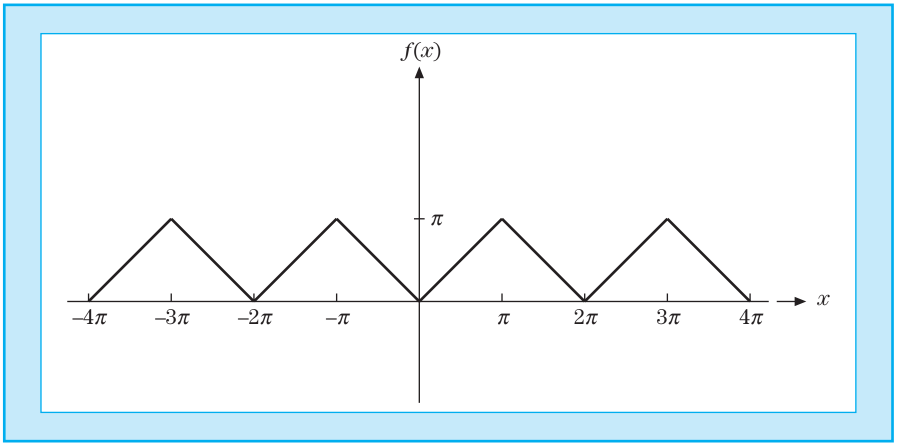
$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ -x, & -\pi \le x \le 0. \end{cases}$$

Represent $f(x)$ by a Fourier series.

$$\textit{ANS.} \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{\cos nx}{n^2}.$$

Figure 14.9

Triangular Wave



14.3.5 Expand

$$f(x) = \begin{cases} 1, & x^2 < x_0^2 \\ 0, & x^2 > x_0^2 \end{cases}$$

in the interval $[-\pi, \pi]$.

Note. This variable-width square wave is important in electronic music.

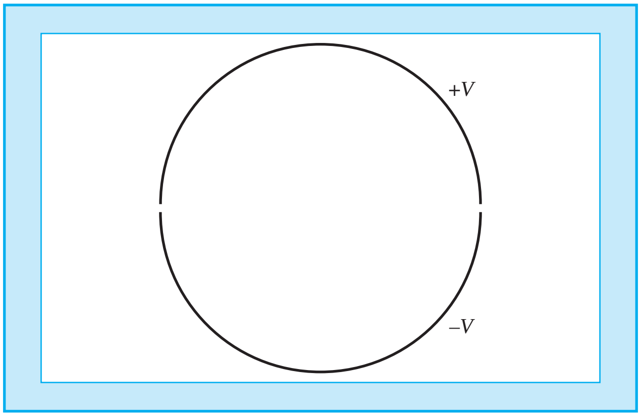


Figure 14.10

Cross Section of Split Tube

14.3.6 A metal cylindrical tube of radius a is split lengthwise into two non-touching halves. The top half is maintained at a potential $+V$ and the bottom half at a potential $-V$ (Fig. 14.10). Separate the variables in Laplace's equation and solve for the electrostatic potential for $r \leq a$. Observe the resemblance between your solution for $r = a$ and the Fourier series for a square wave.

14.3.7 A metal cylinder is placed in a (previously) uniform electric field, E_0, with the axis of the cylinder perpendicular to that of the original field.
(a) Find the perturbed electrostatic potential.
(b) Find the induced surface charge on the cylinder as a function of angular position.

14.3.8 Expand $\delta(x-t)$ in a Fourier series.

ANS.
$$\delta(x-t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}(\cos nx \cos nt + \sin nx \sin nt)$$
$$= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos n(x-t).$$

14.3.9 Verify that

$$\delta(\varphi_1 - \varphi_2) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi_1-\varphi_2)}$$

is a Dirac delta function by showing that it satisfies the definition of a Dirac delta function:

$$\int_{-\pi}^{\pi} f(\varphi_2)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi_1-\varphi_2)}d\varphi_2 = f(\varphi_1).$$

Hint. Represent $f(\varphi_2)$ by an exponential Fourier series.

Note. The continuum analog of this expression is developed in Section 15.2.

14.3.10 (a) Find the Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 \le x < \pi. \end{cases}$$

(b) From the Fourier expansion show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

14.3.11 Let $f(z) = \ln(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}z^n/n$. (This series converges to $\ln(1+z)$ for $|z| \le 1$, except at the point $z = -1$.)

(a) From the imaginary parts (item 5 of Table 14.1) show that

$$\ln\left(2\cos\frac{\theta}{2}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos n\theta}{n}, \quad -\pi < \theta < \pi.$$

(b) Using a change of variable, transform part (a) into

$$-\ln\left(2\sin\frac{\theta}{2}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos n\theta}{n}, \quad 0 < \theta < 2\pi.$$

14.3.12 A symmetric triangular pulse of adjustable height and width is described by

$$f(x) = \begin{cases} a(1 - x/b), & 0 \le |x| \le b \\ 0, & b \le |x| \le \pi. \end{cases}$$

(a) Show that the Fourier coefficients are

$$a_0 = \frac{ab}{\pi}, \quad a_n = \frac{2ab}{\pi}(1 - \cos nb)/(nb)^2.$$

Sum the finite Fourier series through $n = 10$ and through $n = 100$ for $x/\pi = 0(l/9)1$. Take $a = 1$ and $b = \pi/2$.

(b) Call a Fourier analysis subroutine (if available) to calculate the Fourier coefficients of $f(x)$, a_0 through a_{10}.

14.3.13 (a) Using a Fourier analysis subroutine, calculate the Fourier cosine coefficients a_0 through a_{10} of

$$f(x) = [1 - (x/\pi)^2]^{1/2}, \quad x \in [-\pi, \pi].$$

(b) Spot check by calculating some of the preceding coefficients by direct numerical quadrature.

Check values. $a_0 = 0.785$, $a_2 = 0.284$.

14.3.14 A function $f(x)$ is expanded in an exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

If $f(x)$ is real, $f(x) = f^*(x)$, what restriction is imposed on the coefficients c_n?

14.4 Properties of Fourier Series

Convergence

Note that our Fourier series should not be expected to be uniformly convergent if it represents a discontinuous function. A uniformly convergent series of continuous functions ($\sin nx$, $\cos nx$) always yields a continuous function $f(x)$ (compare Section 5.5). If, however, (i) $f(x)$ is continuous, $-\pi \leq x \leq \pi$, (ii) $f(-\pi) = f(+\pi)$, and (iii) $f'(x)$ is sectionally continuous, the Fourier series for $f(x)$ will converge uniformly. These restrictions do not demand that $f(x)$ be periodic, but they will be satisfied by continuous, differentiable, periodic functions (period of 2π). For a proof of uniform convergence we refer to the literature.[8] With or without a discontinuity in $f(x)$, the Fourier series will yield convergence in the mean (Section 9.4).

If a function is square integrable, then Sturm–Liouville theory implies the validity of the Bessel inequality [Eq. (9.73)]

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)dx \tag{14.42}$$

so that at least the sum of squares of its Fourier coefficients converges.

EXAMPLE 14.4.1

Absolutely Convergent Fourier Series If the periodic function $f(x)$ has a bounded second derivative $f''(x)$, then its Fourier series converges absolutely. Note that the converse is not valid, as the full-wave rectifier (Example 14.1.2) demonstrates.

To show this, let us integrate by parts the Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \left. \frac{f(x) \sin nx}{n\pi} \right|_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx,$$

where the integrated term vanishes. Because the first derivative $f'(x)$ is bounded a fortiori, $|a_n| = \mathcal{O}(\frac{1}{n})$, and similarly $b_n \to 0$ for $n \to \infty$, at least as fast as $1/n$, which is not sufficient for absolute convergence. However, another integration by parts can be done, yielding

$$a_n = \left. \frac{f'(x) \cos nx}{n^2\pi} \right|_{-\pi}^{\pi} - \frac{1}{n^2\pi} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx, \tag{14.43}$$

[8] See, for instance, Churchill, R. V. (1993). *Fourier Series and Boundary Value Problems*, 5th ed., Section 38. McGraw-Hill, New York.

where the integrated terms cancel each other because f' is continuous and periodic; that is, $f'(\pi) = f'(-\pi)$. Since $|f''(x)| \le M$, we find the upper bound

$$|a_n| \le \frac{M}{n^2\pi}\int_{-\pi}^{\pi} dx = \frac{2M}{n^2}, \tag{14.44}$$

which implies absolute convergence (by the integral test of Chapter 5), and the same applies to b_n. ■

If the $|a_n|$, $|b_n| \le n^{-\alpha}$ with $0 < \alpha \le 1$, then we have at least conditional convergence, and the function $f(x)$ may have discontinuities. If $\alpha > 1$, then there is absolute convergence by the integral test of Chapter 5.

Integration

Term-by-term integration of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{14.45}$$

yields

$$\int_{x_0}^{x} f(x)dx = \left.\frac{a_0 x}{2}\right|_{x_0}^{x} + \sum_{n=1}^{\infty}\left.\frac{a_n}{n}\sin nx\right|_{x_0}^{x} - \sum_{n=1}^{\infty}\left.\frac{b_n}{n}\cos nx\right|_{x_0}^{x}. \tag{14.46}$$

Clearly, the effect of integration is to place an additional power of n in the denominator of each coefficient. This results in more rapid convergence than before. Consequently, a convergent Fourier series may always be integrated term by term, with the resulting series converging uniformly to the integral of the original function. Indeed, term-by-term integration may be valid even if the original series [Eq. (14.45)] is not convergent. The function $f(x)$ need only be integrable. A discussion will be found in Jeffreys and Jeffreys, Section 14.06 (see Additional Reading).

Strictly speaking, Eq. (14.46) may not be a Fourier series; that is, if $a_0 \ne 0$, there will be a term $\frac{1}{2}a_0 x$. However,

$$\int_{x_0}^{x} f(x)dx - \frac{1}{2}a_0 x \tag{14.47}$$

will still be a Fourier series.

EXAMPLE 14.4.2

Integration of Fourier Series Consider the sawtooth series for

$$f(x) = x, \quad -\pi < x < \pi. \tag{14.48}$$

Comparing with Exercise 14.1.1, the Fourier series is

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}, \quad -\pi < x < \pi,$$

which converges conditionally, but not absolutely, because the harmonic series diverges. Now we integrate it and obtain item 7 of Table 14.1

$$\begin{aligned}2\int_0^x \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n}dx &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^x \sin nx\,dx \\ &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}(1-\cos nx) \\ &= \frac{\pi^2}{6}+2\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}=\frac{x^2}{2}\end{aligned}$$

using Exercise 14.4.1. Because $|\cos nx| \le 1$ is bounded and the series $\sum_n 1/n^2$ converges, our integrated Fourier series converges absolutely to the limit $\int_0^x x\,dx = x^2/2$. ■

Differentiation

The situation regarding differentiation is quite different from that of integration. Here the word is caution.

EXAMPLE 14.4.3

Differentiation of Fourier Series Consider again the series of Example 14.4.2. Differentiating term by term, we obtain

$$1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nx, \tag{14.49}$$

which is not convergent for any value of x. **Warning:** Check the convergence of your derivative.

For a triangular wave (Exercise 14.3.4), in which the convergence is more rapid (and uniform),

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{\cos nx}{n^2}. \tag{14.50}$$

Differentiating term by term,

$$f'(x) = \frac{4}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{\sin nx}{n}, \tag{14.51}$$

which is the Fourier expansion of a square wave

$$f'(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases} \tag{14.52}$$

Inspection of Fig. 14.4 verifies that this is indeed the derivative of our triangular wave. ■

- As the inverse of integration, the operation of differentiation has placed an additional factor n in the numerator of each term. This reduces the rate of convergence and may, as in the first case mentioned, render the differentiated series divergent.
- In general, term-by-term differentiation is permissible under the same conditions listed for uniform convergence in Chapter 5.

From the expansion of x and expansions of other powers of x, numerous other infinite series can be evaluated. A few are included in the subsequent exercises.

EXERCISES

14.4.1 Show that integration of the Fourier expansion of $f(x) = x$, $-\pi < x < \pi$, leads to

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots.$$

14.4.2 Parseval's identity.

(a) Assuming that the Fourier expansion of $f(x)$ is uniformly convergent, show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

This is Parseval's identity. It is actually a special case of the completeness relation [Eq. (9.73)].

(b) Given

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \qquad -\pi \le x \le \pi,$$

apply Parseval's identity to obtain $\zeta(4)$ in closed form.

(c) The condition of uniform convergence is not necessary. Show this by applying the Parseval identity to the square wave

$$\begin{aligned} f(x) &= \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}. \end{aligned}$$

14.4.3 Show that integrating the Fourier expansion of the Dirac delta function (Exercise 14.3.8) leads to the Fourier representation of the square wave [Eq. (14.21)], with $h = 1$.
Note. Integrating the constant term $(1/2\pi)$ leads to a term $x/2\pi$. What are you going to do with this?

14.4.4 Integrate the Fourier expansion of the unit step function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi. \end{cases}$$

Show that your integrated series agrees with Exercise 14.3.12.

14.4.5 In the interval $[-\pi, \pi]$,

$$\delta_n(x) = \begin{array}{ll} n, & \text{for } |x| < 1/(2n), \\ 0, & \text{for } |x| > 1/(2n) \end{array}$$

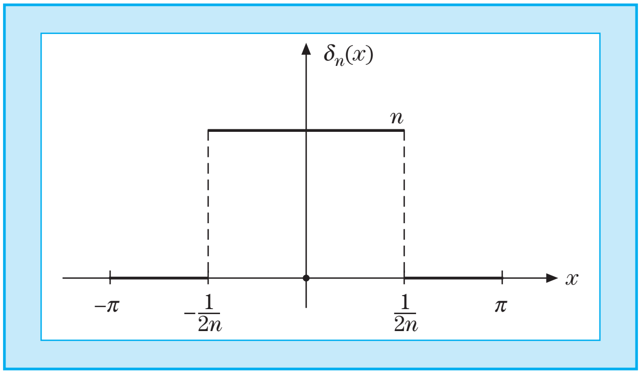
(Fig. 14.11).

(a) Expand $\delta_n(x)$ as a Fourier cosine series.

(b) Show that your Fourier series agrees with a Fourier expansion of $\delta(x)$ in the limit as $n \to \infty$.

Figure 14.11

Rectangular Pulse



14.4.6 Confirm the delta function nature of your Fourier series of Exercise 14.4.5 by showing that for any $f(x)$ that is finite in the interval $[-\pi, \pi]$ and continuous at $x = 0$,

$$\int_{-\pi}^{\pi} f(x)[\text{Fourier expansion of } \delta_\infty(x)]\, dx = f(0).$$

14.4.7 Find the charge distribution over the interior surfaces of the semicircles of Exercise 14.3.6.

Note. You obtain a divergent series and this Fourier approach fails. Using conformal mapping techniques, we may show the charge density to be proportional to $\csc\theta$. Does $\csc\theta$ have a Fourier expansion?

14.4.8 Given

$$\varphi_1(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} -(\pi + x)/2, & -\pi \le x < 0 \\ (\pi - x)/2 & 0 < x \le \pi, \end{cases}$$

show by integrating that

$$\varphi_2(x) \equiv \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \begin{cases} (\pi + x)^2/4 - \pi^2/(12), & -\pi \le x \le 0 \\ (\pi - x)^2/4 - \pi^2/(12), & 0 \le x \le \pi. \end{cases}$$

Additional Reading

Carslaw, H. S. (1921). *Introduction to the Theory of Fourier's Series and Integrals*, 2nd ed. Macmillan, London. Paperback, 3rd ed., Dover, New York (1952). This is a detailed and classic work.

Hamming, R. W. (1973). *Numerical Methods for Scientists and Engineers*, 2nd ed. McGraw-Hill, New York. Reprinted, Dover, New York (1987).

Jeffreys, H., and Jeffreys, B. S. (1972). *Methods of Mathematical Physics*, 3rd ed. Cambridge Univ. Press, Cambridge, UK.

Kufner, A., and Kadlec, J. (1971). *Fourier Series*. Iliffe, London. This book is a clear account of Fourier series in the context of Hilbert space.

Lanczos, C. (1956). *Applied Analysis*. Prentice-Hall, Englewood Cliffs, NJ. Reprinted, Dover, New York (1988). The book gives a well-written presentation of the Lanczos convergence technique (which suppresses the Gibbs phenomenon oscillations). This and several other topics are presented from the point of view of a mathematician who wants useful numerical results and not just abstract existence theorems.

Oberhettinger, F. (1973). *Fourier Expansions, A Collection of Formulas*. Academic Press, New York.

Zygmund, A. (1988). *Trigonometric Series*. Cambridge Univ. Press, Cambridge, UK. The volume contains an extremely complete exposition, including relatively recent results in the realm of pure mathematics.