

Section6: Electromagnetic Radiation

Potential formulation of Maxwell equations

Now we consider a *general* solution of Maxwell's equations. Namely we are interested how the sources (charges and currents) generate electric and magnetic fields. For simplicity we restrict our considerations to the vacuum. In this case Maxwell's equations have the form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (6.1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (6.2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (6.3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (6.4)$$

Maxwell's equations consist of a set of coupled first-order partial differential equations relating the various components of electric and magnetic fields. They can be solved as they stand in simple situations. But it is often convenient to introduce potentials, obtaining a smaller number of second-order equations, while satisfying some of Maxwell's equations identically. We are already familiar with this concept in electrostatics and magnetostatics, where we used the scalar potential Φ and the vector potential \mathbf{A} .

Since $\nabla \cdot \mathbf{B} = 0$ still holds, we can define \mathbf{B} in terms of a vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (6.5)$$

Then the Faraday's law (6.3) can be written

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \quad (6.6)$$

This means that the quantity with vanishing curl in (6.6) can be written as the gradient of some scalar function, namely, a scalar potential Φ .

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \quad (6.7)$$

or

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}. \quad (6.8)$$

The definition of \mathbf{B} and \mathbf{E} in terms of the potentials \mathbf{A} and Φ according to (6.5) and (6.8) satisfies identically the two homogeneous Maxwell equations (6.2) and (6.3). The dynamic behavior of \mathbf{A} and Φ will be determined by the two inhomogeneous equations (6.1) and (6.4). Putting Eq. (6.8) into (6.1) we find that

$$\nabla^2 \Phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}. \quad (6.9)$$

This equation replaces Poisson equation (to which it reduces in the static case). Substituting Eqs. (6.5) and (6.8) into (6.4) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \frac{1}{c^2} \nabla \left(\frac{\partial \Phi}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}. \quad (6.10)$$

Now using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, and rearranging the terms we find

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \frac{1}{c^2} \nabla \left(\frac{\partial \Phi}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}, \quad (6.11)$$

or

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J}. \quad (6.12)$$

We have now reduced the set of four Maxwell equations to two equations (6.9) and (6.12). But they are still coupled equations. The uncoupling can be accomplished by exploiting the arbitrariness involved in the definition of the potentials. Since \mathbf{B} is defined through (6.5) in terms of \mathbf{A} , the vector potential is arbitrary to the extent that the gradient of some scalar function Λ can be added. Thus \mathbf{B} is left unchanged by the transformation,

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \Lambda. \quad (6.13)$$

For the electric field (6.8) to be unchanged as well, the scalar potential must be simultaneously transformed,

$$\Phi \rightarrow \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}. \quad (6.14)$$

The freedom implied by (6.13) and (6.14) means that we can choose a set of potentials (\mathbf{A} , Φ) such that

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0. \quad (6.15)$$

This will uncouple the pair of equations (6.9) and (6.12) and leave two inhomogeneous wave equations, one for Φ and one for \mathbf{A} :

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \quad (6.16)$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \quad (6.17)$$

Equations (6.16) and (6.17), plus (6.15), form a set of equations equivalent in all respects to Maxwell's equations.

Gauge Transformations

The transformation (6.13) and (6.14) is called a *gauge transformation*, and the invariance of the fields under such transformations is called gauge invariance. The relation (6.15) between \mathbf{A} and Φ is called the *Lorentz condition*. To see that potentials can always be found to satisfy the Lorentz condition, suppose that the potentials \mathbf{A} and Φ that satisfy (6.9) and (6.12) do not satisfy (6.15). Then let us make a gauge transformation to potentials \mathbf{A}' and Φ' and demand that \mathbf{A}' and Φ' satisfy the Lorentz condition:

$$\nabla \cdot \mathbf{A}' + \frac{1}{c^2} \frac{\partial \Phi'}{\partial t} = 0 = \nabla \cdot \mathbf{A} + \nabla^2 \Lambda + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2}. \quad (6.18)$$

Thus, provided a gauge function Λ can be found to satisfy

$$\nabla^2 \Lambda - \frac{1}{c^2} \frac{\partial^2 \Lambda}{\partial t^2} = - \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} \right), \quad (6.19)$$

the new potentials \mathbf{A}' and Φ' will satisfy the Lorentz condition and wave equations (6.16) and (6.17).

Even for potentials that satisfy the Lorentz condition (6.15) there is arbitrariness. Evidently the *restricted gauge transformation*,

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\Lambda, \quad (6.20)$$

$$\Phi \rightarrow \Phi - \frac{\partial\Lambda}{\partial t}, \quad (6.21)$$

where

$$\nabla^2\Lambda - \frac{1}{c^2} \frac{\partial^2\Lambda}{\partial t^2} = 0 \quad (6.22)$$

preserves the Lorentz condition, provided \mathbf{A} and Φ satisfy it initially. All potentials in this restricted class are said to belong to the *Lorentz gauge*. The Lorentz gauge is commonly used, first because it leads to the wave equations (6.9) and (6.12), which treat \mathbf{A} and Φ on equivalent footings.

Another useful gauge for the potentials is the so-called *Coulomb gauge*. This is the gauge in which

$$\nabla \cdot \mathbf{A} = 0. \quad (6.23)$$

From (6.16) we see that the scalar potential satisfies the Poisson equation,

$$\nabla^2\Phi - \frac{1}{c^2} \frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad (6.24)$$

with solution,

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (6.25)$$

The scalar potential is just the *instantaneous* Coulomb potential due to the charge density $\rho(\mathbf{r}, t)$. This is the origin of the name “*Coulomb gauge*”. There is a peculiar thing about the scalar potential (6.25) in the Coulomb gauge: it is determined by the distribution of charge *right now*. That sounds particularly odd in the light of special relativity, which allows no message to travel faster than the speed of light. The point is that Φ by itself is not a physically measurable quantity – all we can measure is \mathbf{E} , and that involves \mathbf{A} as well. Somehow it is built into the vector potential, in the Coulomb gauge, that whereas Φ instantaneously reflects all changes in ρ , the combination $-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$ does not; \mathbf{E} will change only after sufficient time has elapsed for the “news” to arrive.

The *advantage* of the Coulomb gauge is that the *scalar* potential is particularly simple to calculate; the *disadvantage* is that \mathbf{A} is particularly *difficult* to calculate. The differential equation for \mathbf{A} (6.12) in the Coulomb gauge reads

$$\nabla^2\mathbf{A} - \frac{1}{c^2} \frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \frac{1}{c^2} \nabla \left(\frac{\partial\Phi}{\partial t} \right). \quad (6.26)$$

Retarded potentials

In the Lorentz gauge \mathbf{A} and Φ satisfy the *inhomogeneous wave equation*, with a “source” term (in place of zero) on the right:

$$\nabla^2\Phi - \frac{1}{c^2} \frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \quad (6.27)$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \quad (6.28)$$

From now we will use the Lorentz gauge exclusively, and the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equations for specified sources*.

In the static case Eqs. (6.27), (6.28) reduce to Poisson's equation

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \quad (6.29)$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \quad (6.30)$$

with familiar solutions

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r', \quad (6.31)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3r'. \quad (6.32)$$

In dynamic case, electromagnetic “news” travel at the speed of light. In the *nonstatic* case, therefore, it's not the status of the source *right now* that matters, but rather its condition at some earlier time t_r (called the *retarded time*) when the “message” left. Since this message must travel a distance $|\mathbf{r}-\mathbf{r}'|$, the delay is $|\mathbf{r}-\mathbf{r}'|/c$:

$$t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \quad (6.33)$$

The natural generalization of Eqs. (6.31), (6.32) to the nonstatic case is therefore

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3r', \quad (6.34)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3r'. \quad (6.35)$$

Here $\rho(\mathbf{r}', t_r)$ and $\mathbf{J}(\mathbf{r}', t_r)$ is the charge and current density that prevailed at point \mathbf{r}' at the retarded time t_r . Because the integrands are evaluated at the retarded time, these are called *retarded potentials*. Note that the retarded potentials reduce properly to Eqs. (6.31), (6.32) in the static case, for which ρ and \mathbf{J} are independent of time.

That all sounds reasonable – and surprisingly simple. But so far we did not prove that all this is correct. To prove this, we must show that the potentials in the form (6.34), (6.35) satisfy the inhomogeneous wave equations (6.27), (6.28) and meet the Lorentz condition (6.15). In calculating the Laplacian we have to take into account that the integrands in Eqs. (6.34), (6.35) depend on \mathbf{r} in two places: explicitly, in the denominator $|\mathbf{r}-\mathbf{r}'|$, and implicitly, through $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$. Thus,

$$\nabla \Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla \rho(\mathbf{r}', t_r) + \rho(\mathbf{r}', t_r) \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right] d^3r', \quad (6.36)$$

and

$$\nabla \rho(\mathbf{r}', t_r) = \sum_i \hat{\mathbf{x}}_i \frac{\partial \rho}{\partial x_i} = \sum_i \hat{\mathbf{x}}_i \frac{\partial \rho}{\partial t_r} \frac{\partial t_r}{\partial x_i} = \frac{\partial \rho}{\partial t_r} \nabla t_r = -\frac{1}{c} \frac{\partial \rho}{\partial t_r} \nabla |\mathbf{r}-\mathbf{r}'| = -\frac{1}{c} \frac{\partial \rho}{\partial t_r} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \quad (6.37)$$

Taking into account that

$$\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \quad (6.38)$$

we find

$$\nabla \Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[-\frac{1}{c} \frac{\partial \rho}{\partial t_r} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} - \rho(\mathbf{r}', t_r) \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \right] d^3 r', \quad (6.39)$$

Taking the divergence we obtain:

$$\nabla^2 \Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left\{ -\frac{1}{c} \left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \cdot \nabla \left(\frac{\partial \rho}{\partial t_r} \right) + \frac{\partial \rho}{\partial t_r} \nabla \cdot \left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \right) \right] - \left[\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \cdot \nabla \rho + \rho \nabla \cdot \left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \right) \right] \right\} d^3 r' \quad (6.40)$$

Similar to (6.37):

$$\nabla \left(\frac{\partial \rho}{\partial t_r} \right) = \frac{\partial^2 \rho}{\partial t_r^2} \nabla t_r = -\frac{1}{c} \frac{\partial^2 \rho}{\partial t_r^2} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \quad (6.41)$$

and

$$\nabla \cdot \left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|^2} \nabla \cdot (\mathbf{r}-\mathbf{r}') + \nabla \cdot \frac{1}{|\mathbf{r}-\mathbf{r}'|^2} = \frac{3}{|\mathbf{r}-\mathbf{r}'|^2} + (\mathbf{r}-\mathbf{r}') \cdot (-2) \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^4} = \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}. \quad (6.42)$$

In addition, we know that

$$\nabla \cdot \left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \right) = 4\pi \delta^3(\mathbf{r}-\mathbf{r}'). \quad (6.43)$$

Due to (6.37) and (6.42) the second and third terms in Eq. (6.40) are canceled out. Thus,

$$\nabla^2 \Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left\{ \frac{1}{c^2} \frac{\partial^2 \rho(\mathbf{r}, t_r)}{\partial t_r^2} \frac{1}{|\mathbf{r}-\mathbf{r}'|} - 4\pi \rho(\mathbf{r}, t_r) \delta^3(\mathbf{r}-\mathbf{r}') \right\} d^3 r' = \frac{1}{c^2} \frac{\partial^2 \Phi(\mathbf{r}, t)}{\partial t^2} - \frac{1}{\epsilon_0} \rho(\mathbf{r}, t), \quad (6.44)$$

where we took into account that $\frac{\partial^2}{\partial t_r^2} = \frac{\partial^2}{\partial t^2}$ and that t_r in the second term should be replaced by t due

to the delta function (6.43). Eq.(6.44) confirms that the retarded potential (6.34) satisfies the inhomogeneous wave equation (6.27). Similar derivation can be performed for the vector potential.

Now let us demonstrate that the retarded potentials satisfy the Lorentz gauge condition

$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t} = 0$. We need to calculate divergence \mathbf{A} given by eq. (6.32) and therefore we have to find

$\nabla \cdot \left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right)$. Let us rewrite it in the following way:

$$\begin{aligned} \nabla \cdot \left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right) &= \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\nabla \cdot \mathbf{J}) + \mathbf{J} \cdot \left(\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\nabla \cdot \mathbf{J}) - \mathbf{J} \cdot \left(\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = \\ &= \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\nabla \cdot \mathbf{J}) + \frac{1}{|\mathbf{r}-\mathbf{r}'|} (\nabla' \cdot \mathbf{J}) - \nabla' \cdot \left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right) \end{aligned} \quad (6.45)$$

Here ∇' denotes differentiation with respect to \mathbf{r}' and we took into account that $\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}$.

Then we find

$$\nabla \cdot \mathbf{J}(\mathbf{r}', t_r) = \sum_i \frac{\partial J_i}{\partial x_i} = \sum_i \frac{\partial J_i}{\partial t_r} \frac{\partial t_r}{\partial x_i} = \frac{\partial \mathbf{J}}{\partial t_r} \cdot \nabla t_r = -\frac{1}{c} \frac{\partial \mathbf{J}}{\partial t_r} \cdot \nabla |\mathbf{r}-\mathbf{r}'|. \quad (6.46)$$

Similarly

$$\nabla' \cdot \mathbf{J}(\mathbf{r}', t_r) = -\frac{\partial \rho}{\partial t} - \frac{1}{c} \frac{\partial \mathbf{J}}{\partial t_r} \cdot \nabla' |\mathbf{r}-\mathbf{r}'|. \quad (6.47)$$

The first term arises when we differentiate with respect to the explicit \mathbf{r}' and use the continuity equation. Thus,

$$\begin{aligned} \nabla \cdot \left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right) &= -\frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{1}{c} \frac{\partial \mathbf{J}}{\partial t_r} \cdot \nabla |\mathbf{r}-\mathbf{r}'| + \frac{1}{|\mathbf{r}-\mathbf{r}'|} \left(\frac{\partial \rho}{\partial t} - \frac{1}{c} \frac{\partial \mathbf{J}}{\partial t_r} \cdot \nabla' |\mathbf{r}-\mathbf{r}'| \right) - \nabla' \cdot \left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right) = \\ &= -\frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \rho}{\partial t} - \nabla' \cdot \left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right), \end{aligned} \quad (6.48)$$

where we took into account that $\nabla |\mathbf{r}-\mathbf{r}'| = -\nabla' |\mathbf{r}-\mathbf{r}'|$. Finally we find

$$\begin{aligned} \nabla \cdot \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \nabla \cdot \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|} \right] d^3 r' = \frac{\mu_0}{4\pi} \left\{ -\int \frac{1}{|\mathbf{r}-\mathbf{r}'|} \frac{\partial \rho}{\partial t} d^3 r' - \int \nabla' \cdot \left(\frac{\mathbf{J}}{|\mathbf{r}-\mathbf{r}'|} \right) d^3 r' \right\} = \\ &= \frac{\mu_0}{4\pi} \left\{ -\frac{\partial}{\partial t} \int \frac{\rho}{|\mathbf{r}-\mathbf{r}'|} d^3 r' - \oint \frac{\mathbf{J} \cdot \mathbf{n}}{|\mathbf{r}-\mathbf{r}'|} da \right\} = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left\{ \frac{1}{4\pi \varepsilon_0} \int \frac{\rho}{|\mathbf{r}-\mathbf{r}'|} d^3 r' \right\} = -\frac{1}{c^2} \frac{\partial \Phi}{\partial t} \end{aligned} \quad (6.49)$$

Here we assumed that $\mathbf{J} = 0$ at infinity and therefore the surface integral vanished. Thus, we have proved that the Lorentz gauge condition is satisfied for the retarded potentials.

Incidentally, this proof applies equally well to the *advanced potentials*,

$$\Phi_a(\mathbf{r}, t) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\mathbf{r}', t_a)}{|\mathbf{r}-\mathbf{r}'|} d^3 r', \quad (6.50)$$

$$\mathbf{A}_a(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_a)}{|\mathbf{r}-\mathbf{r}'|} d^3 r'. \quad (6.51)$$

in which the charge and the current densities are evaluated at the *advanced time*

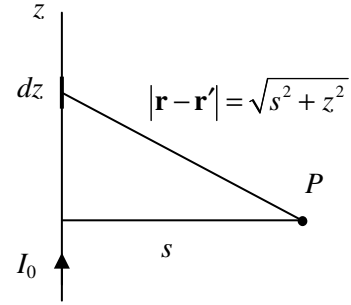
$$t_a = t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \quad (6.52)$$

A few signs are changed, but the final result is unaffected. Although the advanced potentials are entirely consistent with Maxwell's equations, they violate the most sacred tenet in all of physics: the principle of *causality*. They suggest that the potentials *now* depend on what the charge and the current distribution *will* be at some time in the future – the effect, in other words, precedes the cause. Although the advanced potentials are of some theoretical interest, they have no direct physical significance.

Example: An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & t < 0 \\ I_0, & t > 0 \end{cases} \quad (6.53)$$

That is, a constant current I_0 is turned on abruptly at $t=0$. Find the resulting electric and magnetic fields.



Solution: The wire is electrically neutral, so that the scalar potential is zero. Let the wire lie along the z axis as is shown in figure. The retarded potential at point P is

$$\mathbf{A}(s,t) = \frac{\mu_0}{4\pi} \hat{\mathbf{z}} \int_{-\infty}^{\infty} \frac{I(t_r)}{|\mathbf{r}-\mathbf{r}'|} dz' \quad (6.54)$$

For $t < s/c$, the information about current flowing in the wire has not yet reached P , and the potential is zero. For $t > s/c$, only the segment $|z| \leq \sqrt{(ct)^2 - s^2}$ contributes (outside this range t_r is negative, so $I(t_r) = 0$). Thus

$$\begin{aligned} \mathbf{A}(s,t) &= \left(\frac{\mu_0 I_0}{4\pi} \hat{\mathbf{z}} \right) 2 \int_0^{\sqrt{(ct)^2 - s^2}} \frac{dz}{\sqrt{s^2 + z^2}} = \frac{\mu_0 I_0}{2\pi} \hat{\mathbf{z}} \ln \left(\sqrt{s^2 + z^2} + z \right) \Big|_0^{\sqrt{(ct)^2 - s^2}} = \\ &= \frac{\mu_0 I_0}{2\pi} \ln \left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s} \right) \hat{\mathbf{z}} \end{aligned} \quad (6.55)$$

The electric field is

$$\mathbf{E}(s,t) = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 I_0}{2\pi} \frac{s}{ct + \sqrt{(ct)^2 - s^2}} \frac{1}{s} \left(c + \frac{2c^2 t}{2\sqrt{(ct)^2 - s^2}} \right) \hat{\mathbf{z}} = \frac{\mu_0 I_0 c}{2\pi \sqrt{(ct)^2 - s^2}} \hat{\mathbf{z}} \quad (6.56)$$

The magnetic field is

$$\begin{aligned} \mathbf{B}(s,t) &= \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s} \hat{\phi} = \frac{\mu_0 I_0}{2\pi} \frac{s}{ct + \sqrt{(ct)^2 - s^2}} \left(-\frac{ct + \sqrt{(ct)^2 - s^2}}{s^2} - \frac{1}{\sqrt{(ct)^2 - s^2}} \right) \hat{\phi} = \\ &= \frac{\mu_0 I_0}{2\pi s} \frac{ct}{\sqrt{(ct)^2 - s^2}} \hat{\phi} \end{aligned} \quad (6.57)$$

Notice that as $t \rightarrow \infty$ we recover the static case:

$$\mathbf{E} = 0, \quad (6.58)$$

$$\mathbf{B} = \frac{\mu_0 I_0}{2\pi s} \hat{\phi}. \quad (6.59)$$

Jefimenko's Equations

Given the retarded potentials

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (6.60)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (6.61)$$

it is straightforward matter to calculate the fields

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}. \end{aligned} \quad (6.62)$$

For electric fields we have already calculated the gradient of Φ (eq. (6.39)); the time derivative of \mathbf{A} is easy

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi} \int \frac{\dot{\mathbf{J}}}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (6.63)$$

where overdot indicates differentiation with respect to t . Putting together we find

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\rho(\mathbf{r}', t_r) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{1}{c} \dot{\rho}(\mathbf{r}', t_r) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} + \frac{1}{c^2} \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} \right] d^3r', \quad (6.64)$$

This is the time-dependent generalization of Coulomb's law, to which it reduces in the static case (where the second and third terms drop out and the first term loses its dependence on t_r).

As for \mathbf{B} , the curl of \mathbf{A} contains two terms:

$$\nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} (\nabla \times \mathbf{J}) - \mathbf{J} \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] d^3r'. \quad (6.65)$$

Now

$$(\nabla \times \mathbf{J})_i = \sum_{jk} \epsilon_{ijk} \frac{\partial J_k}{\partial x_j} = \sum_{jk} \epsilon_{ijk} \frac{\partial J_k}{\partial t_r} \frac{\partial t_r}{\partial x_j} = -\frac{1}{c} \sum_{jk} \epsilon_{ijk} j_k \frac{\partial |\mathbf{r} - \mathbf{r}'|}{\partial x_j} = \frac{1}{c} (\mathbf{J} \times \nabla |\mathbf{r} - \mathbf{r}'|)_i \quad (6.66)$$

Here we used the *Levi-Civita* symbol ϵ_{ijk} which is defined as follows:

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } ikj = 123, 231, \text{ or } 312 \\ -1, & \text{if } ikj = 132, 213, \text{ or } 321 \\ 0, & \text{otherwise} \end{cases} \quad (6.67)$$

in terms of which the cross product can be written as

$$(\mathbf{a} \times \mathbf{b})_i = \sum_{jk} \epsilon_{ijk} a_j b_k. \quad (6.68)$$

We therefore find

$$\nabla \times \mathbf{J} = \frac{1}{c} \mathbf{J} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \quad (6.69)$$

Taking into account $\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$ we obtain

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c|\mathbf{r}-\mathbf{r}'|^2} \right] \times (\mathbf{r}-\mathbf{r}') d^3 r'. \quad (6.70)$$

This is time dependent generalization of Bio-Savart law, to which it reduces in the static case. Equations (6.64) and (6.70) are known as Jefimenko's equations. In practice Jefimenko's equations are of limited utility, since it is typically easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields. Nevertheless, they provide a satisfying sense of closure to the theory. They also help to clarify the following observation: To get to the retarded *potentials*, all you do is replace t by t_r in the electrostatic and magnetostatic formulas, but in the case of *the fields* not only is time replaced by retarded time, but completely new terms (involving derivatives of ρ and \mathbf{J}) appear.

EM radiation

We discussed the propagation of plane electromagnetic waves through various media, but we were not interested in a way of how the waves were generated. Like all electromagnetic fields, their source is some arrangement of electric charge. But a charge at rest does not generate electromagnetic waves; nor does a steady current. The waves are due to *accelerating* charges, and *changing* currents. Here we consider how such charges and currents produce electromagnetic waves – that is, how they **radiate**.

The *signature* of radiation is irreversible flow of energy away from the source. We assume that the source is *localize* near the origin. If we now imagine a gigantic spherical shell, out at radius r , the total power passing out through this surface is the integral of the Poynting vector:

$$P = \lim_{r \rightarrow \infty} P(r) = \lim_{r \rightarrow \infty} \oint \mathbf{S} \cdot \mathbf{n} da = \frac{1}{\mu_0} \lim_{r \rightarrow \infty} \oint (\mathbf{E} \times \mathbf{B}) \cdot \mathbf{n} da \quad (6.71)$$

The power *radiated* is the limit of this quantity as r goes to infinity. This is the energy (per unit time) that is transported out to infinity, and never comes back. Now, the area of the sphere is $4\pi r^2$, so for radiation to occur the Poynting vector must decrease (at large r) no faster than $1/r^2$ (if it went like $1/r^3$, for example, then $P(r)$ would go like $1/r$, and P would be zero). According to Coulomb's law, electrostatic fields fall off like $1/r^2$ (or even faster, if the total charge is zero), and the Biot-Savart law says that magnetostatic fields go like $1/r^2$ (or faster), which means that $S \sim 1/r^4$, for static configurations. So *static* sources do not radiate. But Jefimenko's equations (6.64) and (6.70) indicate that *time-dependent* fields include terms that go like $1/r$; it is *these* terms that are responsible for electromagnetic radiation.

The study of radiation, then, involves picking out the parts of \mathbf{E} and \mathbf{B} that go like $1/r$ at large distances from the source, constructing from them the $1/r^2$ term in S , integrating over a large spherical surface, and taking the limit as $r \rightarrow \infty$.

We consider a general situation, in which electromagnetic radiation is produced by an *arbitrary distribution of charges and currents*, with an *arbitrary time dependence* (not necessarily oscillating with a single frequency ω). Our only restrictions are that

- (i) the source is confined to a bounded region V of space;
- (ii) the charges are moving slowly.

These conditions will allow us to formulate useful approximations for the behavior of the electric and magnetic fields.

To make the slow-motion approximation precise, and to define near and wave zones in the general case, we introduce the following scaling quantities:

r_s – characteristic length scale of the charge and current distribution,

t_s – characteristic time scale over which the distribution changes,

$v_s = \frac{r_s}{t_s}$ – characteristic velocity of the source,

$\omega_s = \frac{2\pi}{t_s}$ – characteristic frequency of the source,

$\lambda_s = \frac{2\pi c}{\omega_s} = ct_s$ – characteristic wave length of radiation.

The characteristic length scale is defined such that the distribution of charge and current is localized within a region whose volume is of the order of r_s^3 . The characteristic time scale is defined such that $\partial\rho/\partial t$ is of order ρ/t_s throughout the source.

The slow-motion approximation means that $v_s = r_s/t_s$ is much smaller than the speed of light:

$$v_s \ll c \quad (6.72)$$

This condition gives us $r_s = v_s t_s \ll ct_s = \lambda_s$, or

$$r_s \ll \lambda_s \quad (6.73)$$

The source is therefore confined to a region that is *much smaller* than a typical wavelength of the radiation.

There are three spatial regions of interest:

The near (static) zone: $r \ll \lambda_s$; (6.74)

The intermediate (induction) zone: $r_s \ll r \sim \lambda_s$; (6.75)

The far (radiation) zone: $r_s \ll \lambda_s \ll r$. (6.76)

We will see that the fields have very different properties in these zones. In the near zone the fields have the character of static fields, with radial components and variation with the distance that depend in detail on the properties of the source. In the far zone, on the other hand, the fields are transverse to the radius vector and fall off as $1/r$ which is typical for radiation fields. We will compute the potentials and fields in the near and far zones.

Electric dipole radiation

We begin by calculating the scalar potential,

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad (6.77)$$

where V is the region of space occupied by the source and $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time. In the *near zone* we can treat $|\mathbf{r} - \mathbf{r}'|/c$ as a small quantity and Taylor-expand the charge density about the current time t . We have

$$\rho\left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) = \rho(t) - \frac{1}{c} \dot{\rho}(t) |\mathbf{r} - \mathbf{r}'| + \dots \quad (6.78)$$

where an overdot indicates differentiation with respect to t . Relative to the first term, the second term is of order $r/(ct_s) = r/\lambda_s$, and by virtue of Eq.(6.74), this is small in the near zone. The third term would be smaller still, and we neglect it. We then have

$$\Phi(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0} \left\{ \int_V \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3r' - \frac{1}{c} \int_V \dot{\rho}(\mathbf{r}', t) d^3r' \right\} = \frac{1}{4\pi\epsilon_0} \left\{ \int_V \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3r' - \frac{1}{c} \frac{d}{dt} \int_V \rho(\mathbf{r}', t) d^3r' \right\}, \quad (6.79)$$

The second term vanishes, because it involves the time derivative of the total charge $\int_V \rho(\mathbf{r}', t) d^3r'$, which is conserved. We have therefore obtained

$$\Phi(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (6.80)$$

We see that the near-zone potential is similar to its usual static expression, except the fact that charge density depends on time. The time delay between the source and the potential has disappeared, and what we have is a potential that adjusts instantaneously to the changes within the distribution. The electric field it produces is then a “time-changing electrostatic field”. This near-zone field does not behave as radiation.

To witness radiative effects we must go to the *radiation zone*. Here $|\mathbf{r} - \mathbf{r}'|$ is large and we can no longer Taylor-expand the density as we did previously. Instead we must introduce another approximation technique. We use the fact that in the induction zone, r is much larger than r' , so that

$$|\mathbf{r} - \mathbf{r}'| = r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots, \quad (6.81)$$

This gives

$$\rho\left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) \approx \rho\left(t - \frac{r}{c} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) = \rho\left(t_0 + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) + \dots \quad (6.82)$$

where we defined *retarded time at the origin*

$$t_0 = t - \frac{r}{c}. \quad (6.83)$$

Let us now Taylor-expand the charge density about the retarded time t_0 instead of the current time t . We have

$$\rho\left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) \approx \rho\left(t_0 + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) = \rho(t_0) + \dot{\rho}(t_0) \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} + \dots \quad (6.84)$$

where an overdot now indicates differentiation with respect to t_0 .

Inside the integral for Φ we approximate

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \dots \quad (6.85)$$

It is sufficient to keep only the leading term $1/r$ because the higher order terms do not contribute to radiation. The radiation zone potential is therefore

$$\begin{aligned} \Phi(\mathbf{r}, t) &\approx \frac{1}{4\pi\epsilon_0 r} \left\{ \int_V \rho(\mathbf{r}', t_0) d^3r' + \int_V \dot{\rho}(\mathbf{r}', t_0) \left(\frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) d^3r' \right\} = \\ &\frac{1}{4\pi\epsilon_0 r} \left\{ \int_V \rho(\mathbf{r}', t_0) d^3r' + \frac{\hat{\mathbf{r}}}{c} \cdot \frac{d}{dt_0} \int_V \rho(\mathbf{r}', t_0) \mathbf{r}' d^3r' \right\}. \end{aligned} \quad (6.86)$$

In the first integral we recognize the total charge of the distribution:

$$q = \int_V \rho(\mathbf{r}', t_0) d^3 r' . \quad (6.87)$$

It is actually independent of t_0 by virtue of charge conservation. In the second integral we recognize the dipole moment vector of the charge distribution:

$$\mathbf{p}(t_0) = \int_V \rho(\mathbf{r}', t_0) \mathbf{r}' d^3 r' , \quad (6.88)$$

This does depend on retarded time t_0 because the charge density is time dependent. Our final expression for the potential is therefore

$$\Phi(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0} \frac{q}{r} + \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0)}{cr} , \quad (6.89)$$

The first term on the right hand side of eq. (6.89) is the static, monopole potential associated with the total charge q . This term does not depend on time and is not associated with the propagation of radiation; we shall simply omit it in later calculations. The second term, on the other hand, is radiative: it depends on retarded time t_0 and decays as $1/r$. We see that the radiative part of the scalar potential is produced by a time-changing dipole moment of the charge distribution; it is nonzero whenever $d\mathbf{p}/dt$ is nonzero.

An exact expression for the vector potential is

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' . \quad (6.90)$$

In the *near zone* we approximate the current density as follows

$$\mathbf{J}(\mathbf{r}', t_r) \approx \mathbf{J}(\mathbf{r}', t) \quad (6.91)$$

and we obtain

$$\mathbf{A}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r' . \quad (6.92)$$

We see that here also, the vector potential takes its static form. The potential responds virtually instantaneously to changes in the distribution, and there are no radiative effects in the near zone.

In the *radiation zone* we have instead

$$\mathbf{J}\left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right) \approx \mathbf{J}\left(t_0 + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) = \mathbf{J}(t_0) + \dot{\mathbf{J}}(t_0) \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c} + \dots \quad (6.93)$$

For now we will keep only first term in this equation, then

$$\mathbf{A}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi r} \int \mathbf{J}(\mathbf{r}', t_0) d^3 r' . \quad (6.94)$$

In static situations, the volume integral of \mathbf{J} vanishes. But here the current density depends on time, and we have instead

$$\int \mathbf{J}(\mathbf{r}', t_0) d^3 r' = \dot{\mathbf{p}}(t_0) . \quad (6.95)$$

To prove this we write the i component of $\dot{\mathbf{p}}(t)$ as follows

$$\begin{aligned}
\dot{p}_i(t) &= \frac{d}{dt} \int_V \rho(\mathbf{r}, t) x_i d^3 r = \int_V \frac{\partial \rho}{\partial t} x_i d^3 r = - \int_V (\nabla \cdot \mathbf{J}) x_i d^3 r = \\
&= - \int_V \nabla \cdot (\mathbf{J} x_i) d^3 r + \int_V \mathbf{J} \cdot (\nabla x_i) d^3 r = - \oint_S \mathbf{J} \cdot \mathbf{n} x_i da + \int_V \mathbf{J} \cdot \hat{\mathbf{x}}_i d^3 r = \int_V J_i d^3 r
\end{aligned} \tag{6.96}$$

Here we took into account the statement of charge conservation, $\partial \rho / \partial t = -\nabla \cdot \mathbf{J}$ and the fact that no current is crossing surface S bounding volume V . The vector potential is therefore

$$\mathbf{A}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \frac{\dot{\mathbf{p}}(t_0)}{r}. \tag{6.97}$$

This has the structure of a spherical wave, and we see that the radiative part of the vector potential is produced by a time-changing dipole moment.

The potentials

$$\Phi(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0)}{cr}, \tag{6.98}$$

$$\mathbf{A}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \frac{\dot{\mathbf{p}}(t_0)}{r}, \tag{6.99}$$

are generated by time variations of the dipole moment vector \mathbf{p} of the charge and current distribution. They therefore give rise to *electric-dipole radiation*, the leading-order contribution, in our slow-motion approximation, to the radiation emitted by an arbitrary source. We now compute the electric and magnetic fields in this approximation.

To get the electric field we keep only those terms that decay as $1/r$, and neglect terms that decay faster. For example, when computing the gradient of the scalar potential we can neglect $\nabla r^{-1} = -\hat{\mathbf{r}}/r^2$ so that

$$\begin{aligned}
\nabla \Phi(\mathbf{r}, t) &\approx \frac{1}{4\pi\epsilon_0} \frac{1}{cr} \nabla [\hat{\mathbf{r}} \cdot \dot{\mathbf{p}}(t_0)] = \frac{1}{4\pi\epsilon_0} \frac{1}{cr} \hat{\mathbf{r}} \nabla \dot{\mathbf{p}}(t_0) = \frac{1}{4\pi\epsilon_0} \frac{1}{cr} \hat{\mathbf{r}} \sum_i \frac{d}{dx_i} \dot{p}_i(t_0) = \\
&\frac{1}{4\pi\epsilon_0} \frac{1}{cr} \hat{\mathbf{r}} \sum_i \ddot{p}_i(t_0) \frac{dt_0}{dx_i} = \frac{1}{4\pi\epsilon_0} \frac{1}{cr} \hat{\mathbf{r}} \sum_i \ddot{p}_i \left(-\frac{1}{c} \right) \frac{x_i}{r} = -\frac{1}{4\pi\epsilon_0} \frac{1}{c^2 r} (\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}) \hat{\mathbf{r}}
\end{aligned} \tag{6.100}$$

The electric field is then

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{1}{c^2 r} (\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}) \hat{\mathbf{r}} - \frac{\mu_0}{4\pi} \frac{\ddot{\mathbf{p}}}{r} = \frac{\mu_0}{4\pi r} [(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}) \hat{\mathbf{r}} - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r} [\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})]. \tag{6.101}$$

Notice that the radiation-zone electric field behaves as a spherical wave, and that it is transverse to $\hat{\mathbf{r}}$, the direction in which the wave propagates.

To get the magnetic field we need to compute $\nabla \times \mathbf{A}(\mathbf{r}, t)$. Similarly to (6.100) we can write

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi r} \nabla \times \dot{\mathbf{p}}(t_0) = -\frac{\mu_0}{4\pi r c} [\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t_0)]. \tag{6.102}$$

This because

$$[\nabla \times \dot{\mathbf{p}}(t_0)]_i = \sum_{ijk} \epsilon_{ijk} \hat{\mathbf{x}}_j \frac{\partial \dot{p}_k(t_0)}{\partial x_j} = -\frac{1}{c} \sum_{ijk} \epsilon_{ijk} \hat{\mathbf{x}}_j \ddot{p}_k \frac{x_j}{r} = -\frac{1}{c} \sum_{ijk} \epsilon_{ijk} \hat{\mathbf{r}}_j \ddot{p}_k = -\frac{1}{c} [\hat{\mathbf{r}} \times \ddot{\mathbf{p}}]_i. \tag{6.103}$$

Notice that the radiation-zone magnetic field behaves as a spherical wave, and that it is orthogonal to both $\hat{\mathbf{r}}$ and the electric field. Notice finally that the fields are in phase - they both depend on $\ddot{\mathbf{p}}$ and are related

as follows

$$\mathbf{E} = -c(\mathbf{B} \times \hat{\mathbf{r}}). \quad (6.104)$$

so that their magnitudes are $|\mathbf{B}|/|\mathbf{E}| = 1/c$.

Energy radiated

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = -\frac{1}{\mu_0} [c(\mathbf{B} \times \hat{\mathbf{r}}) \times \mathbf{B}] = \frac{c}{\mu_0} [B^2 \hat{\mathbf{r}} - (\mathbf{B} \cdot \hat{\mathbf{r}}) \cdot \mathbf{B}] = \frac{c}{\mu_0} B^2 \hat{\mathbf{r}}. \quad (6.105)$$

The fact that the Poynting vector is directed along $\hat{\mathbf{r}}$ shows that the electromagnetic field energy travels along with the wave.

The energy crossing a sphere of radius r per unit time is given by $P = \oint \mathbf{S} \cdot \mathbf{n} da$, where $\mathbf{n} da = \hat{\mathbf{r}} r^2 d\Omega$ and $d\Omega = \sin \theta d\theta d\phi$. Substituting Eqs. (6.105) and (6.102) yields

$$P = \oint \mathbf{S} \cdot \mathbf{n} da = \frac{\mu_0}{(4\pi)^2 c} \oint [\hat{\mathbf{r}} \times \ddot{\mathbf{p}}]^2 d\Omega. \quad (6.106)$$

To evaluate the integral we use the trick of momentarily aligning the z axis with the instantaneous direction of $\ddot{\mathbf{p}}$ - we must do this for each particular value of t_0 . Then $\hat{\mathbf{r}} \times \ddot{\mathbf{p}} = |\ddot{\mathbf{p}}| \sin \theta$ and

$$P = \frac{\mu_0}{(4\pi)^2 c} |\ddot{\mathbf{p}}|^2 \oint \sin^2 \theta d\Omega = \frac{\mu_0}{(4\pi)^2 c} |\ddot{\mathbf{p}}|^2 2\pi \frac{4}{3} = \frac{\mu_0}{6\pi c} |\ddot{\mathbf{p}}|^2. \quad (6.107)$$

This is the total power radiated by a slowly-moving distribution of charge and current. The power's angular distribution is described by

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c} |\ddot{\mathbf{p}}|^2 \sin^2 \theta. \quad (6.108)$$

Center-fed linear antenna

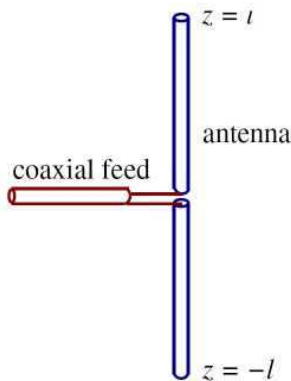


Figure: Center-fed linear antenna. The oscillating current is provided by a coaxial feed.

As an example of a radiating system we consider a thin wire of total length $2l$ which is fed an oscillating current through a small gap at its midpoint. The wire runs along the z axis, from $z = -l$ to $z = l$, and the gap is located at $z = 0$ (see figure above). For such antennas, the current typically oscillates both in time and in space, and it is usually represented by

$$\mathbf{J}(\mathbf{r}, t) = I_m \sin[k(l - |z|)] \delta(x) \delta(y) \hat{\mathbf{z}} \cos \omega t, \quad (6.109)$$

where $k = \omega/c$. The current is an even function of z (it is the same in both arms of the antenna) and it goes to zero at both ends (at $z = \pm l$). The current at the gap (at $z = 0$) is $I_0 = I_m \sin(kl)$, and I_0 is the current's peak value.

We want to calculate the total power radiated by this antenna, using the electric-dipole approximation. To be consistent we must be sure that $v_s \ll c$ for this distribution of current, or equivalently, that $l \equiv r_s \ll \lambda_s \equiv 2\pi/k$. In other words, we must demand that $kl \ll 1$, which means that $k|z|$ is small throughout the antenna. We can therefore approximate $\sin[k(l - |z|)]$ by $k(l - |z|)$ and Eq. (6.109) becomes

$$\mathbf{J}(\mathbf{r}, t) = I_0(1 - |z|/l) \delta(x) \delta(y) \hat{\mathbf{z}} \cos \omega t, \quad (6.110)$$

where $I_0 = I_m kl$ is the value of the current at the gap. In this approximation the current no longer oscillates in space: it simply goes from its peak value I_0 at the gap to zero at the two ends of the wire.

To compute the power radiated by our simplified antenna we first need to calculate $\dot{\mathbf{p}}(t)$, the second time derivative of the dipole moment vector. For this it is efficient to turn to Eq.(6.95),

$$\dot{\mathbf{p}}(t) = \int \mathbf{J}(\mathbf{r}, t) d^3 r. \quad (6.111)$$

in which we substitute Eq.(6.110). We have

$$\dot{\mathbf{p}}(t) = I_0 \hat{\mathbf{z}} \cos \omega t \int_{-l}^l (1 - |z|/l) dz. \quad (6.112)$$

and evaluating the integral gives

$$\dot{\mathbf{p}}(t) = -I_0 l \hat{\mathbf{z}} \cos \omega t. \quad (6.113)$$

Taking the second derivative yields

$$\ddot{\mathbf{p}}(t) = I_0 l \omega \hat{\mathbf{z}} \sin \omega t = (I_0 c)(kl) \hat{\mathbf{z}} \sin \omega t. \quad (6.114)$$

By introducing a vector θ between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ we find for the angular distribution of the power

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c} (I_0 c)^2 (kl)^2 \sin^2 \omega t \sin^2 \theta. \quad (6.115)$$

After averaging over a complete wave cycle, this reduces to

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 c}{32\pi^2} (I_0 kl)^2 \sin^2 \theta. \quad (6.116)$$

To obtain the total power radiated we must integrate over the angles. Using $\oint \sin^2 \theta d\Omega = 8\pi/3$, we arrive at our final result

$$\langle P \rangle = \frac{\mu_0 c}{12\pi} (I_0 kl)^2. \quad (6.117)$$

For a fixed frequency ω , the power increases like the square of the feed current I_0 . For a fixed current, the power increases like the square of the frequency, so long as the condition $kl \ll 1$ is satisfied. From Eq. (6.116) we learn that most of the energy is radiated in the directions perpendicular to the antenna; none of the energy propagates along the axis.