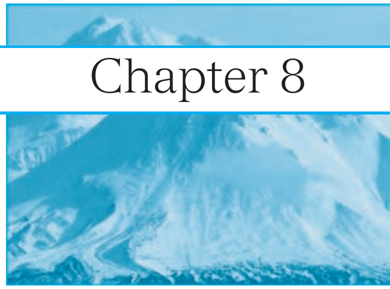


Chapter 8

Differential Equations

8.1 Introduction

In physics, the knowledge of the force in an equation of motion usually leads to a differential equation, with time as the independent variable, that governs dynamical changes in space. Almost all the elementary and numerous advanced parts of theoretical physics are formulated in terms of differential equations. Sometimes these are ordinary differential equations in one variable (**ODE**). More often, the equations are partial differential equations (**PDE**) in combinations of space and time variables. In fact, PDEs motivate physicists' interest in ODEs. The term **ordinary** is applied when the only derivatives dy/dx, $d^2y/dx^2, \ldots$ are ordinary or total derivatives. An ODE is **first order** if it contains the first and no higher derivatives of the unknown function $y(x)$, **second order** if it contains d^2y/dx^2 and no higher derivatives, etc.

Recall from calculus that the operation of taking an ordinary derivative is a **linear operation** ($\mathcal{L}$)[1]

$$\frac{d(a\varphi(x) + b\psi(x))}{dx} = a\frac{d\varphi}{dx} + b\frac{d\psi}{dx}.$$

In general,

$$\mathcal{L}(a\varphi + b\psi) = a\mathcal{L}(\varphi) + b\mathcal{L}(\psi), \tag{8.1}$$

where a and b are constants. An ODE is called **linear** if it is **linear in the unknown function and its derivatives**. Thus, linear ODEs appear as linear operator equations

$$\mathcal{L}\psi = F,$$

[1] We are especially interested in linear operators because in quantum mechanics physical quantities are represented by linear operators operating in a complex, infinite dimensional Hilbert space.

where ψ is the unknown function or general solution, the source F is a known function of one variable (for ODEs) and independent of ψ, and $\mathcal{L}$ is a linear combination of derivatives acting on ψ. If $F \neq 0$, the ODE is called **inhomogeneous**; if $F \equiv 0$, the ODE is called **homogeneous**. The solution of the homogeneous ODE can be multiplied by an arbitrary constant. If ψ_p is a **particular solution** of the inhomogeneous ODE, then $\psi_h = \psi - \psi_p$ is a solution of the homogeneous ODE because $\mathcal{L}(\psi - \psi_p) = F - F = 0$. Thus, the general solution is given by $\psi = \psi_p + \psi_h$. For the homogeneous ODE, any linear combination of solutions is again a solution, provided the differential equation is linear in the unknown function ψ_h; this is the **superposition principle**. We usually have to solve the homogeneous ODE first before searching for particular solutions of the inhomogeneous ODE.

Since the dynamics of many physical systems involve second-order derivatives (e.g., acceleration in classical mechanics and the kinetic energy operator, $\sim\nabla^2$, in quantum mechanics), differential equations of **second order** occur most frequently in physics. [Maxwell's and Dirac's equations are first order but involve two unknown functions. Eliminating one unknown yields a second-order differential equation for the other (compare Section 1.9).] Similarly, any higher order (linear) ODE can be reduced to a system of coupled first-order ODEs.

Nonetheless, there are many physics problems that involve first-order ODEs. Examples are resistance–inductance electrical circuits, radioactive decays, and special second-order ODEs that can be reduced to first-order ODEs. These cases and **separable ODEs** will be discussed first. ODEs of second order are more common and treated in subsequent sections, involving the special class of **linear ODEs with constant coefficients**. The important power-series expansion method of solving ODEs is demonstrated using second-order ODEs.

8.2 First-Order ODEs

Certain physical problems involve first-order differential equations. Moreover, sometimes second-order ODEs can be reduced to first-order ODEs, which then have to be solved. Thus, it seems desirable to start with them. We consider here differential equations of the general form

$$\frac{dy}{dx} = f(x, y) = -\frac{P(x, y)}{Q(x, y)}. \tag{8.2}$$

Equation (8.2) is clearly a first-order ODE; it may or may not be **linear**, although we shall treat the linear case explicitly later, starting with Eq. (8.12).

Separable Variables

Frequently, Eq. (8.2) will have the special form

$$\frac{dy}{dx} = f(x, y) = -\frac{P(x)}{Q(y)}. \tag{8.3}$$

Then it may be rewritten as

$$P(x)dx + Q(y)dy = 0.$$

Integrating from (x_0, y_0) to (x, y) yields

$$\int_{x_0}^{x} P(X)dX + \int_{y_0}^{y} Q(Y)dY = 0. \tag{8.4}$$

Here we have used capitals to distinguish the integration variables from the upper limits of the integrals, a practice that we will continue without further comment. Since the lower limits x_0 and y_0 contribute constants, we may ignore the lower limits of integration and write a constant of integration on the right-hand side instead of zero, which can be used to satisfy an initial condition. Note that this separation of variables technique does **not** require that the differential equation be linear.

EXAMPLE 8.2.1

Radioactive Decay The decay of a radioactive sample involves an event that is repeated at a constant rate λ. If the observation time dt is small enough so that the emission of two or more particles is negligible, then the probability that one particle is emitted is $\lambda\, dt$, with $\lambda\, dt \ll 1$. The decay law is given by

$$\frac{dN(t)}{dt} = -\lambda N(t), \tag{8.5}$$

where $N(t)$ is the number of radioactive atoms in the sample at time t. This ODE is separable

$$dN/N = -\lambda\, dt \tag{8.6}$$

and can be integrated to give

$$\ln N = -\lambda t + \ln N_0, \qquad \text{or} \quad N(t) = N_0 e^{-\lambda t}, \tag{8.7}$$

where we have written the integration constant in logarithmic form for convenience; N_0 is fixed by an initial condition $N(0) = N_0$. ■

In the next example from classical mechanics, the ODE is separable but not linear in the unknown, which poses no problem.

EXAMPLE 8.2.2

Parachutist We want to find the velocity of the falling parachutist as a function of time and are particularly interested in the constant limiting velocity, v_0, that comes about by air resistance taken to be quadratic, $-bv^2$, and opposing the force of the gravitational attraction, mg, of the earth. We choose a coordinate system in which the positive direction is downward so that the gravitational force is positive. For simplicity we assume that the parachute opens immediately, that is, at time $t = 0$, where $v(t = 0) = 0$, our initial condition. Newton's law applied to the falling parachutist gives

$$m\dot{v} = mg - bv^2,$$

where m includes the mass of the parachute.

The terminal velocity v_0 can be found from the equation of motion as $t \to \infty$, when there is no acceleration, $\dot{v} = 0$, so that

$$bv_0^2 = mg, \quad \text{or} \quad v_0 = \sqrt{mg/b}.$$

The variables t and v separate

$$\frac{dv}{g - \frac{b}{m}v^2} = dt,$$

which we integrate by decomposing the denominator into partial fractions. The roots of the denominator are at $v = \pm v_0$. Hence,

$$\left(g - \frac{b}{m}v^2\right)^{-1} = \frac{m}{2v_0 b}\left(\frac{1}{v + v_0} - \frac{1}{v - v_0}\right).$$

Integrating both terms yields

$$\int^v \frac{dV}{g - \frac{b}{m}V^2} = \frac{1}{2}\sqrt{\frac{m}{gb}} \ln \frac{v_0 + v}{v_0 - v} = t.$$

Solving for the velocity yields

$$v = \frac{e^{2t/T} - 1}{e^{2t/T} + 1} v_0 = v_0 \frac{\sinh \frac{t}{T}}{\cosh \frac{t}{T}} = v_0 \tanh \frac{t}{T},$$

where $T = \sqrt{\frac{m}{gb}}$ is the time constant governing the asymptotic approach of the velocity to the limiting velocity v_0.

Putting in numerical values, $g = 9.8$ m/sec^2 and taking $b = 700$ kg/m, $m = 70$ kg, gives $v_0 = \sqrt{9.8/10} \sim 1$ m/sec, ~ 3.6 km/hr, or ~ 2.23 miles/hr, the walking speed of a pedestrian at landing, and $T = \sqrt{\frac{m}{bg}} = 1/\sqrt{10 \cdot 9.8} \sim 0.1$ sec. Thus, the constant speed v_0 is reached within 1 sec. Finally, because **it is always important to check the solution**, we verify that our solution satisfies

$$\dot{v} = \frac{\cosh t/T}{\cosh t/T} \frac{v_0}{T} - \frac{\sinh^2 t/T}{\cosh^2 t/T} \frac{v_0}{T} = \frac{v_0}{T} - \frac{v^2}{Tv_0} = g - \frac{b}{m}v^2,$$

that is, Newton's equation of motion. The more realistic case, in which the parachutist is in free fall with an initial speed $v_i = v(0) \neq 0$ before the parachute opens, is addressed in Exercise 8.2.16. ■

Exact Differential Equations

We rewrite Eq. (8.2) as

$$P(x, y)dx + Q(x, y)dy = 0. \tag{8.8}$$

This equation is said to be **exact** if we can match the left-hand side of it to a differential $d\varphi$,

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy. \tag{8.9}$$

Since Eq. (8.8) has a zero on the right, we look for an unknown function $\varphi(x, y) = \text{constant}$ and $d\varphi = 0$.

We have [if such a function $\varphi(x, y)$ exists]

$$P(x, y)dx + Q(x, y)dy = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy \tag{8.10a}$$

and

$$\frac{\partial\varphi}{\partial x} = P(x, y), \qquad \frac{\partial\varphi}{\partial y} = Q(x, y). \tag{8.10b}$$

The necessary and sufficient condition for our equation to be exact is that the second, mixed partial derivatives of $\varphi(x, y)$ (assumed continuous) are independent of the order of differentiation:

$$\frac{\partial^2\varphi}{\partial y\,\partial x} = \frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x} = \frac{\partial^2\varphi}{\partial x\,\partial y}. \tag{8.11}$$

Note the resemblance to Eq. (1.124) of Section 1.12. If Eq. (8.8) corresponds to a curl (equal to zero), then a potential, $\varphi(x, y)$, must exist.

If $\varphi(x, y)$ exists then, from Eqs. (8.8) and (8.10a), our solution is

$$\varphi(x, y) = C.$$

We may construct $\varphi(x, y)$ from its partial derivatives, just as we construct a magnetic vector potential from its curl. See Exercises 8.2.7 and 8.2.8.

It may well turn out that Eq. (8.8) is not exact and that Eq. (8.11) is not satisfied. However, there always exists at least one and perhaps many more integrating factors, $\alpha(x, y)$, such that

$$\alpha(x, y)P(x, y)dx + \alpha(x, y)Q(x, y)dy = 0$$

is exact. Unfortunately, an integrating factor is not always obvious or easy to find. Unlike the case of the linear first-order differential equation to be considered next, there is no systematic way to develop an integrating factor for Eq. (8.8).

A differential equation in which the variables have been separated is automatically exact. An exact differential equation is not necessarily separable.

Linear First-Order ODEs

If $f(x, y)$ in Eq. (8.2) has the form $-p(x)y + q(x)$, then Eq. (8.2) becomes

$$\frac{dy}{dx} + p(x)y = q(x). \tag{8.12}$$

Equation (8.12) is the most general **linear** first-order ODE. If $q(x) = 0$, Eq. (8.12) is **homogeneous** (in y). A nonzero $q(x)$ may be regarded as a **source** or a **driving term** for the **inhomogeneous ODE**. In Eq. (8.12), each term is linear in y or dy/dx. There are no higher powers, such as y^2, and no products, such as $y(dy/dx)$. Note that the **linearity** refers to the y and dy/dx; $p(x)$ and $q(x)$ need not be linear in x. Equation (8.12), the most important for physics of these first-order ODEs, may be solved exactly.

Let us look for an **integrating factor** $\alpha(x)$ so that

$$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y = \alpha(x)q(x) \tag{8.13}$$

may be rewritten as

$$\frac{d}{dx}[\alpha(x)y] = \alpha(x)q(x). \tag{8.14}$$

The purpose of this is to make the left-hand side of Eq. (8.12) a derivative so that it can be integrated by inspection. It also, incidentally, makes Eq. (8.12) exact. Expanding Eq. (8.14), we obtain

$$\alpha(x)\frac{dy}{dx} + \frac{d\alpha}{dx}y = \alpha(x)q(x).$$

Comparison with Eq. (8.13) shows that we must require

$$\frac{d\alpha}{dx} = \alpha(x)p(x). \tag{8.15}$$

Here is a differential equation for $\alpha(x)$, with the variables α and x **separable**. We separate variables, integrate, and obtain

$$\alpha(x) = \exp\left[\int^x p(X)dX\right] \tag{8.16}$$

as our integrating factor. The lower limit is not written because it only multiplies α and the ODE by a constant, which is irrelevant.

With $\alpha(x)$ known we proceed to integrate Eq. (8.14). This, of course, was the point of introducing α in the first place. We have

$$\int^x \frac{d}{dX}[\alpha(X)y(X)]dX = \int^x \alpha(X)q(X)dX.$$

Now integrating by inspection, we have

$$\alpha(x)y(x) = \int^x \alpha(X)q(X)dX + C.$$

The constants from a constant lower limit of integration are absorbed in the constant C. Dividing by $\alpha(x)$, we obtain

$$y(x) = [\alpha(x)]^{-1}\left\{\int^x \alpha(X)q(X)dX + C\right\}.$$

Finally, substituting in Eq. (8.16) for α yields

$$y(x) = \exp\left[-\int^x p(X)dX\right]\left\{\int^x \exp\left[\int^Z p(Y)dY\right]q(Z)dZ + C\right\}. \tag{8.17}$$

Here the (dummy) variables of integration have been rewritten as capitals. Equation (8.17) is the complete general solution of the linear, first-order ODE, Eq. (8.12). The portion

$$y_h(x) = C \exp\left[-\int^x p(X)dX\right] \tag{8.18}$$

corresponds to the case $q(x) = 0$ and is a **general solution of the homogeneous ODE** because it contains the integration constant. The other term in Eq. (8.17),

$$y_p(x) = \exp\left[-\int^x p(X)dX\right]\int^x \exp\left[\int^Z p(Y)dY\right]q(Z)dZ, \tag{8.19}$$

is a **particular solution of the inhomogeneous ODE corresponding to the specific source term** $q(x)$.

Let us summarize this solution of the inhomogeneous ODE in terms of a method called **variation of the constant** as follows. In the first step, we solve the homogeneous ODE by separation of variables as before, giving

$$\frac{y'}{y} = -p, \quad \ln y = -\int^x p(X)dX + \ln C, \quad y(x) = Ce^{-\int^x p(X)dX}.$$

In the second step, we let the integration constant become x-dependent, that is, $C \to C(x)$. This is the variation of the constant used to solve the inhomogeneous ODE. Differentiating $y(x)$ we obtain

$$y' = -pCe^{-\int p(x)dx} + C'(x)e^{-\int p(x)dx} = -py(x) + C'(x)e^{-\int p(x)dx}.$$

Comparing with the inhomogeneous ODE we find the ODE for C:

$$C'e^{-\int p(x)dx} = q, \quad \text{or} \quad C(x) = \int^x e^{\int^X p(Y)dY}q(X)dX.$$

Substituting this C into $y = C(x)e^{-\int^x p(X)dX}$ reproduces Eq. (8.19).

Now we prove the *theorem* that **the solution of the inhomogeneous ODE is unique up to an arbitrary multiple of the solution of the homogeneous ODE**.

To show this, suppose y_1, y_2 both solve the inhomogeneous ODE [Eq. (8.12)]; then

$$y_1' - y_2' + p(x)(y_1 - y_2) = 0$$

follows by subtracting the ODEs and states that $y_1 - y_2$ is a solution of the homogeneous ODE. The solution of the homogeneous ODE can always be multiplied by an arbitrary constant. ■

We also prove the *theorem* that **a first-order linear homogeneous ODE has only one linearly independent solution**. This is meant in the following sense. If two solutions are **linearly dependent**, by definition they satisfy $ay_1(x) + by_2(x) = 0$ with nonzero constants a, b for all values of x. If the only solution of this linear relation is $a = 0 = b$, then our solutions y_1 and y_2 are said to be **linearly independent**.

To prove this theorem, suppose y_1, y_2 both solve the homogeneous ODE. Then

$$\frac{y_1'}{y_1} = -p(x) = \frac{y_2'}{y_2} \text{ implies } W(x) \equiv y_1' y_2 - y_1 y_2' \equiv 0. \tag{8.20}$$

The functional determinant W is called the **Wronskian** of the pair y_1, y_2. We now show that $W \equiv 0$ is the condition for them to be linearly dependent. Assuming linear dependence, that is,

$$a y_1(x) + b y_2(x) = 0$$

with nonzero constants a, b for all values of x, we differentiate this linear relation to get another linear relation

$$a y_1'(x) + b y_2'(x) = 0.$$

The condition for these two homogeneous linear equations in the unknowns a, b to have a nontrivial solution is that their determinant be zero, which is $W = 0$.

Conversely, from $W = 0$, there follows linear dependence because we can find a nontrivial solution of the relation

$$\frac{y_1'}{y_1} = \frac{y_2'}{y_2}$$

by integration, which gives

$$\ln y_1 = \ln y_2 + \ln C, \quad \text{or} \quad y_1 = C y_2.$$

Linear dependence and the Wronskian are generalized to three or more functions in Section 8.3. ■

EXAMPLE 8.2.3

Linear Independence The solutions of the linear oscillator equation $y'' + \omega^2 y(x) = 0$ are $y_1 = \sin\omega x$, $y_2 = \cos\omega x$, which we check by differentiation. The Wronskian becomes

$$\begin{vmatrix} \sin\omega x & \cos\omega x \\ \omega\cos\omega x & -\omega\sin\omega x \end{vmatrix} = -\omega \neq 0.$$

These two solutions, y_1 and y_2, are therefore linearly independent. For just two functions this means that one is not a multiple of the other, which is obviously true in this case.

You know that

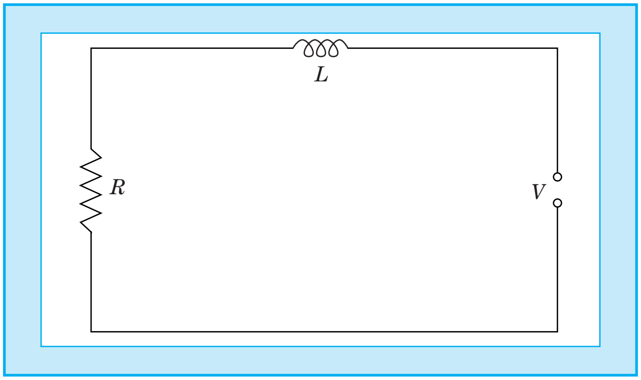
$$\sin\omega x = \pm(1 - \cos^2\omega x)^{1/2},$$

but this is **not** a **linear** relation. ■

Note that if our linear first-order differential equation is homogeneous ($q = 0$), then it is separable. Otherwise, apart from special cases such as $p =$ constant, $q =$ constant, or $q(x) = ap(x)$, Eq. (8.12) is not separable.

Figure 8.1

Circuit with Resistance *R* and Inductance *L* in Series



EXAMPLE 8.2.4

RL Circuit For a resistance–inductance circuit (Fig. 8.1 and Example 6.1.6) Kirchhoff's first law leads to

$$L\frac{dI(t)}{dt} + RI(t) = V(t) \tag{8.21}$$

for the current $I(t)$, where L is the inductance and R the resistance, both constant. Here, $V(t)$ is the time-dependent input voltage.

From Eq. (8.16), our integrating factor $\alpha(t)$ is

$$\alpha(t) = \exp\int^t \frac{R}{L}dT = e^{Rt/L}.$$

Then by Eq. (8.17),

$$I(t) = e^{-Rt/L}\left[\int^t e^{RT/L}\frac{V(T)}{L}dT + C\right], \tag{8.22}$$

with the constant C to be determined by an initial condition (a boundary condition).

For the special case $V(t) = V_0$, a constant,

$$I(t) = e^{-Rt/L}\left[\frac{V_0}{L}\cdot\frac{L}{R}e^{Rt/L} + C\right] = \frac{V_0}{R} + Ce^{-Rt/L}.$$

For a first-order ODE one initial condition has to be given. If it is $I(0) = 0$, then $C = -V_0/R$ and

$$I(t) = \frac{V_0}{R}[1 - e^{-Rt/L}]. \quad ■$$

ODEs of Special Type

Let us mention a few more types of ODEs that can be integrated analytically.

EXAMPLE 8.2.5

First-Order ODEs, with *y/x* Dependence The ODE $y' = f(y/x)$ is not of the form of Eq. (8.12) in general but is homogeneous in y. The substitution $z(x) = y(x)/x$, suggested by the form of the ODE, leads via $y' = xz' + z$ to the ODE $xz' + z = f(z)$, which is not of the type in Eq. (8.12). However, it is separable and can be integrated as follows:

$$z' = \frac{f(z) - z}{x}, \qquad \int \frac{dz}{f(z) - z} = \int \frac{dx}{x} = \ln x + \ln C.$$

An explicit case is the ODE

$$xyy' = y^2 - x^2, \qquad \text{or} \qquad y' = \frac{y}{x} - \frac{x}{y}.$$

In terms of $z(x) = y/x$, we obtain $xz' + z = z - \frac{1}{z}$, or $zdz = -dx/x$, which has separated variables. We integrate it to get $z^2 = C - 2\ln x$, where C is the integration constant. We check that our solution $y = x\sqrt{C - 2\ln x}$ satisfies

$$y' = \sqrt{C - 2\ln x} - 1\big/\sqrt{C - 2\ln x}, \qquad \text{or} \qquad \frac{yy'}{x} = \frac{y^2}{x^2} - 1.$$

The constant C is determined by the initial condition. If, for example, $y(1) = 1$, we obtain $C = 1$. ■

Clairaut's ODE $y = xy' + f(y')$ can be solved in closed form despite the general nature of the function f in it.

Replacing y' by a constant C, we verify that each straight line $y = Cx + f(C)$ is a solution. The slope of each straight line coincides with the direction of the tangent prescribed by the ODE. A systematic method to find this class of solutions starts by setting $y' = u(x)$ in the ODE so that $y = xu + f(u)$ with the differential $udx = dy = udx + xdu + f'(u)du$. Dropping the udx term we find

$$[x + f'(u)]du = 0.$$

Setting each factor equal to zero, $du = 0$ yields $u = C = \text{const.}$ and the straight lines again. Next, eliminating u from the other factor set to zero,

$$x + f'(u) = 0, \qquad \text{and} \qquad y = xu + f(u)$$

generates another solution of Clairaut's ODE, a curve $(x(u), y(u))$ that no longer contains the arbitrary constant C. From $y' = u$, we verify $y = xu + f(u) = xy' + f(y')$. The pair of coordinates $x(u), y(u)$ given previously represents a curve parameterized by the variable u; it represents the **envelope of the class of straight lines** $y = Cx + f(C)$ for various values of C that are tangents to this curve. The envelope of a class of solutions of an ODE is called its **singular solution**; it does not involve an integration constant (and cannot be adapted to initial conditions).

In general, **geometric problems in which a curve is to be determined from properties of its tangent at** x, y **lead to Clairaut's ODE** as follows. The **tangent equation** is given by

$$Y - y = y'(X - x), \qquad \text{or} \qquad Y = y'X + (y - xy'),$$

where X, Y are the coordinates of the tangent and y' is its slope. A property of the tangent can be expressed as some functional relation $F(y', y - xy') = 0$. Solving this relation for $y - xy'$ yields Clairaut's ODE. Let us illustrate this by the following example.

EXAMPLE 8.2.6

Envelope of Tangents as Singular Solution of Clairaut's ODE Determine a curve so that the length of the line segment T_1T_2 in Fig. 8.2 cut out of its tangent by the coordinate axes X, Y is a constant a. Setting $X = 0$ in the previous tangent equation gives the length OT_1 from the origin to T_1 on the Y-axis as $y - xy'$, and setting $Y = 0$ gives the OT_2 length on the X-axis as $(xy' - y)/y'$. The right-angle triangle with corners OT_1T_2 yields the tangent condition

$$(y - xy')^2 + \frac{(y - xy')^2}{y'^2} = a^2, \quad \text{or} \quad y = xy' \pm \frac{ay'}{\sqrt{y'^2 + 1}},$$

a Clairaut ODE with the general solution $y = xC \pm \frac{aC}{\sqrt{C^2+1}}$, which are straight lines. The envelope of this class of straight lines is obtained by eliminating u from

$$y = xu \pm \frac{au}{\sqrt{u^2 + 1}}, \qquad x \pm a\left(\frac{1}{\sqrt{u^2 + 1}} - \frac{u^2}{\sqrt{u^2 + 1}^3}\right) = 0.$$

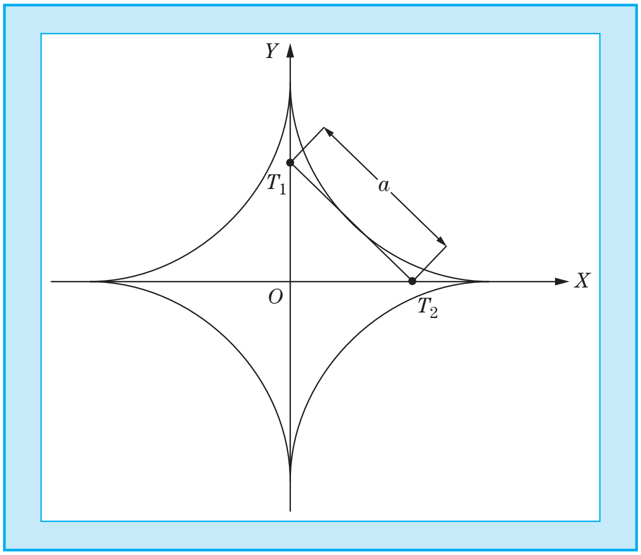
The second equation simplifies to $x \pm \frac{a}{\sqrt{u^2+1}^3} = 0$. Substituting $u = \tan\varphi$ yields $x \pm a\cos^3\varphi = 0$ and

$$y = \mp a\cos^3\varphi \pm a\sin\varphi = \pm a\sin^3\varphi$$

from the first equation. Eliminating the parameter φ from $x(\varphi)$, $y(\varphi)$ yields the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, plotted in Fig. 8.2. ■

Figure 8.2

Astroid as Envelope of Tangents of Constant Length $T_1T_2 = a$



First-order differential equations will be discussed again in Chapter 15 in connection with Laplace transforms, in Chapter 18 with regard to the Euler equation of the calculus of variations, and in Chapter 19 with regard to nonlinear (Riccati and Bernoulli's) ODEs. Numerical techniques for solving first-order differential equations are examined in Section 8.7.

SUMMARY

In summary, **first-order** ODEs of the implicit form $F(x, y, y') = 0$ (as discussed in the context of Clairaut's ODE) or explicit form $y' = f(x, y)$ contain the variable x, the unknown function $y(x)$, and its derivative $\frac{dy}{dx} = y'(x)$. The general solution contains one arbitrary constant, called the integration constant, which often is determined by an initial condition $y(x_0) = y_0$ involving given constants x_0, y_0. Such ODEs are sometimes called initial value problems.

Among the simplest ODEs are separable equations $y' = f(x, y) = -\frac{P(x)}{Q(y)}$ of Section 8.2. Their general solution is obtained by the integration $\int_{x_0}^{x} P(X)dX + \int_{y_0}^{y} Q(Y)dY = \text{const.}$

Closely related are the more general **exact** differential equations

$$P(x, y)dx + Q(x, y)dy = d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy$$

with the integrability condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. If the **integrability condition** is not satisfied, a solution $\varphi(x, y)$ does not exist. In that case, one has to search for an **integrating factor** $\alpha(x, y)$ so that $\frac{\partial(\alpha P)}{\partial y} = \frac{\partial(\alpha Q)}{\partial x}$ holds.

Linear first-order equations $y' + p(x)y = q(x)$ are common ODEs. The radioactive decay law and electrical circuits are prime examples. The homogeneous ODE $y' + py = 0$ is separable and integrated first, yielding $\ln y + \int^{x} p\,dX = \ln C$; then the integration constant $C \to C(x)$ is varied to find the solution of the inhomogeneous ODE.

EXERCISES

8.2.1 From Kirchhoff's law, the current I in an RC (resistance–capacitance) circuit [change L to C in Fig. 8.1 and remove $V(t)$; that is, short out the circuit] obeys the equation

$$R\frac{dI}{dt} + \frac{1}{C}I = 0.$$

(a) Find $I(t)$.

(b) For a capacitance of 10,000 μF charged to 100 V and discharging through a resistance of 1 mΩ, find the current I for $t = 0$ and for $t = 100$ sec.

Note. The initial voltage is $I_0 R$ or Q/C, where $Q = \int_0^{\infty} I(t)dt$.

8.2.2 The Laplace transform of Bessel's equation ($n = 0$) leads to

$$(s^2 + 1)f'(s) + sf(s) = 0.$$

Solve for $f(s)$.

8.2.3 The decay of a population by catastrophic two-body collisions is described by

$$\frac{dN}{dt} = -kN^2$$

for $t \geq 0$. This is a first-order, **nonlinear** differential equation. Derive the solution

$$N(t) = N_0\left(1 + \frac{t}{\tau_0}\right)^{-1},$$

where $\tau_0 = (kN_0)^{-1}$ and N_0 is the population at time $t = 0$. This implies an infinite population at $t = -\tau_0$, which is irrelevant because the initial value problem starts at $t = 0$ with $N(0) = N_0$.

8.2.4 The rate of a particular chemical reaction $A + B \rightarrow C$ is proportional to the concentrations of the reactants A and B:

$$\frac{dC(t)}{dt} = \alpha[A(0) - C(t)][B(0) - C(t)],$$

where $A(0) - C(t)$ is the amount of A left to react at time t, and similarly for B.

(a) Find $C(t)$ for $A(0) \neq B(0)$.

(b) Find $C(t)$ for $A(0) = B(0)$.

The initial condition is that $C(0) = 0$.

8.2.5 A boat coasting through the water experiences a resisting force proportional to v^n, where v is the boat's instantaneous velocity and n an integer. Newton's second law leads to

$$m\frac{dv}{dt} = -kv^n.$$

With $v(t = 0) = v_0$, $x(t = 0) = 0$, integrate to find v as a function of time and v as a function of distance.

8.2.6 The differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

is **exact**. Verify that

$$\varphi(x, y) = \int_{x_0}^{x} P(X, Y)dX + \int_{y_0}^{y} Q(X, Y)dY = \text{constant}$$

is a solution.

8.2.7 The differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

is **exact**. If

$$\varphi(x, y) = \int_{x_0}^{x} P(X, Y)dX + \int_{y_0}^{y} Q(x, Y)dY,$$

show that

$$\frac{\partial\varphi}{\partial x} = P(x, y), \qquad \frac{\partial\varphi}{\partial y} = Q(x, y).$$

Hence, $\varphi(x, y) =$ constant is a solution of the original differential equation.

8.2.8 Prove that Eq. (8.13) is exact in the sense of Eq. (8.8), provided that $\alpha(x)$ satisfies Eq. (8.15).

8.2.9 A certain differential equation has the form

$$f(x)dx + g(x)h(y)dy = 0,$$

with none of the functions $f(x)$, $g(x)$, $h(y)$ identically zero. Show that a necessary and sufficient condition for this equation to be exact is that $g(x) =$ constant.

8.2.10 Show that

$$y(x) = \exp\left[-\int^x p(t)dt\right]\left\{\int^x \exp\left[\int^s p(t)dt\right] q(s)ds + C\right\}$$

is a solution of

$$\frac{dy}{dx} + p(x)y(x) = q(x)$$

by differentiating the expression for $y(x)$ and substituting into the differential equation.

8.2.11 The motion of a body falling in a resisting medium may be described by

$$m\frac{dv}{dt} = mg - bv$$

when the retarding force is proportional to the velocity, v. Find the velocity. Evaluate the constant of integration by demanding that $v(0) = 0$. Explain the signs of the terms mg and bv.

8.2.12 The rate of evaporation from a particular spherical drop of liquid (constant density) is proportional to its surface area. Assuming this to be the sole mechanism of mass loss, find the radius of the drop as a function of time.

8.2.13 In the linear homogeneous differential equation

$$\frac{dv}{dt} = -av$$

the variables are separable. When the variables are separated the equation is exact. Solve this differential equation subject to $v(0) = v_0$ by the following three methods:

(a) separating variables and integrating;
(b) treating the separated variable equation as exact; and
(c) using the result for a linear homogeneous differential equation.

ANS. $v(t) = v_0 e^{-at}$.

8.2.14 Bernoulli's equation,

$$\frac{dy}{dt} + f(x)y = g(x)y^n$$

is nonlinear for $n \neq 0$ or 1. Show that the substitution $u = y^{1-n}$ reduces Bernoulli's equation to a linear equation.

ANS. $\dfrac{du}{dx} + (1-n)f(x)u = (1-n)g(x)$.

8.2.15 Solve the linear, first-order equation, Eq. (8.12), by assuming $y(x) = u(x)v(x)$, where $v(x)$ is a solution of the corresponding homogeneous equation $[q(x) = 0]$.

8.2.16 (a) Rework Example 8.2.2 with an initial speed $v_i = 60$ miles/hr, when the parachute opens. Find $v(t)$.
(b) For a skydiver in free fall (no parachute) use the much smaller friction coefficient $b = 0.25$ kg/m and $m = 70$ kg. What is the limiting velocity in this case?

ANS. $v_0 = 52$ m/sec $= 187$ km/hr.

8.2.17 The flow lines of a fluid are given by the hyperbolas $xy = C =$ const. Find the orthogonal trajectories (equipotential lines) and plot them along with the flow lines using graphical software.
Hint. Start from $y' = \tan\alpha$ for the hyperbolas.

8.2.18 Heat flows in a thin plate in the xy-plane along the hyperbolas $xy =$ const. What are the lines of constant temperature (isotherms)?

8.2.19 Solve the ODE $y' = ay/x$ for real a and initial condition $y(0) = 1$.

8.2.20 Solve the ODE $y' = y + y^2$ with $y(0) = 1$.

8.2.21 Solve the ODE $y' = \frac{1}{x+y}$ with $y(0) = 0$.

ANS. $x(y) = e^y - 1 - y$.

8.2.22 Find the general solution of $y'^3 - 4xyy' + 8y^2 = 0$ and its singular solution. Plot them.

ANS. $y = C(x - C)^2$. The singular solution is $y = \frac{4}{27}x^3$.

8.3 Second-Order ODEs

Linear ODEs of second order are most common in physics and engineering applications because of dynamics: In classical mechanics the acceleration is a second-order derivative and so is the kinetic energy in quantum mechanics. Thus, any problem of classical mechanics, where we describe the motion of a particle subject to a force, involves an ODE. Specifically, a force or driving term leads to an inhomogeneous ODE. In quantum mechanics we are led to the Schrödinger equation, a PDE. We will develop methods to find **particular solutions of the inhomogeneous ODE** and the **general solution of the homogeneous ODE**, such as the variation of constants, power-series expansion, and Green's functions. Special classes of ODEs are those with **constant**

coefficients that occur in RLC electrical circuits and harmonic oscillators in classical mechanics. The simple harmonic oscillator of quantum mechanics is treated in Chapter 13. Nonlinear ODEs are addressed in Chapter 19. We start this section with examples of special classes of ODEs. We use the standard notation[2] $dy/dx = y'$, $d^2y/dx^2 = y''$.

In Examples 8.3.1 and 8.3.2, we encounter a general feature. **Because the solution of a second-order ODE involves two integrations, the general solution will contain two integration constants that may be adjusted to initial or boundary conditions.**

When one variable is missing in the ODE, such as x or y, the ODE can be reduced to a first-order ODE.

EXAMPLE 8.3.1

Second-Order ODEs, Missing Variable y If the unknown function y **is absent from the ODE**, as in

$$y'' = f(y', x), \tag{8.23}$$

then it becomes a first-order ODE for $z(x) = y'(x)$, $z' = f(z, x)$. If $z(x, C_1)$ is a solution of this ODE depending on an integration constant C_1, then

$$y(x) = \int^x z(X, C_1)dX + C_2$$

is the general solution of the second-order ODE.

A simple example is the ODE $y'' = y'$ with boundary conditions $y(0) = 1$, $y(-\infty) = 0$.

Setting $z = y'$, we solve $\frac{dz}{dx} = z$ by integrating

$$\int^z \frac{dZ}{Z} = \ln z = \int^x dX = x + \ln C_1.$$

Exponentiating we obtain

$$z = C_1 e^x = \frac{dy}{dx}.$$

Integrating again we find

$$y = C_1 \int^x e^X dX + C_2 = C_1 e^x + C_2.$$

We check our solution by differentiating it twice: $y' = C_1 e^x$, $y'' = C_1 e^x = y'$. The boundary conditions $y(0) = 1$, $y(-\infty) = 0$ determine the integration constants C_1, C_2. They give $C_1 + C_2 = 1$ and $C_2 = 0$ so that $C_1 = 1$ results, and the solution is $y = e^x$.

Another specific case is $y'' = y'^2$ with initial conditions $y(0) = 2$, $y'(0) = -1$.

[2]This prime notation y' was introduced by Lagrange in the late 18th century as an abbreviation for Leibniz's more explicit but more cumbersome dy/dx.

We start by integrating $z' = z^2$, or

$$\int^z \frac{dZ}{Z^2} = -1/z = \int^x dX + C_1.$$

This yields $z = y' = \frac{-1}{x+C_1}$. Integrating again we find

$$y(x) = -\ln(x + C_1) + C_2.$$

Checking this solution gives $y'' = (x+C_1)^{-2} = y'^2$. The initial conditions yield $2 = -\ln C_1 + C_2$, $-1 = -1/C_1$ so that $C_1 = 1$, implying $C_2 = 2$. The solution is $y = -\ln(x + 1) + 2$.

A third case is the ODE $y'' = (xy')^2$. We solve $z' = (xz)^2$ by separating variables:

$$\int^z \frac{dZ}{Z^2} = \frac{-1}{z} = \int^x X^2\, dX = \frac{1}{3}(x^3 - C_1^3).$$

We have chosen the integration constant in this special cubic form so that we can factorize the third-order polynomial

$$x^3 - C_1^3 = (x - C_1)(x^2 + C_1 x + C_1^2)$$

and, in the ODE,

$$z = y' = \frac{-3}{x^3 - C_1^3}$$

decompose the inverse polynomial into partial fractions

$$\frac{1}{x^3 - C_1^3} = \frac{1}{x - C_1} + \frac{i}{C_1\sqrt{3}}\left(\frac{1}{x + \frac{C_1}{2}(1 + i\sqrt{3})} - \frac{1}{x + \frac{C_1}{2}(1 - i\sqrt{3})}\right).$$

Integrating the ODE yields the solution

$$\begin{aligned} y(x) &= -3\ln(x - C_1) + \ln C_2 \\ &\quad - i\frac{\sqrt{3}}{C_1}[\ln(x + C_1(1 + i\sqrt{3})/2) - \ln(x + C_1(1 - i\sqrt{3})/2)]. \end{aligned}$$ ■

EXAMPLE 8.3.2 **Second-Order ODEs, Missing Variable x** If the variable x does not appear in the ODE, as in

$$y'' = f(y', y), \tag{8.24}$$

then we seek a solution $y' = z(y)$ instead of searching for $y(x)$ directly. Using the chain rule we obtain

$$y'' = \frac{dz}{dy}\frac{dy}{dx} = z\frac{dz}{dy} = f(z, y),$$

which is a first-order ODE for $z(y)$. If we can find a solution $z(y, C_1)$, then we can integrate $y' = z(y)$ to get

$$\int^y \frac{dY}{z(Y, C_1)} = \int^x dX = x + C_2.$$ ■

EXAMPLE 8.3.3

$y'' + f(x)y' + g(y)y'^2 = 0$ This more general and nonlinear ODE is a combination of the types treated in Examples 8.3.1 and 8.3.2 so that we try a product solution $y' = v(x)w(y)$ incorporating the previous solution types. Differentiating this ansatz (trial solution) and substituting into the ODE we find

$$y'' = v'w + v\frac{dw}{dy}y' = v'w + v^2 w\frac{dw}{dy} = -fvw - gv^2w^2.$$

Here, we divide by an overall factor v^2w without loss of generality because we reject $y' = 0$ as a trivial solution. We can solve the resulting ODE

$$\frac{v' + f(x)v}{v^2} + \frac{dw}{dy} + g(y)w(y) = 0$$

by choosing $v(x)$ as a solution of the first-order ODE $v' + f(x)v(x) = 0$ from the first term alone and $w(y)$ as a solution of the first-order ODE $\frac{dw}{dy} + g(y)w(y) = 0$ from the second term alone. Both ODEs can be solved by separating variables

$$\int^v \frac{dV}{V} = -\int^x f(X)dX = \ln v, \qquad \int^w \frac{dW}{W} = -\int^y g(Y)dY = \ln w.$$

Alternatively, integrating the ODE written in the form

$$\frac{y''}{y'} + f(x) + g(y)y' = 0$$

yields

$$\ln y' + \int^x f(X)dX + \int^y g(Y)dY = C,$$

where C is an integration constant. Exponentiating this result gives the same solution.

Let us illustrate a more specific example:

$$xyy'' + yy' - xy'^2 = 0.$$

where $f(x) = \frac{1}{x}$ and $g(y) = -\frac{1}{y}$ so that $\ln v = -\ln x + \ln C_1$ [i.e., $v(x) = \frac{C_1}{x}$] and $\ln w = \ln y + \ln C_2$ [i.e., $w(y) = C_2 y$]. Therefore, $y' = C_1C_2y/x$, which we integrate as

$$\ln y = C_1C_2 \ln x + \ln C_3$$

so that finally $y(x) = C_3 x^{C_1C_2}$, a power law that indeed satisfies the ODE. ■

EXAMPLE 8.3.4

Euler's ODE Euler's ODE,

$$ax^2y'' + bxy' + cy = 0, \tag{8.25}$$

is a **homogeneous linear** ODE that can be solved with a power ansatz $y = x^p$. This power law is a natural guess because the reduction of the exponent by differentiation is restored by the coefficients x, x^2 of the y' and y'' terms, each producing the same power.

Substituting $y' = px^{p-1}$, $y'' = p(p-1)x^{p-2}$ into the ODE yields

$$[ap(p-1) + bp + c]x^p = 0,$$

an **algebraic equation for the exponent** but only for the homogeneous ODE. Now we drop the factor x^p to find two roots p_1, p_2 from the quadratic equation. If both exponents p_i are real, the **general solution** is

$$C_1 x^{p_1} + C_2 x^{p_2}.$$

If the **exponents are complex conjugates** $p_{1,2} = r \pm iq$, then the Euler identity for $x^{iq} = e^{iq \ln x}$ yields the **general solution**

$$y(x) = x^r [C_1 \cos(q \ln x) + C_2 \sin(q \ln x)].$$

If there is a **degenerate** solution $p_1 = p_2 = p$ for the exponent, we approach the degenerate case by letting the exponents become equal in the linear combination $(x^{p+\varepsilon} - x^p)/\varepsilon$, which is a solution of the ODE for $\varepsilon \to 0$. This may be achieved by slightly varying the coefficients a, b, c of the ODE so that the degenerate exponent p splits into $p+\varepsilon$ and p. Thus, we are led to differentiate x^p with respect to p. This yields the second solution $x^p \ln x$ and the **general solution**

$$y = x^p (C_1 + C_2 \ln x).$$

A specific example is the ODE

$$x^2 y'' + 3xy' + y = 0 \quad \text{with} \quad p(p-1) + 3p + 1 = 0 = (p+1)^2$$

so that $p = -1$ is a degenerate exponent. Thus, the solution is $y(x) = \frac{C_1}{x} + C_2 \frac{\ln x}{x}$. ■

EXAMPLE 8.3.5

ODEs with Constant Coefficients ODEs with constant coefficients

$$ay'' + by' + cy = 0 \tag{8.26}$$

are solved with the exponential ansatz $y = e^{px}$. This is a natural guess because differentiation reproduces the exponential up to a multiplicative constant $y' = py$ and $y'' = p^2 y$. Hence, substituting **the exponential ansatz reduces the ODE to the quadratic equation**

$$ap^2 + bp + c = 0$$

for the exponent. If there are **two real roots** p_1, p_2, then

$$y = C_1 e^{p_1 x} + C_2 e^{p_2 x}$$

is the general solution. If $p_1 > 0$, or $p_2 > 0$, we have an exponentially growing solution. When $p_1 < 0$, and $p_2 < 0$, we have the **overdamped** solution displayed in Fig. 8.3.

Figure 8.3

Typical Solution of ODE with Constant Coefficients: Two Negative Exponents

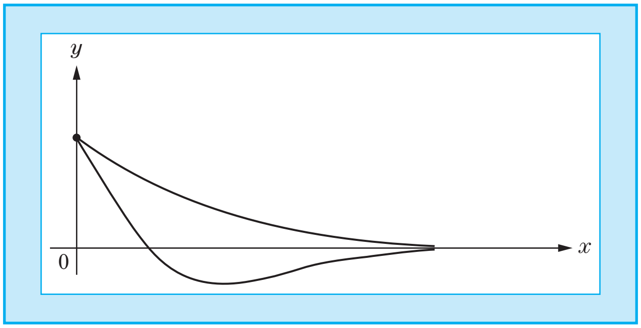
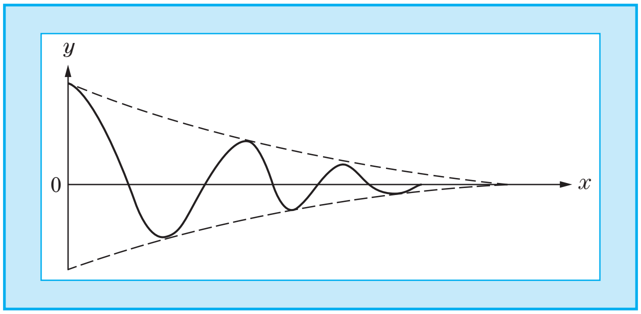


Figure 8.4

Typical Solution of ODE with Constant Coefficients: Two Complex Conjugate Exponents



If there are **two complex conjugate roots**, then Euler's identity yields

$$p_{1,2} = r \pm iq, \quad y(x) = e^{rx}(C_1 \cos qx + C_2 \sin qx)$$

as the **general oscillatory or underdamped solution** (Fig. 8.4).

If there is **one degenerate exponent**, we approach the degenerate case with two slightly different exponents $p + \varepsilon$ and p for $\varepsilon \to 0$ in the solution $[e^{(p+\varepsilon)x} - e^{px}]/\varepsilon$ of the ODE. Again, as in Example 8.3.4, this leads us to differentiate e^{px} with respect to p to find the **second solution** xe^{px}, giving the general **critically damped solution** $y = e^{px}(C_1 + C_2 x)$ for the double-root case (Fig. 8.5). See also Examples 15.8.1, 15.9.1, and 15.10.1 for a solution by Laplace transform. ■

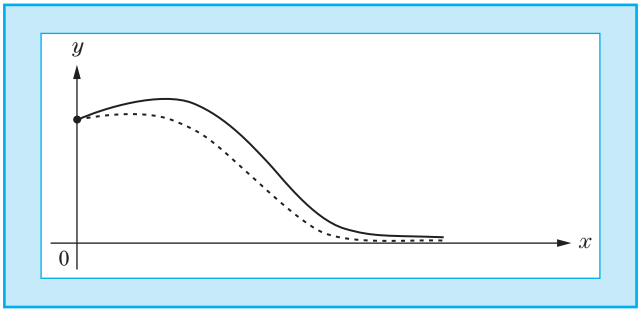
Because ODEs with constant coefficients and Euler's ODEs are linear in the unknown function y and homogeneous, we have used the **superposition principle** in Examples 8.3.4 and 8.3.5: If y_1, y_2 are two solutions of the homogeneous ODE, so is the linear combination $C_1 y_1 + C_2 y_2$ with constants C_1, C_2 that are fixed by initial or boundary conditions as usual.

The same exponential form $y(x) = e^{px}$ leads to the solutions of nth-order ODEs

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

Figure 8.5

Typical Solutions of ODE with Constant Coefficients: Two Equal Exponents



with constant coefficients a_i in terms of exponents p_i that are roots of the polynomial equation

$$a_0 p^n + a_1 p^{n-1} + \cdots + a_{n-1} p + a_n = 0.$$

The general solution is the linear combination

$$y(x) = \sum_{i=1}^{n} b_i e^{p_i x},$$

where the constants b_i are determined by initial or boundary conditions.

Other generalizations are coupled ODEs with constant coefficients. Several cases are treated in Chapter 19 (Examples 19.4.6–19.4.10) in the context of linear approximations to nonlinear ODEs.

Inhomogeneous Linear ODEs and Particular Solutions

We have already discussed inhomogeneous first-order ODEs, such as Eq. (8.12). The general solution $y(x) = y_h(x) + y_p(x)$ is a sum of the general solution y_h of the homogeneous ODE and a particular solution y_p of the inhomogeneous ODE, which can be immediately verified by substituting y into the inhomogeneous ODE. This theorem generalizes to nth-order linear ODEs, the general solution being $y(x) = y_p(x) + \sum_{i=1}^{n} c_i y_i(x)$, where y_i are the independent solutions of the homogeneous ODE with constants c_i. The particular solution y_p usually inherits its form from the driving term $q(x)$ provided differentiations produce the same types of functions that $q(x)$ contains. The next few examples are cases in point, where we treat special types of functions $q(x)$, such as power laws, periodic functions, exponentials, and their combinations.

Inhomogeneous Euler ODE

Let us look at the inhomogeneous Euler ODE with a power law driving term

$$ax^2 y'' + bxy' + cy = Dx^d,$$

where the exponent d and strength D are known numbers. The power law is the natural form for the Euler ODE because each term retains its exponent. Substituting the ansatz $y_p = Ax^d$ into the Euler ODE, we realize that each term contains the same power x^d, which we can drop. We obtain

$$A[ad(d-1)+bd+c]=D,$$

which determines A provided d is not an exponent of the homogeneous ODE.

If d is an exponent of the homogeneous ODE, that is, $ad(d-1)+bd+c=0$, then our solution $y_p=x^d(A+B\ln x)$ is a linear combination of both contributions of the degenerate case in Example 8.3.4. Substituting this trial solution into Euler's ODE yields

$$\begin{aligned}
Dx^d &= ax^2y''+bxy'+cy\\
&= x^d[a(d-1)dA+a(d-1)dB\ln x+a(2d-1)B+bdA\\
&\quad +bdB\ln x+bB+cA+cB\ln x]\\
&= x^d(A[a(d-1)d+bd+c]+B[a(2d-1)+b]\\
&\quad +B[a(d-1)d+bd+c]\ln x),
\end{aligned}$$

where the terms containing a come from y_p'', those containing b from y_p', and those containing c from y_p. Now we drop x^d and use $ad(d-1)+bd+c=0$, obtaining

$$D=B[a(2d-1)+b],$$

thereby getting B in terms of D, whereas A is not determined by the source term; A can be used to satisfy an initial or boundary condition. The source can also have the more general form $x^d(D+E\ln x)$ in the degenerate case.

For an exponential driving term

$$ax^2y''+bxy'+cy=De^{-x},$$

the powers of x in the ODE force us to a more complicated trial solution $y_p=e^{-x}\sum_{n=0}^{\infty}a_nx^n$. Substituting this ansatz into Euler's ODE yields recursion relations for the coefficients a_n. Such power series solutions are treated more systematically in Section 8.5. Similar complications arise for a periodic driving term, such as $\sin\omega x$, which shows that these forms are not natural for Euler's ODE.

Inhomogeneous ODE with Constant Coefficients

We start with a natural driving term of exponential form

$$ay''+by'+cy=De^{-dx},$$

where the strength D and exponent d are known numbers. We choose a particular solution $y_p=Ae^{-dx}$ of the same form as the source, because the derivatives preserve it. Substituting this y_p into the ODE with constant coefficients a, b, c yields

$$A[ad^2-bd+c]=D,$$

determining A in terms of D, provided d is not an exponent of the homogeneous ODE.

If the latter is the case, that is, $ad^2 - bd + c = 0$, we have to start from the more general form $y_p = e^{-dx}(A + Bx)$ appropriate for the degenerate case of Example 8.3.5. Substituting this y_p into the ODE yields

$$D = ad^2(A + Bx) - 2adB - bd(A + Bx) + bB + c(A + Bx)$$
$$= A[ad^2 - bd + c] + Bx[ad^2 - bd + c] + B(b - 2ad),$$

where the terms containing a come from y_p'', those containing b from y_p' and c from y_p. Now we drop the terms containing $ad^2 - bd + c = 0$ to obtain

$$B(b - 2ad) = D,$$

determining B in terms of D, while A remains free to be adjusted to an initial or boundary condition.

A source term of polynomial form is solved by a particular solution of polynomial form of the same degree if the coefficient of the y term in the ODE is nonzero; if not, the degree of y increases by one, etc.

Periodic source terms, such as $\cos\omega x$ or $\sin\omega x$, are also natural and lead to particular solutions of the form $y_p = A\cos\omega x + B\sin\omega x$, where both the sine and cosine have to be included because the derivative of the sine gives the cosine and vice versa. We deal with such a case in the next example.

EXAMPLE 8.3.6

Electrical Circuit Let us take Example 8.2.4, include a capacitance C and an external AC voltage $V(t) = V_0 \sin\omega t$ in series to form an RLC circuit (Fig. 8.6). Here, the $\sin\omega t$ driving term leads to a particular solution $y_p \sim \sin(\omega t - \varphi)$, a sine shape with the same frequency ω as the driving term.

The voltage drop across the resistor is RI, across the inductor it is given by the instantaneous rate of change of the current $L\frac{dI}{dt}$, and across the capacitor it is given by Q/C with the charge $Q(t)$ giving

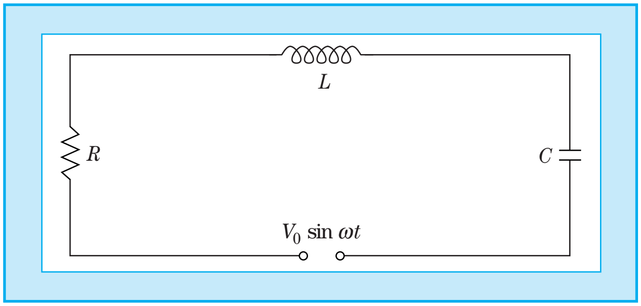
$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V_0 \sin\omega t.$$

Because $I(t) = \frac{dQ}{dt}$, we differentiate both sides of this equation to obtain the ODE with constant coefficients

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \omega V_0 \cos\omega t.$$

Figure 8.6

Electrical Circuit: Resistance, Inductance, and Capacitance in Series



Comparing this ODE with the harmonic oscillator ODE in classical mechanics, we see that the inductance L is the electrical analog of the mass, the resistance R is the analog of the damping, and the inverse of the capacitance $1/C$ is the analog of a spring constant, whereas the current I is the analog of the mechanical displacement $x(t)$. The general solution of the **homogeneous** ODE is

$$I_h = C_1 e^{p_1 t} + C_2 e^{p_2 t},$$

where $p = p_1$ and $p = p_2$ are the roots of the quadratic equation

$$p^2 + \frac{R}{L}p + \frac{1}{LC} = 0, \quad p = -\frac{R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Because of the dominant negative term $-R/2L$ in p (note the negative sign in the radicand), I_h is a **transient current that decays exponentially with time**.

We now look for the **particular solution** with the same harmonic form as the driving voltage $I_p = A\cos\omega t + B\sin\omega t$. This is called the **steady-state current** with the same frequency as the input, which survives after a sufficiently long time ($-p_2 t \gg 1$). This is seen from the general solution $I = I_p + I_h$. In this sense, the steady-state current is an asymptotic form, but it is a particular solution that is present from the initial time onward. We differentiate I_p twice, substitute into the ODE, and compare the coefficients of the $\sin\omega t$ and $\cos\omega t$ terms. This yields

$$-\omega^2 L(A\cos\omega t + B\sin\omega t) + R\omega(-A\sin\omega t + B\cos\omega t) + \frac{1}{C}(A\cos\omega t + B\sin\omega t) = \omega V_0 \cos\omega t$$

so that

$$-\omega^2 LA + \omega RB + \frac{A}{C} = \omega V_0, \quad -\omega^2 LB - \omega RA + \frac{B}{C} = 0.$$

From the second of these equations we find

$$A = -B\frac{S}{R}, \quad S = \omega L - \frac{1}{\omega C},$$

where S is defined as the reactance by electrical engineers. Substituting this expression A into the first equation yields

$$B = \frac{V_0 R}{R^2 + S^2} \quad \text{so that} \quad A = -\frac{V_0 S}{R^2 + S^2}.$$

The steady-state current may also be written as

$$I_p = I_0 \sin(\omega t - \varphi), \quad I_0 = \sqrt{A^2 + B^2} = \frac{V_0}{\sqrt{R^2 + S^2}}, \quad \tan\varphi = -\frac{A}{B} = \frac{S}{R},$$

where $\sqrt{R^2 + S^2}$ is the impedance. ■

More examples of coupled and nonlinear ODEs are given in Chapter 19, particularly Examples 19.4.6–19.4.10.

Finally, let us address the **uniqueness and generality** of our solutions. If we have found **a particular solution of a linear inhomogeneous second-order ODE**

$$y'' + P(x)y' + Q(x)y = f(x), \tag{8.27}$$

then it **is unique up to an additive solution of the homogeneous ODE.** To show this theorem, suppose y_1, y_2 are two solutions. Subtracting both ODEs it follows that $y_1 - y_2$ is a solution of the homogeneous ODE

$$y'' + P(x)y' + Q(x)y = 0 \tag{8.28}$$

because of linearity of the ODE in y, y', y'' and $f(x)$ cancels. ■

The general solution of the homogeneous ODE [Eq. (8.28)] **is a linear combination of two linearly independent solutions.** To prove this theorem we assume there are three solutions and show that there is a linear relation between them. The analysis will lead us to the generalization of the Wronskian of two solutions of a first-order ODE in Section 8.2. Therefore, now we consider the question of linear independence of a set of functions.

Linear Independence of Solutions

Given a set of functions, φ_λ, the criterion for linear dependence is the existence of a relation of the form

$$\sum_\lambda k_\lambda \varphi_\lambda = 0, \tag{8.29}$$

in which **not all the coefficients** k_λ **are zero**. On the other hand, if the only solution of Eq. (8.29) is $k_\lambda = 0$ for all λ, the set of functions φ_λ is said to be linearly **independent**. In other words, functions are linearly independent if they cannot be obtained as solutions of linear relations that hold for all x.

It may be helpful to think of linear dependence of vectors. Consider **A, B**, and **C** in three-dimensional space with $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} \neq 0$. Then no nontrivial relation of the form

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = 0 \tag{8.30}$$

exists. **A**, **B**, and **C** are linearly independent. On the other hand, any fourth vector **D** may be expressed as a linear combination of **A**, **B**, and **C** (see Section 2.1). We can always write an equation of the form

$$\mathbf{D} - a\mathbf{A} - b\mathbf{B} - c\mathbf{C} = 0, \tag{8.31}$$

and the four vectors are **not** linearly independent. The three noncoplanar vectors **A**, **B**, and **C** span our real three-dimensional space.

Let us assume that the functions φ_λ are differentiable as needed. Then, differentiating Eq. (8.29) repeatedly, we generate a set of equations

$$\sum_\lambda k_\lambda \varphi'_\lambda(x) = 0, \tag{8.32}$$

$$\sum_\lambda k_\lambda \varphi''_\lambda(x) = 0, \tag{8.33}$$

and so on. This gives us a set of homogeneous linear equations in which k_λ are the unknown quantities. By Section 3.1 there is a solution $k_\lambda \neq 0$ only if the determinant of the coefficients of the k_λs vanishes for all values of x. This means that the **Wronskian** of $\varphi_1, \varphi_2, \ldots, \varphi_n$,

$$W(\varphi_1, \varphi_2, \ldots, \varphi_n) \equiv \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}, \tag{8.34}$$

a function of x, vanishes for all x.

1. If the Wronskian is not equal to zero, then Eq. (8.29) has no solution other than $k_\lambda = 0$. The set of functions φ_λ is therefore linearly independent.
2. If the Wronskian vanishes at isolated values of the argument, this does not necessarily prove linear dependence (unless the set of functions has only two functions). However, if the Wronskian is zero over the entire range of the variable, the functions φ_λ are linearly dependent over this range.[3]

EXAMPLE 8.3.7

Linear Dependence For an illustration of linear dependence of three functions, consider the solutions of the one-dimensional diffusion equation $y'' = y$. We have $\varphi_1 = e^x$ and $\varphi_2 = e^{-x}$, and we add $\varphi_3 = \cosh x$, also a solution. The Wronskian is

$$\begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = 0.$$

The determinant vanishes for all x because the first and third rows are identical. Hence, e^x, e^{-x}, and $\cosh x$ are linearly dependent, and indeed, we have a relation of the form of Eq. (8.29):

$$e^x + e^{-x} - 2\cosh x = 0 \qquad \text{with} \quad k_\lambda \neq 0. \quad ■$$

Now we are ready to prove the **theorem that a second-order homogeneous ODE has two linearly independent solutions**.

Suppose y_1, y_2, y_3 are three solutions of the homogeneous ODE [Eq. (8.28)]. Then we form the Wronskian $W_{jk} = y_j y_k' - y_j' y_k$ of any pair y_j, y_k of them and recall that $W_{jk}' = y_j y_k'' - y_j'' y_k$. Next we divide each ODE by y, getting $-Q$ on the right-hand side so that

$$\frac{y_j''}{y_j} + P\frac{y_j'}{y_j} = -Q(x) = \frac{y_k''}{y_k} + P\frac{y_k'}{y_k}.$$

Multiplying by $y_j y_k$, we find

$$(y_j y_k'' - y_j'' y_k) + P(y_j y_k' - y_j' y_k) = 0, \qquad \text{or} \qquad W_{jk}' = -PW_{jk} \tag{8.35}$$

[3] For proof, see H. Lass (1957), *Elements of Pure and Applied Mathematics*, p. 187. McGraw-Hill, New York. It is assumed that the functions have continuous derivatives and that at least one of the minors of the bottom row of Eq. (8.34) (Laplace expansion) does not vanish in $[a, b]$, the interval under consideration.

for any pair of solutions. Finally, we evaluate the Wronskian of all three solutions expanding it along the second row and using the ODEs for the W_{jk}:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -y_1' W_{23}' + y_2' W_{13}' - y_3' W_{12}'$$

$$= P(y_1' W_{23} - y_2' W_{13} + y_3' W_{12}) = -P \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \end{vmatrix} = 0.$$

The vanishing Wronskian, $W = 0$, because of two identical rows is the condition for linear dependence of the solutions y_j. Thus, there are at most two linearly independent solutions of the homogeneous ODE. Similarly, one can prove that a linear homogeneous nth-order ODE has n linearly independent solutions y_j so that the general solution $y(x) = \sum c_j y_j(x)$ is a linear combination of them. ■

Biographical Data
Wrónski, Jozef Maria. Wrónski, a Polish mathematician (1778–1853) who changed his name from Höne, introduced the determinants named after him.

SUMMARY

In summary, second-order ODEs require two integrations and therefore contain two integration constants, and there are two linearly independent solutions. The general solution $y_p + c_1 y_1 + c_2 y_2$ of the inhomogeneous ODE consists of a particular solution y_p and the general solution of the homogeneous ODE. If an ODE $y'' = f(y', y)$ does not contain the variable x, then a solution of the form $y' = z(y)$ reduces the second-order ODE to a first-order ODE. An ODE where the unknown function $y(x)$ does not appear can be reduced to first order similarly, and combinations of these types can also be treated. Euler's ODE involving $x^2 y''$, xy', y linearly is solved by a linear combination of the power x^p, where the exponent p is a solution of a quadratic equation, to which the ODE reduces. ODEs with constant coefficients are solved by exponential functions e^{px}, where the exponent p is a solution of a quadratic equation, to which the ODE reduces.

EXERCISES

8.3.1 You know that the three unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are mutually perpendicular (orthogonal). Show that $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are linearly independent. Specifically, show that no relation of the form of Eq. (8.30) exists for $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

8.3.2 The criterion for the linear **independence** of three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ is that the equation

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = 0$$

[analogous to Eq. (8.30)] has no solution other than the trivial $a = b = c = 0$. Using components $\mathbf{A} = (A_1, A_2, A_3)$, and so on, set up the determinant criterion for the existence or nonexistence of a nontrivial solution for the coefficients a, b, and c. Show that your criterion is equivalent to the scalar product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} \neq 0$.

8.3.3 Using the Wronskian determinant, show that the set of functions

$$\left\{1, \frac{x^n}{n!}(n = 1, 2, \ldots, N)\right\}$$

is linearly independent.

8.3.4 If the Wronskian of two functions y_1 and y_2 is identically zero, show by direct integration that

$$y_1 = cy_2;$$

that is, y_1 and y_2 are linearly dependent. Assume the functions have continuous derivatives and that at least one of the functions does not vanish in the interval under consideration.

8.3.5 The Wronskian of two functions is found to be zero at $x = x_0$ and all x in a small neighborhood of x_0. Show that this Wronskian vanishes for all x and that the functions are linearly dependent. If x_0 is an isolated zero of the Wronskian, show by giving a counterexample that linear dependence is not a valid conclusion in general.

8.3.6 The three functions $\sin x$, e^x, and e^{-x} are linearly independent. No one function can be written as a linear combination of the other two. Show that the Wronskian of $\sin x$, e^x, and e^{-x} vanishes but only at isolated points.

$$\textit{ANS.} \quad W = 4\sin x,$$
$$W = 0 \text{ for } x = \pm n\pi, \quad n = 0, 1, 2, \ldots.$$

8.3.7 Consider two functions $\varphi_1 = x$ and $\varphi_2 = |x| = x \operatorname{sgn} x$ (Fig. 8.7). The function $\operatorname{sgn} x$ is the sign of x. Since $\varphi_1' = 1$ and $\varphi_2' = \operatorname{sgn} x$, $W(\varphi_1, \varphi_2) = 0$ for any interval including $[-1, +1]$. Does the vanishing of the Wronskian over $[-1, +1]$ prove that φ_1 and φ_2 are linearly dependent? Clearly, they are not. What is wrong?

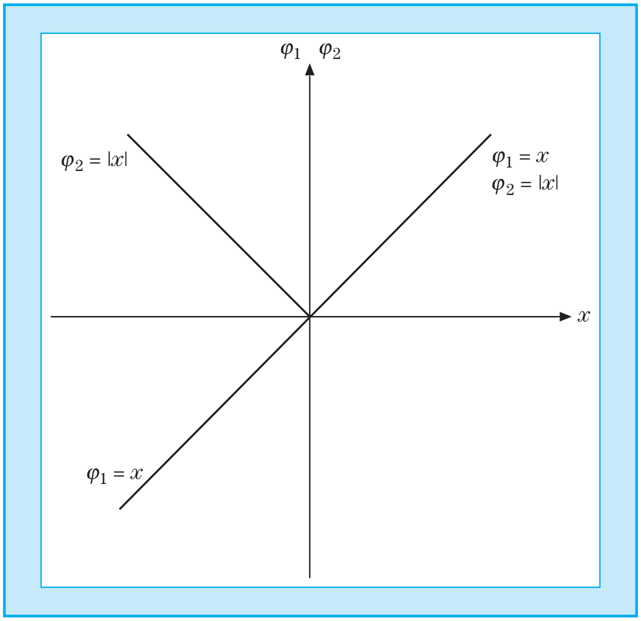
8.3.8 Explain that **linear independence** does not mean the absence of any dependence. Illustrate your argument with $y_1 = \cosh x$ and $y_2 = e^x$.

8.3.9 Find and plot the solution of the ODE satisfying the given initial conditions:
1. $y'' + 3y' - 4y = 0$ with $y(0) = 1$, $y'(0) = 0$,
2. $y'' + 2y' - 3y = 0$ with $y(0) = 0$, $y'(0) = 1$,
3. $y'' + 2y' + 3y = 0$ with $y(0) = 0$, $y'(0) = 1$.

8.3.10 Find the general solution of the ODEs in Exercise 8.3.9.

Figure 8.7

***x* and |*x*|**



8.3.11 Find and plot the solution of the ODE satisfying the given boundary conditions:
1. $y'' + 3y' - 4y = 0$ with $y(0) = 1$, $y(\infty) = 0$,
2. $y'' + 2y' - 3y = 0$ with $y(0) = 1$, $y(-\infty) = 0$,
3. $y'' + 4y' - 12y = 0$ with $y(0) = 1$, $y(1) = 2$.

8.3.12 Find and plot a particular solution of the inhomogeneous ODE
1. $y'' + 3y' - 4y = \sin \omega x$,
2. $y'' + 3y' - 4y = \cos \omega x$.

8.3.13 Find the general solution of the ODE $x^2 y'' + xy' - n^2 y = 0$ for integer n.

8.3.14 Solve the ODE $y'' + 9y = 0$ using the ansatz $y' = z(y)$ as the ODE does not contain the variable x. Compare your result with the standard solution of an ODE with constant coefficients.

8.3.15 Find and plot a particular solution of the following ODEs and give all details for the general solution of the corresponding homogeneous ODEs

1. $y'' + 3y = 2 \cos x - 3 \sin 2x$,
2. $y'' + 4y' + 20y = \sin x + \frac{1}{19} \cos x$,
3. $y'' + y' - 2y = e^x / x$.

8.3.16 The sun moves along the x-axis with constant velocity $c \neq 0$. A planet moves around it so that its velocity is always perpendicular to the radius vector from the sun to the planet, but no other force is acting (i.e., no gravitational force). Show that Kepler's area law is valid and the planet's orbit is an ellipse with the sun in a focus.

8.3.17 A small massive sphere is elastically coupled to the origin moving in a straight line through the origin in a massless glass tube that rotates at a constant angular velocity ω around the origin. Describe the orbit of the mass if it is at $r = a$, $\dot{r} = 0$ at time $t = 0$.

ANS. The rosetta curve $r = a\cos N\varphi$, $N = \sqrt{(\omega_0/\omega)^2 - 1}$ for $\omega_0^2 = k/m > \omega^2$; for $\omega_0 = \omega$ a circle, and for $\omega_0 < \omega$ a hyperbolic cosine spiral $r = a\cosh n\varphi$, $n = \sqrt{(1 - (\omega_0/\omega)^2}$.

8.3.18 A charged particle of mass m and charge e is moving in a constant electric field in the positive x-direction and a constant magnetic field in the positive z-direction. At time $t = 0$ the particle is located at the origin with velocity v in the y-direction. Determine the motion $\mathbf{r}(t)$ and orbits for the cases $B = 0$, $E \neq 0$; $E = 0$, $B \neq 0$; $E \neq 0$, $B \neq 0$, $v = 0$; $E \neq 0$, $B \neq 0$, $v \neq 0$. Plot the orbits.

8.3.19 Two small masses m_1, m_2 are suspended at the ends of a rope of constant length L over a pulley. Find their motion $z_i(t)$ under the influence of the constant gravitational acceleration $g = 9.8$ m/sec^2. Discuss various initial conditions.

8.3.20 Find the general solution of the ODE $x^2y'' - 4xy' + 6y = 14x^{-4}$, showing all steps of your calculations.

8.3.21 Find the steady-state current of the RLC circuit in Example 8.3.6 for $R = 7\,\Omega$, $L = 10$ H, $C = 10^{-4}$ F, $V = 220\sin 60t$ V.

8.3.22 Find the transient current for Exercise 8.3.21.

8.4 Singular Points

In this section, the concept of a singular point or singularity (as applied to a differential equation) is introduced. The interest in this concept stems from its usefulness in (i) classifying ODEs and (ii) investigating the feasibility of a series solution. This feasibility is the topic of Fuchs's theorem (Section 8.5). First, we give a definition of ordinary and singular points of ODEs.

All the ODEs listed in Sections 8.2 and 8.3 may be solved for d^2y/dx^2. We have

$$y'' = f(x, y, y'). \tag{8.36}$$

Now, if in Eq. (8.36), y and y' can take on all finite values at $x = x_0$ and y'' remains finite, point $x = x_0$ is an **ordinary** point. On the other hand, if y'' **becomes infinite** for any finite choice of y and y', point $x = x_0$ is labeled a **singular point**. We need to understand if the solution $y(x_0)$ is still well defined at such a point.

Another way of presenting this definition of a singular point is to write our second-order, homogeneous, linear differential equation (in y) as

$$y'' + P(x)y' + Q(x)y = 0. \tag{8.37}$$

Now, if the functions $P(x)$ and $Q(x)$ remain finite at $x = x_0$, point $x = x_0$ is an **ordinary point**. However, if $P(x)$ or $Q(x)$ (or both) diverges as $x \to x_0$, point x_0 is a **singular point**. Using Eq. (8.37), we may distinguish between two kinds of singular points.

1. If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$ but $(x-x_0)P(x)$ and $(x-x_0)^2 Q(x)$ remain finite as $x \to x_0$, then $x = x_0$ is a **regular** or nonessential singular point. We shall see that a power series solution is possible at ordinary points and regular singularities.
2. If $P(x)$ diverges faster than $1/(x-x_0)$ so that $(x-x_0)P(x)$ goes to infinity as $x \to x_0$, or $Q(x)$ diverges faster than $1/(x-x_0)^2$ so that $(x-x_0)^2 Q(x)$ goes to infinity as $x \to x_0$, then point $x = x_0$ is an **irregular or essential singularity**. We shall see that at such essential singularities a solution usually does not exist.

These definitions hold for all finite values of x_0. The analysis of point $x \to \infty$ is similar to the treatment of functions of a complex variable (Chapters 6 and 7). We set $x = 1/z$, substitute into the differential equation, and then let $z \to 0$. By changing variables in the derivatives, we have

$$\frac{dy(x)}{dx} = \frac{dy(z^{-1})}{dz}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy(z^{-1})}{dz} = -z^2\frac{dy(z^{-1})}{dz} \tag{8.38}$$

$$\begin{aligned}\frac{d^2y(x)}{dx^2} &= \frac{d}{dz}\left[\frac{dy(x)}{dx}\right]\frac{dz}{dx} = (-z^2)\left[-2z\frac{dy(z^{-1})}{dz} - z^2\frac{d^2y(z^{-1})}{dz^2}\right]\\ &= 2z^3\frac{dy(z^{-1})}{dz} + z^4\frac{d^2y(z^{-1})}{dz^2}.\end{aligned} \tag{8.39}$$

Using these results, we transform Eq. (8.37) into

$$z^4\frac{d^2y}{dz^2} + [2z^3 - z^2P(z^{-1})]\frac{dy}{dz} + Q(z^{-1})y = 0. \tag{8.40}$$

The behavior at $x = \infty\,(z = 0)$ then depends on the behavior of the new coefficients

$$\frac{2z - P(z^{-1})}{z^2} \quad \text{and} \quad \frac{Q(z^{-1})}{z^4},$$

as $z \to 0$. If these two expressions remain finite, point $x = \infty$ is an ordinary point. If they diverge no more rapidly than $1/z$ and $1/z^2$, respectively, point $x = \infty$ is a **regular singular point**; otherwise, it is an **irregular singular point** (an essential singularity).

EXAMPLE 8.4.1

Bessel Singularity Bessel's equation is

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{8.41}$$

Comparing it with Eq. (8.37) we have

$$P(x) = \frac{1}{x}, \quad Q(x) = 1 - \frac{n^2}{x^2},$$

which shows that point $x = 0$ is a regular singularity. By inspection we see that there are no other singular points in the finite range. As $x \to \infty$ ($z \to 0$), from Eq. (8.41) we have the coefficients

$$\frac{2z - z}{z^2} \quad \text{and} \quad \frac{1 - n^2z^2}{z^4}.$$

Since the latter expression diverges as z^4, point $x = \infty$ is an irregular or essential singularity.

More examples of ODEs with regular and irregular singularities are discussed in Section 8.5. ■

EXERCISES

8.4.1 Show that Legendre's equation has regular singularities at $x = -1$, 1, and ∞.

8.4.2 Show that Laguerre's equation, like the Bessel equation, has a regular singularity at $x = 0$ and an irregular singularity at $x = \infty$.

8.4.3 Show that the substitution

$$x \to \frac{1-x}{2}, \quad a = -l, \quad b = l + 1, \quad c = 1$$

converts the hypergeometric equation into Legendre's equation.

8.5 Series Solutions—Frobenius's Method

In this section, we develop a method of obtaining one solution of the linear, second-order, homogeneous ODE. The method, a power series expansion, will always work, provided the point of expansion is no worse than a regular singular point, a gentle condition that is almost always satisfied in physics.

A **linear, second-order, homogeneous** ODE may be put in the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0. \tag{8.42}$$

The equation is **homogeneous** because each term contains $y(x)$ or a derivative, and it is **linear** because each y, dy/dx, or d^2y/dx^2 appears as the first power—and no products.

Here, we develop (at least) one solution of Eq. (8.42). In Section 8.6, we develop the **second, independent solution. We have proved that no third, independent solution exists**. Therefore, the **most general solution of the homogeneous ODE**, Eq. (8.42), may be written as

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) \tag{8.43}$$

as a consequence of the superposition principle for linear ODEs. Our physical problem may involve a driving term and lead to a **nonhomogeneous**, linear, second-order ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x). \tag{8.44}$$

The function on the right, $F(x)$, represents a source (such as electrostatic charge) or a driving force (as in a driven oscillator). These are also explored in detail in Chapter 15 with a Laplace transform technique. Calling this a particular solution y_p, we may add to it any solution of the corresponding homogeneous equation [Eq. (8.42)]. Hence, the **most general solution of the inhomogeneous ODE** [Eq. (8.44)] is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \tag{8.45}$$

The constants c_1 and c_2 will eventually be fixed by boundary or initial conditions.

For now, we assume that $F(x) = 0$—that our differential equation is homogeneous. We shall attempt to develop a solution of our linear, second-order, homogeneous differential equation [Eq. (8.42)] by substituting in a power series with undetermined coefficients. Also available as a parameter is the power of the lowest nonvanishing term of the series. To illustrate, we apply the method to two important differential equations. First, the linear oscillator equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \tag{8.46}$$

with known solutions $y = \sin \omega x$, $\cos \omega x$. Now we try

$$\begin{aligned} y(x) &= x^k(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_0 \neq 0, \end{aligned} \tag{8.47}$$

with the exponent k and all the coefficients a_λ still undetermined. Note that k need not be an integer. By differentiating twice, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1}, \\ \frac{d^2y}{dx^2} &= \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2}. \end{aligned}$$

By substituting the series for y and y'' into the ODE [Eq. (8.46)], we have

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0. \tag{8.48}$$

From our analysis of the uniqueness of power series (Chapter 5) the coefficients of each power of x on the left-hand side of Eq. (8.48) must vanish individually.

The lowest power of x appearing in Eq. (8.48) is x^{k-2}, for $\lambda = 0$ in the first summation. The requirement that the coefficient vanish[4] yields

$$a_0 k(k-1) = 0.$$

We had chosen a_0 as the coefficient of the lowest nonvanishing term of the series [Eq. (8.48)]; hence, by definition, $a_0 \neq 0$. Therefore, we have

$$k(k-1) = 0. \tag{8.49}$$

This equation, coming from the coefficient of the lowest power of x, we call the **indicial equation**. The indicial equation and its roots are of critical importance to our analysis. The coefficient $a_1(k+1)k$ of x^{k-1} must also vanish. This is satisfied if $k = 0$; if $k = 1$, then $a_1 = 0$. Clearly, in this example we must require that either $k = 0$ or $k = 1$.

Before considering these two possibilities for k, we return to Eq. (8.48) and demand that the remaining coefficients, viz., the coefficient of x^{k+j} $(j \geq 0)$, vanish. We set $\lambda = j+2$ in the first summation and $\lambda = j$ in the second. (They are independent summations and λ is a dummy index.) This results in

$$a_{j+2}(k+j+2)(k+j+1) + \omega^2 a_j = 0$$

or

$$a_{j+2} = -a_j \frac{\omega^2}{(k+j+2)(k+j+1)}. \tag{8.50}$$

This is a two-term **recurrence relation**.[5] Given a_j, we may compute a_{j+2} and then a_{j+4}, a_{j+6}, and so on as far as desired. Note that for this example, if we start with a_0, Eq. (8.50) leads to the even coefficients a_2, a_4, and so on and ignores a_1, a_3, a_5, and so on. Since a_1 is arbitrary if $k = 0$ and necessarily zero if $k = 1$, let us set it equal to zero (compare Exercises 8.5.3 and 8.5.4) and then by Eq. (8.50)

$$a_3 = a_5 = a_7 = \cdots = 0,$$

and all the odd-numbered coefficients vanish. The odd powers of x will actually reappear when the **second** root of the indicial equation is used.

[4]See the uniqueness of power series (Section 5.7).

[5]Recurrence relations may involve three or more terms; that is, a_{j+2}, depending on a_j and a_{j-2}, etc. An unusual feature is that it goes in steps of two rather than the more common steps of one. This feature will be explained by a symmetry of the ODE called parity.

Returning to Eq. (8.49), our indicial equation, we first try the solution $k = 0$. The recurrence relation [Eq. (8.50)] becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+2)(j+1)}, \tag{8.51}$$

which leads to

$$\begin{aligned} a_2 &= -a_0 \frac{\omega^2}{1 \cdot 2} = -\frac{\omega^2}{2!} a_0, \\ a_4 &= -a_2 \frac{\omega^2}{3 \cdot 4} = +\frac{\omega^4}{4!} a_0, \\ a_6 &= -a_4 \frac{\omega^2}{5 \cdot 6} = -\frac{\omega^6}{6!} a_0, \quad \text{and so on.} \end{aligned}$$

By inspection (or mathematical induction),

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0, \tag{8.52}$$

and our solution is

$$y(x)_{k=0} = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots \right] = a_0 \cos \omega x. \tag{8.53}$$

If we choose the indicial equation root $k = 1$ [Eq. (8.49)], the recurrence relation becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)}. \tag{8.54}$$

Substituting in $j = 0, 2, 4$, successively, we obtain

$$\begin{aligned} a_2 &= -a_0 \frac{\omega^2}{2 \cdot 3} = -\frac{\omega^2}{3!} a_0, \\ a_4 &= -a_2 \frac{\omega^2}{4 \cdot 5} = +\frac{\omega^4}{5!} a_0, \\ a_6 &= -a_4 \frac{\omega^2}{6 \cdot 7} = -\frac{\omega^6}{7!} a_0, \quad \text{and so on.} \end{aligned}$$

Again, by inspection and mathematical induction,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0. \tag{8.55}$$

For this choice, $k = 1$, we obtain

$$\begin{aligned} y(x)_{k=1} &= a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots \right] \\ &= \frac{a_0}{\omega} \left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots \right] \\ &= \frac{a_0}{\omega} \sin \omega x. \end{aligned} \tag{8.56}$$

Figure 8.8

Schematic Power Series

$$\overset{\mathrm{I}}{\underbrace{\boxed{a_0k(k-1)}}_{=0}}\,x^{k-2} + \overset{\mathrm{II}}{\underbrace{\boxed{a_1(k+1)k}}_{=0}}\,x^{k-1} + \overset{\mathrm{III}}{\underbrace{\boxed{\begin{matrix}a_2(k+2)(k+1)\,x^k \\ +\,a_0\omega^2\,x^k\end{matrix}}}_{=0}} + \overset{\mathrm{IV}}{\underbrace{\boxed{\begin{matrix}a_3(k+3)(k+2)\,x^{k+1} \\ +\,a_1\omega^2\,x^{k+1}\end{matrix}}}_{=0}} + \cdots = 0$$

SUMMARY

To summarize this power series approach, we may write Eq. (8.48) schematically as shown in Fig. 8.8. From the uniqueness of power series (Section 5.7), the total coefficient of each power of x must vanish all by itself. The requirement that the first coefficient vanish leads to the indicial equation [Eq. (8.49)]. The second coefficient is handled by setting $a_1 = 0$. The vanishing of the coefficient of x^k (and higher powers, taken one at a time) leads to the recurrence relation [Eq. (8.50)].

This series substitution, known as Frobenius's method, has given us two series solutions of the linear oscillator equation. However, there are two points about such series solutions that must be strongly emphasized:

- The series solution should always be substituted back into the differential equation, to see if it works, as a precaution against algebraic and logical errors. If it works, it is a solution.
- The acceptability of a series solution depends on its convergence (including asymptotic convergence). It is quite possible for Frobenius's method to give a series solution that satisfies the original differential equation, when substituted in the equation, but that does **not** converge over the region of interest.

Expansion about x_0

Equation (8.47) is an expansion about the origin, $x_0 = 0$. It is perfectly possible to replace Eq. (8.47) with

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{k+\lambda}, \qquad a_0 \neq 0. \tag{8.57}$$

The point x_0 should not be chosen at an essential singularity—or our Frobenius method will probably fail. The resultant series (x_0 an ordinary point or regular singular point) will be valid where it converges. You can expect a divergence of some sort when $|x - x_0| = |z_s - x_0|$, where z_s is the closest singularity to x_0 in the complex plane.

Symmetry of ODE and Solutions

Note that for the ODE in Eq. (8.46) we obtained one solution of **even symmetry**, defined as $y_1(x) = y_1(-x)$, and one of **odd symmetry**, defined as $y_2(x) = -y_2(-x)$. This is not just an accident but a direct consequence of the form of the ODE. Writing a general ODE as

$$\mathcal{L}(x)y(x) = 0, \tag{8.58}$$

where $\mathcal{L}(x)$ is the differential operator, we see that for the linear oscillator equation [Eq. (8.46)], upon reversing the coordinate $x \to -x$ (defined as parity transformation),

$$\mathcal{L}(x) = \mathcal{L}(-x) \tag{8.59}$$

is **even under parity**. Whenever the differential operator has a specific parity or symmetry, either even or odd, we may interchange $+x$ and $-x$, and Eq. (8.58) becomes

$$\pm\mathcal{L}(x)y(-x) = 0. \tag{8.60}$$

It is + if $\mathcal{L}(x)$ is even and − if $\mathcal{L}(x)$ is odd. Clearly, if $y(x)$ is a solution of the differential equation, $y(-x)$ is also a solution. Then any solution may be resolved into even and odd parts,

$$y(x) = \frac{1}{2}[y(x) + y(-x)] + \frac{1}{2}[y(x) - y(-x)], \tag{8.61}$$

the first bracket on the right giving an even solution and the second an odd solution. Such a combination of solutions of definite parity has no definite parity.

Many other ODEs of importance in physics exhibit this even parity; that is, their $P(x)$ in Eq. (8.42) is odd and $Q(x)$ even. Solutions of all of them may be presented as series of even powers of x and separate series of odd powers of x. Parity is particularly important in quantum mechanics. We find that wave functions are usually either even or odd, meaning that they have a definite parity. For example, the Coulomb potential in the Schrödinger equation for hydrogen has positive parity. As a result, its solutions have definite parity.

Limitations of Series Approach—Bessel's Equation

The power series solution for the linear oscillator equation was perhaps a bit too easy. By substituting the power series [Eq. (8.47)] into the differential equation [Eq. (8.46)], we obtained two independent solutions with no trouble at all.

To get some idea of what can happen, we try to solve Bessel's equation,

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{8.62}$$

Again, assuming a solution of the form

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda},$$

we differentiate and substitute into Eq. (8.62). The result is

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2} - \sum_{\lambda=0}^{\infty} a_\lambda n^2 x^{k+\lambda} = 0. \tag{8.63}$$

By setting $\lambda = 0$, we get the coefficient of x^k, the lowest power of x appearing on the left-hand side,

$$a_0[k(k-1) + k - n^2] = 0, \tag{8.64}$$

and again $a_0 \neq 0$ by definition. Equation (8.64) therefore yields the **indicial** equation

$$k^2 - n^2 = 0, \tag{8.65}$$

with solutions $k = \pm n$.

It is of interest to examine the coefficient of x^{k+1}. Here, we obtain

$$a_1[(k+1)k + k + 1 - n^2] = 0$$

or

$$a_1(k+1-n)(k+1+n) = 0. \tag{8.66}$$

For $k = \pm n$, neither $k+1-n$ nor $k+1+n$ vanishes and we **must** require $a_1 = 0$.[6]

Proceeding to the coefficient of x^{k+j} for $k = n$, we set $\lambda = j$ in the first, second, and fourth terms of Eq. (8.63) and $\lambda = j-2$ in the third term. By requiring the resultant coefficient of x^{k+1} to vanish, we obtain

$$a_j[(n+j)(n+j-1) + (n+j) - n^2] + a_{j-2} = 0.$$

When j is replaced by $j+2$, this can be rewritten for $j \geq 0$ as

$$a_{j+2} = -a_j \frac{1}{(j+2)(2n+j+2)}, \tag{8.67}$$

which is the desired recurrence relation. Repeated application of this recurrence relation leads to

$$a_2 = -a_0 \frac{1}{2(2n+2)} = -\frac{a_0 n!}{2^2 1!(n+1)!},$$

$$a_4 = -a_2 \frac{1}{4(2n+4)} = \frac{a_0 n!}{2^4 2!(n+2)!},$$

$$a_6 = -a_4 \frac{1}{6(2n+6)} = -\frac{a_0 n!}{2^6 3!(n+3)!}, \quad \text{and so on,}$$

[6] $k = \pm n = -\frac{1}{2}$ are exceptions.

and in general,

$$a_{2p} = (-1)^p \frac{a_0 n!}{2^{2p} p!(n+p)!}. \tag{8.68}$$

Inserting these coefficients in our assumed series solution, we have

$$y(x) = a_0 x^n \left[1 - \frac{n! x^2}{2^2 1!(n+1)!} + \frac{n! x^4}{2^4 2!(n+2)!} - \cdots \right]. \tag{8.69}$$

In summation form

$$\begin{aligned} y(x) &= a_0 \sum_{j=0}^{\infty} (-1)^j \frac{n! x^{n+2j}}{2^{2j} j!(n+j)!} \\ &= a_0 2^n n! \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(n+j)!} \left(\frac{x}{2} \right)^{n+2j}. \end{aligned} \tag{8.70}$$

In Chapter 12, the final summation is identified as the Bessel function $J_n(x)$. Notice that this solution $J_n(x)$ has either even or odd symmetry,[7] as might be expected from the form of Bessel's equation.

When $k = -n$, and n is not an integer, we may generate a second distinct series to be labeled $J_{-n}(x)$. However, when $-n$ is a negative integer, trouble develops. The recurrence relation for the coefficients a_j is still given by Eq. (8.67), but with $2n$ replaced by $-2n$. Then, when $j + 2 = 2n$ or $j = 2(n-1)$, the coefficient a_{j+2} blows up and we have no series solution. This catastrophe can be remedied in Eq. (8.70), as it is done in Chapter 12, with the result that

$$J_{-n}(x) = (-1)^n J_n(x), \qquad n \text{ an integer}. \tag{8.71}$$

The second solution simply reproduces the first. We have failed to construct a second independent solution for Bessel's equation by this series technique when n is an integer.

SUMMARY

By substituting in an infinite series, we have obtained two solutions for the linear oscillator equation and one for Bessel's equation (two if n is not an integer). To the questions "Can we always do this? Will this method always work?" the answer is no. This method of power series solution will not always work, as we explain next.

Biographical Data

Frobenius, Georg. Frobenius, a German mathematician (1849–1917), contributed to matrices, groups, and algebra as well as differential equations.

Regular and Irregular Singularities

The success of the series substitution method depends on the roots of the indicial equation and the degree of singularity of the coefficients in the differential equation. To understand better the effect of the equation coefficients on this

[7] $J_n(x)$ is an even function if n is an even integer, and it is an odd function if n is an odd integer. For nonintegral n the x^n has no such simple symmetry.

naive series substitution approach, consider four simple equations:

$$y'' - \frac{6}{x^2}y = 0, \tag{8.72a}$$

$$y'' - \frac{6}{x^3}y = 0, \tag{8.72b}$$

$$y'' + \frac{1}{x}y' - \frac{a^2}{x^2}y = 0, \tag{8.72c}$$

$$y'' + \frac{1}{x^2}y' - \frac{a^2}{x^2}y = 0. \tag{8.72d}$$

You may show that for Eq. (8.72a) the indicial equation is

$$k^2 - k - 6 = 0,$$

giving $k = 3, -2$. Since the equation is homogeneous in x (counting d^2/dx^2 as x^{-2}), there is no recurrence relation. However, we are left with two perfectly good solutions, x^3 and x^{-2}.

Equation (8.72b) differs from Eq. (8.72a) by only one power of x, but this changes the indicial equation to

$$-6a_0 = 0,$$

with no solution at all because we have agreed that $a_0 \neq 0$. Our series substitution worked for Eq. (8.72a), which had only a regular singularity, but broke down in Eq. (8.72b), which has an **irregular singular point** at the origin.

Continuing with Eq. (8.72c), we have added a term y'/x. The indicial equation is

$$k^2 - a^2 = 0,$$

but again there is no recurrence relation. The solutions are $y = x^a, x^{-a}$—both perfectly acceptable one-term series. Despite the regular singularity at the origin, two independent solutions exist in this case.

When we change the power of x in the coefficient of y' from -1 to -2 [Eq. (8.72d)], there is a drastic change in the solution. The indicial equation (with only the y' term contributing) becomes

$$k = 0.$$

There is a recurrence relation

$$a_{j+1} = +a_j \frac{a^2 - j(j-1)}{j+1}.$$

Unless the parameter a is selected to make the series terminate, we have

$$\lim_{j\to\infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j\to\infty} \frac{j(j+1)}{j+1} = \lim_{j\to\infty} \frac{j^2}{j} = \infty.$$

Hence, our series solution diverges for all $x \neq 0$. Again, our method worked for Eq. (8.72c) with a regular singularity but failed when we had the irregular singularity of Eq. (8.72d).

Fuchs's Theorem

The answer to the basic question as to when the method of series substitution can be expected to work is given by Fuchs's theorem, which asserts that **we can always obtain at least one power series solution, provided we are expanding about a point that is an ordinary point or at worst a regular singular point**.

If we attempt an expansion about an irregular or essential singularity, our method may fail as it did for Eqs. (8.72b) and (8.72d). Fortunately, the more important equations of mathematical physics have no irregular singularities in the finite plane. Further discussion of Fuchs's theorem appears in Section 8.6.

Summary

If we are expanding about an ordinary point or, at worst, about a regular singularity, the series substitution approach will yield at least one solution (**Fuchs's theorem**).

Whether we get one or two distinct solutions depends on the roots of the indicial equation:

- If the two roots of the indicial equation are equal, we can obtain only one solution by this series substitution method.
- If the two roots differ by a nonintegral number, two independent solutions may be obtained.
- If the two roots differ by an integer, the larger of the two will yield a solution.

The smaller may or may not give a solution, depending on the behavior of the coefficients. In the linear oscillator equation we obtain two solutions; for Bessel's equation, only one solution is obtained.

The usefulness of the series solution in terms of what is the solution (i.e., numbers) depends on the rapidity of convergence of the series and the availability of the coefficients. Many ODEs will not yield simple recurrence relations for the coefficients. In general, the available series will probably be useful when $|x|$ (or $|x-x_0|$) is very small. Computers can be used to determine additional series coefficients using a symbolic language, such as Mathematica,[8] Maple,[9] or Reduce.[10] Often, however, for numerical work a direct numerical integration will be preferred (Section 8.7).

[8]Wolfram, S. (1991). *Mathematica, A System for Doing Mathematics by Computer*. Addison Wesley, New York.
[9]Heck, A. (1993). *Introduction to Maple*. Springer, New York.
[10]Rayna, G. (1987). *Reduce Software for Algebraic Computation*. Springer, New York.

EXERCISES

8.5.1 Uniqueness theorem. The function $y(x)$ satisfies a second-order, linear, homogeneous differential equation. At $x = x_0$, $y(x) = y_0$ and $dy/dx = y_0'$. Show that $y(x)$ is unique in that no other solution of this differential equation passes through the points (x_0, y_0) with a slope of y_0'.
Hint. Assume a second solution satisfying these conditions and compare the Taylor series expansions.

8.5.2 A series solution of Eq. (8.47) is attempted, expanding about the point $x = x_0$. If x_0 is an ordinary point, show that the indicial equation has roots $k = 0, 1$.

8.5.3 In the development of a series solution of the simple harmonic oscillator (SHO) equation the second series coefficient a_1 was neglected except to set it equal to zero. From the coefficient of the next to the lowest power of x, x^{k-1}, develop a second indicial-type equation.
(a) SHO equation with $k = 0$: Show that a_1 may be assigned any finite value (including zero).
(b) SHO equation with $k = 1$: Show that a_1 must be set equal to zero.

8.5.4 Analyze the series solutions of the following differential equations to see when a_1 **may** be set equal to zero without irrevocably losing anything and when a_1 **must** be set equal to zero.
(a) Legendre, (b) Bessel, (c) Hermite.

ANS. (a) Legendre and (c) Hermite: For $k = 0$, a_1 **may** be set equal to zero; for $k = 1$, a_1 **must** be set equal to zero.
(b) Bessel: a_1 **must** be set equal to zero (except for $k = \pm n = -\frac{1}{2}$).

8.5.5 Solve the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

by direct series substitution and plot the solution for $n = 0, 1, 2, 3$.
(a) Verify that the indicial equation is

$$k(k - 1) = 0.$$

(b) Using $k = 0$, obtain a series of even powers of x, $(a_1 = 0)$.

$$y_{\text{even}} = a_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 + \cdots \right],$$

where

$$a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)} a_j.$$

(c) Using $k = 1$, develop a series of odd powers of $x(a_1 = 0)$.

$$y_{\text{odd}} = a_0 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 + \cdots \right],$$

where

$$a_{j+2} = \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)} a_j.$$

(d) Show that both solutions, y_{even} and y_{odd}, diverge for $x = \pm 1$ **if the series continue to infinity**.

(e) Finally, show that by an appropriate choice of n, one series at a time may be converted into a polynomial, thereby avoiding the divergence catastrophe. In quantum mechanics this restriction of n to integral values corresponds to **quantization of angular momentum**.

8.5.6 Develop series solutions for Hermite's differential equation

(a) $y'' - 2xy' + 2\alpha y = 0$.

ANS. $k(k-1) = 0$, indicial equation.

For $k = 0$

$$a_{j+2} = 2a_j \frac{j - \alpha}{(j+1)(j+2)} \quad (j \text{ even}),$$

$$y_{\text{even}} = a_0 \left[1 + \frac{2(-\alpha)x^2}{2!} + \frac{2^2(-\alpha)(2-\alpha)x^4}{4!} + \cdots \right].$$

For $k = 1$

$$a_{j+2} = 2a_j \frac{j + 1 - \alpha}{(j+2)(j+3)} \quad (j \text{ even}),$$

$$y_{\text{even}} = a_0 \left[x + \frac{2(1-\alpha)x^3}{3!} + \frac{2^2(1-\alpha)(3-\alpha)x^5}{5!} + \cdots \right].$$

(b) Show that both series solutions are convergent for all x, the ratio of successive coefficients behaving, for large index, like the corresponding ratio in the expansion of $\exp(2x^2)$.

(c) Show that by appropriate choice of α the series solutions may be cut off and converted to finite polynomials. (These polynomials, properly normalized, become the Hermite polynomials in Section 13.1.)

8.5.7 Laguerre's ODE is

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0.$$

Develop a series solution selecting the parameter n to make your series a polynomial and plot the partial series for the three lowest values of n and enough terms to demonstrate convergence.

8.5.8 A quantum mechanical analysis of the Stark effect (parabolic coordinates) leads to the differential equation

$$\frac{d}{d\xi}\left(\xi\frac{du}{d\xi}\right)+\left(\frac{1}{2}E\xi+\alpha-\frac{m^2}{4\xi}-\frac{1}{4}F\xi^2\right)u=0,$$

where α is a separation constant, E is the total energy, and F is a constant; Fz is the potential energy added to the system by the introduction of an electric field.

Using the larger root of the indicial equation, develop a power series solution about $\xi = 0$. Evaluate the first three coefficients in terms of a_0, the lowest coefficient in the power series for $u(\xi)$ below.

$$\text{Indicial equation} \quad k^2-\frac{m^2}{4}=0,$$

$$u(\xi) = a_0\xi^{m/2}\left\{1-\frac{\alpha}{m+1}\xi + \left[\frac{\alpha^2}{2(m+1)(m+2)}-\frac{E}{4(m+2)}\right]\xi^2+\cdots\right\}.$$

Note that the perturbation F does not appear until a_3 is included.

8.5.9 For the special case of no azimuthal dependence, the quantum mechanical analysis of the hydrogen molecular ion leads to the equation

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{du}{d\eta}\right]+\alpha u+\beta\eta^2 u=0.$$

Develop a power series solution for $u(\eta)$. Evaluate the first three nonvanishing coefficients in terms of a_0.

$$\text{Indicial equation} \quad k(k-1)=0.$$

$$u_{k=1}=a_0\eta\left\{1+\frac{2-\alpha}{6}\eta^2+\left[\frac{(2-\alpha)(12-\alpha)}{120}-\frac{\beta}{20}\right]\eta^4+\cdots\right\}.$$

8.5.10 To a good approximation, the interaction of two nucleons may be described by a mesonic potential

$$V=\frac{Ae^{-ax}}{x},$$

attractive for A negative. Develop a series solution of the resultant Schrödinger wave equation

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+(E-V)\psi=0$$

through the first three nonvanishing coefficients:

$$\psi_{k=1}=a_0\left\{x+\frac{1}{2}A'x^2+\frac{1}{6}\left[\frac{1}{2}A'^2-E'-aA'\right]x^3+\cdots\right\},$$

where the prime indicates multiplication by $2m/\hbar^2$. Plot the solution for $a = 0.7\ \text{fm}^{-1}$ and $A = -0.1$.

8.5.11 Near the nucleus of a complex atom the potential energy of one electron is given by

$$V = -\frac{Ze^2}{r}(1 + b_1 r + b_2 r^2),$$

where the coefficients b_1 and b_2 arise from screening effects. For the case of zero angular momentum determine the first three terms of the solution of the Schrödinger equation; that is, write out the first three terms in a series expansion of the wave function. Plot the potential and wave function.

8.5.12 If the parameter a^2 in Eq. (8.72d) is equal to 2, Eq. (8.72d) becomes

$$y'' + \frac{1}{x^2}y' - \frac{2}{x^2}y = 0.$$

From the indicial equation and the recurrence relation, **derive** a solution $y = 1 + 2x + 2x^2$. Verify that this is indeed a solution by substituting back into the differential equation.

8.6 A Second Solution

In Section 8.5, a solution of a second-order homogeneous ODE was developed by substituting in a power series. By Fuchs's theorem this is possible, provided the power series is an expansion about an ordinary point or a nonessential singularity.[11] There is no guarantee that this approach will yield the two independent solutions we expect from a linear second-order ODE. Indeed, the technique gives only one solution for Bessel's equation (n an integer). In this section, we develop two methods of obtaining a second independent solution: an integral method and a power series containing a logarithmic term.

Returning to our linear, second-order, homogeneous ODE of the general form

$$y'' + P(x)y' + Q(x)y = 0, \tag{8.73}$$

let y_1 and y_2 be two independent solutions. Then the Wronskian, by definition, is

$$W = y_1 y_2' - y_1' y_2. \tag{8.74}$$

By differentiating the Wronskian, we obtain

$$\begin{aligned} W' &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\ &= y_1[-P(x)y_2' - Q(x)y_2] - y_2[-P(x)y_1' - Q(x)y_1] \\ &= -P(x)(y_1 y_2' - y_1' y_2). \end{aligned} \tag{8.75}$$

The expression in parentheses is just W, the Wronskian, and we have

$$W' = -P(x)W. \tag{8.76}$$

[11] This is why the classification of singularities in Section 8.4 is of vital importance.

In the special case that $P(x) = 0$, that is,

$$y'' + Q(x)y = 0, \tag{8.77}$$

the Wronskian

$$W = y_1 y_2' - y_1' y_2 = \text{constant}. \tag{8.78}$$

Since our original differential equation is homogeneous, we may multiply the solutions y_1 and y_2 by whatever constants we wish and arrange to have the Wronskian equal to unity (or -1). This case, $P(x) = 0$, appears more frequently than might be expected. Recall that ∇^2 in Cartesian coordinates contains no first derivative. Similarly, the radial dependence of $\nabla^2(r\psi)$ in spherical polar coordinates lacks a first derivative. Finally, every linear second-order differential equation can be transformed into an equation of the form of Eq. (8.77) (compare Exercise 8.6.3).

For the general case, let us assume that we have one solution of Eq. (8.73) by a series substitution (or by guessing). We now proceed to develop a second, independent solution for which $W \neq 0$. Rewriting Eq. (8.76) as

$$\frac{dW}{W} = -P\,dx_1,$$

we integrate from $x_1 = a$ to $x_1 = x$ to obtain

$$\ln \frac{W(x)}{W(a)} = -\int_a^x P(x_1)dx_1$$

or[12]

$$W(x) = W(a)\exp\left[-\int_a^x P(x_1)dx_1\right]. \tag{8.79}$$

However,

$$W(x) = y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right). \tag{8.80}$$

By combining Eqs. (8.79) and (8.80), we have

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = W(a)\frac{\exp\left[-\int_a^x P(x_1)dx_1\right]}{y_1^2(x)}. \tag{8.81}$$

Finally, by integrating Eq. (8.81) from $x_2 = b$ to $x_2 = x$ we get

$$y_2(x) = y_1(x)W(a)\int_b^x \frac{\exp\left[-\int_a^{x_2} P(x_1)dx_1\right]}{[y_1(x_2)]^2}dx_2, \tag{8.82}$$

[12]If $P(x_1)$ remains finite, $a \leq x_1 \leq x$, $W(x) \neq 0$ unless $W(a) = 0$. That is, the Wronskian of our two solutions is either identically zero or never zero.

where a and b are arbitrary constants and a term $y_1(x)y_2(b)/y_1(b)$ has been dropped because it leads to nothing new. Since $W(a)$, the Wronskian evaluated at $x = a$, is a constant and our solutions for the homogeneous differential equation always contain an unknown normalizing factor, we set $W(a) = 1$ and write

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^{x_2} P(x_1)dx_1\right]}{[y_1(x_2)]^2} dx_2. \tag{8.83}$$

Note that the lower limits $x_1 = a$ and $x_2 = b$ have been omitted. If they are retained, they simply make a contribution equal to a constant times the known first solution, $y_1(x)$, and hence add nothing new.

If we have the important special case of $P(x) = 0$, Eq. (8.83) reduces to

$$y_2(x) = y_1(x) \int^x \frac{dx_2}{[y_1(x_2)]^2}. \tag{8.84}$$

This means that by using either Eq. (8.77) or Eq. (8.78), we can take one known solution and by integrating can generate a second independent solution of Eq. (8.73). As we shall see later, this technique to generate the second solution from the power series of the first solution $y_1(x)$ can be tedious.

EXAMPLE 8.6.1

A Second Solution for the Linear Oscillator Equation From $d^2y/dx^2 + y = 0$ with $P(x) = 0$, let one solution be $y_1 = \sin x$. By applying Eq. (8.84), we obtain

$$y_2(x) = \sin x \int^x \frac{dx_2}{\sin^2 x_2} = \sin x(-\cot x) = -\cos x,$$

which is clearly independent (not a linear multiple) of $\sin x$. ■

Series Form of the Second Solution

Further insight into the nature of the second solution of our differential equation may be obtained by the following sequence of operations:

1. Express $P(x)$ and $Q(x)$ in Eq. (8.77) as

$$P(x) = \sum_{i=-1}^{\infty} p_i x^i, \qquad Q(x) = \sum_{j=-2}^{\infty} q_j x^j. \tag{8.85}$$

   The lower limits of the summations are selected to create the strongest possible **regular** singularity (at the origin). These conditions just satisfy Fuchs's theorem and thus help us gain a better understanding of that theorem.
2. Develop the first few terms of a power series solution, as in Section 8.5.
3. Using this solution as y_1, obtain a second series-type solution, y_2, with Eq. (8.77), integrating term by term.

Proceeding with step 1, we have

$$y'' + (p_{-1}x^{-1} + p_0 + p_1 x + \cdots)y' + (q_{-2}x^{-2} + q_{-1}x^{-1} + \cdots)y = 0, \quad (8.86)$$

in which point $x = 0$ is at worst a regular singular point. If $p_{-1} = q_{-1} = q_{-2} = 0$, it reduces to an ordinary point. Substituting

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

(step 2), we obtain

$$\sum_{\lambda=0}^{\infty} (k+\lambda)(k+\lambda-1)a_\lambda k^{k+\lambda-2} + \sum_{i=-1}^{\infty} p_i x^i \sum_{\lambda=0}^{\infty} (k+\lambda)a_\lambda x^{k+\lambda-1}$$
$$+ \sum_{j=-2}^{\infty} q_j x^j \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0. \quad (8.87)$$

Our indicial equation is

$$k(k-1) + p_{-1}k + q_{-2} = 0,$$

which sets the coefficient of x^{k-2} equal to zero. This reduces to

$$k^2 + (p_{-1} - 1)k + q_{-2} = 0. \quad (8.88)$$

We denote the two roots of this indicial equation by $k = \alpha$ and $k = \alpha - n$, where n is zero or a positive integer. (If n is not an integer, we expect two independent series solutions by the methods of Section 8.6 and we are done.) Then

$$(k-\alpha)(k-\alpha+n) = 0$$

or

$$k^2 + (n - 2\alpha)k + \alpha(a - n) = 0, \quad (8.89)$$

and equating coefficients of k in Eqs. (8.88) and (8.89), we have

$$p_{-1} - 1 = n - 2\alpha. \quad (8.90)$$

The known series solution corresponding to the larger root $k = \alpha$ may be written as

$$y_1 = x^\alpha \sum_{\lambda=0}^{\infty} a_\lambda x^\lambda.$$

Substituting this series solution into Eq. (8.77) (step 3), we are faced with the formidable-looking expression,

$$y_2(x) = y_1(x) \int^x \frac{\exp\left(-\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i \, dx_1\right)}{x_2^{2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^2} dx_2, \quad (8.91)$$

where the solutions y_1 and y_2 have been normalized so that the Wronskian $W(a) = 1$. Handling the exponential factor first, we have

$$\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i \, dx_1 = p_{-1} \ln x_2 + \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + f(a), \tag{8.92}$$

where $f(a) = -p_{-1} \ln a$ is an integration constant from the $i = -1$ term that leads to an unimportant overall factor and can be dropped. Hence,

$$\begin{aligned} \exp\left(-\int_a^{x_2} \sum_i p_i x_1^i dx_1\right) &= \exp[-f(a)] x_2^{-p_{-1}} \exp\left(-\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1}\right) \\ &= \exp[-f(a)] x_2^{-p_{-1}} \left[1 - \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + \frac{1}{2!}\left(-\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1}\right)^2 + \cdots\right]. \end{aligned} \tag{8.93}$$

This final series expansion of the exponential is certainly convergent if the original expansion of the coefficient $P(x)$ was convergent.

The denominator in Eq. (8.91) may be handled by writing

$$\left[x_2^{2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^2\right]^{-1} = x_2^{-2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^{-2} = x_2^{-2\alpha} \sum_{\lambda=0}^{\infty} b_\lambda x_2^\lambda, \tag{8.94}$$

provided $a_0 \neq 0$. Neglecting constant factors that will be picked up anyway by the requirement that $W(a) = 1$, we obtain

$$y_2(x) = y_1(x) \int^x x_2^{-p_{-1}-2\alpha}\left(\sum_{\lambda=0}^{\infty} c_\lambda x_2^\lambda\right) dx_2. \tag{8.95}$$

By Eq. (8.90),

$$x_2^{-p_{-1}-2\alpha} = x_2^{-n-1}, \tag{8.96}$$

where $n \geq 0$ is an integer. Substituting this result into Eq. (8.95), we obtain

$$y_2(x) = y_1(x) \int^x \left(c_0 x_2^{-n-1} + c_1 x_2^{-n} + c_2 x_2^{-n+1} + \cdots + c_n x_2^{-1} + \cdots\right) dx_2. \tag{8.97}$$

The integration indicated in Eq. (8.97) leads to a coefficient of $y_1(x)$ consisting of two parts:

1. A power series starting with x^{-n}.
2. A logarithm term from the integration of x^{-1} (when $\lambda = n$). This term always appears when n is an integer **unless** c_n fortuitously happens to vanish.[13]

EXAMPLE 8.6.2

A Second Solution of Bessel's Equation From Bessel's equation, Eq. (8.62) [divided by x^2 to agree with Eq. (8.73)], we have

$$P(x) = x^{-1} \qquad Q(x) = 1 \qquad \text{for the case} \quad n = 0.$$

[13]For parity considerations, $\ln x$ is taken to be $\ln |x|$, even.

Hence, $p_{-1} = 1, q_0 = 1$ in Eq. (8.85); all other p_i and q_j vanish. The Bessel indicial equation is

$$k^2 = 0$$

[Eq. (8.65) with $n = 0$]. Hence, we verify Eqs. (8.88)–(8.90) with n and $\alpha = 0$.

Our first solution is available from Eq. (8.69). Relabeling it to agree with Chapter 12 (and using $a_0 = 1$), we obtain[14]

$$y_1(x) = J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - O(x^6), \tag{8.98a}$$

valid for all x because of the absolute convergence of the series. Now, substituting all this into Eq. (8.83), we have the specific case corresponding to Eq. (8.91):

$$y_2(x) = J_0(x) \int^x \frac{\exp\left[-\int^{x_2} x_1^{-1} dx_1\right]}{\left[1 - x_2^2/4 + x_2^4/64 - \cdots\right]^2} dx_2. \tag{8.98b}$$

From the numerator of the integrand

$$\exp\left[-\int^{x_2} \frac{dx_1}{x_1}\right] = \exp[-\ln x_2] = \frac{1}{x_2}.$$

This corresponds to the $x_2^{-p_{-1}}$ in Eq. (8.93). From the denominator of the integrand, using a binomial expansion, we obtain

$$\left[1 - \frac{x_2^2}{4} + \frac{x_2^4}{64}\right]^{-2} = 1 + \frac{x_2^2}{2} + \frac{5x_2^4}{32} + \cdots.$$

Corresponding to Eq. (8.85), we have

$$\begin{aligned} y_2(x) &= J_0(x) \int^x \frac{1}{x_2}\left[1 + \frac{x_2^2}{2} + \frac{5x_2^4}{32} + \cdots\right] dx_2 \\ &= J_0(x)\left\{\ln x + \frac{x^2}{4} + \frac{5x^4}{128} + \cdots\right\}. \end{aligned} \tag{8.98c}$$

Let us check this result. From Eqs. (12.60) and (12.62), which give the standard form of the second solution,

$$Y_0(x) = \frac{2}{\pi}[\ln x - \ln 2 + \gamma]J_0(x) + \frac{2}{\pi}\left\{\frac{x^2}{4} - \frac{3x^4}{128} + \cdots\right\}. \tag{8.98d}$$

Two points arise. First, since Bessel's equation is homogeneous, we may multiply $y_2(x)$ by any constant. To match $Y_0(x)$, we multiply our $y_2(x)$ by $2/\pi$. Second, to our second solution $(2/\pi)y_2(x)$, we may add any constant multiple

[14] The capital O (order of) as written here means terms proportional to x^6 and possibly higher powers of x.

of the first solution. Again, to match $Y_0(x)$ we add

$$\frac{2}{\pi}[-\ln 2 + \gamma]J_0(x),$$

where γ is the Euler–Mascheroni constant (Section 5.2).[15] Our new, modified second solution is

$$y_2(x) = \frac{2}{\pi}[\ln x - \ln 2 + \gamma]J_0(x) + \frac{2}{\pi}J_0(x)\left\{\frac{x^2}{4} - \frac{5x^4}{128} + \cdots\right\}. \tag{8.98e}$$

Now the comparison with $Y_0(x)$ becomes a simple multiplication of $J_0(x)$ from Eq. (8.98a) and the curly bracket of Eq. (8.98c). The multiplication checks, through terms of order x^2 and x^4, which is all we carried. Our second solution from Eqs. (8.83) and (8.91) agrees with the standard second solution, the Neumann function, $Y_0(x)$. ■

From the preceding analysis, the second solution of Eq. (8.83), $y_2(x)$, may be written as

$$y_2(x) = y_1(x)\ln x + \sum_{j=-n}^{\infty} d_j x^{j+\alpha}, \tag{8.98f}$$

the first solution times ln x and another power series, this one starting with $x^{\alpha-n}$, which means that we may look for a logarithmic term when the indicial equation of Section 8.5 gives only one series solution. With the form of the second solution specified by Eq. (8.98f), we can substitute Eq. (8.98f) into the original differential equation and determine the coefficients d_j exactly as in Section 8.5. It is worth noting that no series expansion of ln x is needed. In the substitution ln x will drop out; its derivatives will survive.

The second solution will usually diverge at the origin because of the logarithmic factor and the negative powers of x in the series. For this reason, $y_2(x)$ is often referred to as the **irregular solution**. The first series solution, $y_1(x)$, which usually converges at the origin, is the regular solution. The question of behavior at the origin is discussed in more detail in Chapters 11 and 12, in which we take up Legendre functions and Bessel functions.

SUMMARY

The two solutions of both sections (together with the exercises) provide a **complete solution** of our linear, homogeneous, second-order ODE—assuming that the point of expansion is no worse than a regular singularity. At least one solution can always be obtained by series substitution (Section 8.5). A **second, linearly independent solution** can be constructed by the **Wronskian** double integral [Eq. (8.83)]. **No third, linearly independent solution exists**.

The **inhomogeneous**, linear, second-order ODE has an **additional** solution: the **particular solution**. This particular solution may be obtained by the method of variation of parameters.

[15] The Neumann function Y_0 is defined as it is in order to achieve convenient asymptotic properties (Section 12.3).

EXERCISES

8.6.1 Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

has a regular solution $P_n(x)$ and an irregular solution $Q_n(x)$. Show that the Wronskian of P_n and Q_n is given by

$$P_n(x)Q_n'(x) - P_n'(x)Q_n(x) = \frac{A_n}{1 - x^2},$$

with A_n **independent** of x.

8.6.2 Show, by means of the Wronskian, that a linear, second-order, homogeneous ODE of the form

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

cannot have three independent solutions. (Assume a third solution and show that the Wronskian vanishes for all x.)

8.6.3 Transform our linear, second-order ODE

$$y'' + P(x)y' + Q(x)y = 0$$

by the substitution

$$y = z \exp\left[-\frac{1}{2} \int^x P(t)\,dt \right]$$

and show that the resulting differential equation for z is

$$z'' + q(x)z = 0,$$

where

$$q(x) = Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x).$$

8.6.4 Use the result of Exercise 8.6.3 to show that the replacement of $\varphi(r)$ by $r\varphi(r)$ may be expected to eliminate the first derivative from the Laplacian in spherical polar coordinates. See also Exercise 2.5.15(b).

8.6.5 By direct differentiation and substitution show that

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^s P(t)\,dt \right]}{[y_1(s)]^2}\,ds$$

satisfies

$$y_2''(x) + P(x)y_2'(x) + Q(x)y_2(x) = 0.$$

Note. The Leibniz formula for the derivative of an integral is

$$\frac{d}{d\alpha} \int_{g(\alpha)}^{h(\alpha)} f(x, \alpha)\,dx = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x, \alpha)}{\partial \alpha}\,dx + f[h(\alpha), \alpha]\frac{dh(\alpha)}{d\alpha} - f[g(\alpha), \alpha]\frac{dg(\alpha)}{d\alpha}.$$

8.6.6 In the equation

$$y_2(x) = y_1(x) \int^x \frac{\exp\left[-\int^s P(t)\,dt \right]}{[y_1(s)]^2}\, ds,$$

$y_1(x)$ satisfies

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0.$$

The function $y_2(x)$ is a linearly **independent** second solution of the same equation. Show that the inclusion of lower limits on the two integrals leads to nothing new; that is, it adds only overall factors and/or a multiple of the known solution $y_1(x)$.

8.6.7 Given that one solution of

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$$

is $R = r^m$, show that Eq. (8.83) predicts a second solution, $R = r^{-m}$.

8.6.8 Using $y_1(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ as a solution of the linear oscillator equation, follow the analysis culminating in Eq. (8.98f) and show that $c_1 = 0$ so that the second solution does not, in this case, contain a logarithmic term.

8.6.9 Show that when n is **not** an integer the second solution of Bessel's equation, obtained from Eq. (8.83), does **not** contain a logarithmic term.

8.6.10 (a) One solution of Hermite's differential equation

$$y'' - 2xy' + 2\alpha y = 0$$

for $\alpha = 0$ is $y_1(x) = 1$. Find a second solution $y_2(x)$ using Eq. (8.83). Show that your second solution is equivalent to y_{odd} (Exercise 8.5.6).

(b) Find a second solution for $\alpha = 1$, where $y_1(x) = x$, using Eq. (8.83). Show that your second solution is equivalent to y_{even} (Exercise 8.5.6).

8.6.11 One solution of Laguerre's differential equation

$$xy'' + (1 - x)y' + ny = 0$$

for $n = 0$ is $y_1(x) = 1$. Using Eq. (8.83), develop a second, linearly independent solution. Exhibit the logarithmic term explicitly.

8.6.12 For Laguerre's equation with $n = 0$

$$y_2(x) = \int^x \frac{e^s}{s}\, ds.$$

(a) Write $y_2(x)$ as a logarithm plus a power series.

(b) Verify that the integral form of $y_2(x)$, previously given, is a solution of Laguerre's equation ($n = 0$) by direct differentiation of the integral and substitution into the differential equation.

(c) Verify that the series form of $y_2(x)$, part (a), is a solution by differentiating the series and substituting back into Laguerre's equation.

8.6.13 The radial Schrödinger wave equation has the form

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + l(l+1)\frac{\hbar^2}{2mr^2} + V(r)\right] y(r) = Ey(r).$$

The potential energy $V(r)$ may be expanded about the origin as

$$V(r) = \frac{b_{-1}}{r} + b_0 + b_1 r + \cdots.$$

(a) Show that there is one (regular) solution starting with r^{l+1}.
(b) From Eq. (8.83) show that the irregular solution diverges at the origin as r^{-l}.

8.6.14 Show that if a second solution, y_2, is assumed to have the form $y_2(x) = y_1(x)f(x)$, substitution back into the original equation

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

leads to

$$f(x) = \int^x \frac{\exp\left[-\int^s P(t)\,dt\right]}{[y_1(s)]^2}\,ds,$$

in agreement with Eq. (8.83).

8.6.15 If our linear, second-order ODE is nonhomogeneous—that is, of the form of Eq. (8.44)—the **most general solution is**

$$y(x) = y_1(x) + y_2(x) + y_p(x).$$

(y_1 and y_2 are independent solutions of the homogeneous equation.) Show that

$$y_p(x) = y_2(x)\int^x \frac{y_1(s)F(s)\,ds}{W\{y_1(s),\,y_2(s)\}} - y_1(x)\int^x \frac{y_2(s)F(s)\,ds}{W\{y_1(s),\,y_2(s)\}},$$

where $W\{y_1(x),\,y_2(x)\}$ is the Wronskian of $y_1(s)$ and $y_2(s)$.
Hint. Let $y_p(x) = y_1(x)v(x)$ and develop a first-order ODE for $v'(x)$.

8.6.16 (a) Show that

$$y'' + \frac{1-\alpha^2}{4x^2}y = 0$$

has two solutions:

$$y_1(x) = a_0 x^{(1+\alpha)/2}$$

$$y_2(x) = a_0 x^{(1-\alpha)/2}.$$

(b) For $\alpha = 0$ the two linearly independent solutions of part (a) reduce to $y_{10} = a_0 x^{1/2}$. Using Eq. (8.83) derive a second solution

$$y_{20}(x) = a_0 x^{1/2}\ln x.$$

Verify that y_{20} is indeed a solution.

(c) Show that the second solution from part (b) may be obtained as a limiting case from the two solutions of part (a):

$$y_{20}(x) = \lim_{\alpha \to 0} \left(\frac{y_1 - y_2}{\alpha} \right).$$

8.7 Numerical Solutions

The analytic solutions and approximate solutions to differential equations in this chapter and in succeeding chapters may suffice to solve the problem at hand, particularly if there is some symmetry present. The power series solutions show how the solution behaves at small values of x. The asymptotic solutions show how the solution behaves at large values of x. These limiting cases and also the possible resemblance of our differential equation to the standard forms with known solutions (Chapters 11–13) are invaluable in helping us gain an understanding of the general behavior of our solution.

However, the usual situation is that we have a **different** equation, perhaps a different potential in the Schrödinger wave equation, and we want a reasonably exact solution. So we turn to numerical techniques.

First-Order Differential Equations

The differential equation involves a continuity of points. The independent variable x is continuous. The (unknown) dependent variable $y(x)$ is assumed continuous. The concept of differentiation demands continuity. Numerical processes replace these continua by discrete sets. We take x to have only specific values on a uniform grid, such as at

$$x_0, \qquad x_0 + h, \qquad x_0 + 2h, \qquad x_0 + 3h, \qquad \text{and so on},$$

where h is some small interval. The smaller h is, the better the approximation is in principle. However, if h is made too small, the demands on machine time will be excessive, and accuracy may actually decline because of accumulated round-off errors. In practice, therefore, one chooses a step size by trial and error or the code adapts the step size optimally that minimizes round-off errors. We refer to the successive discrete values of x as x_n, x_{n+1}, and so on, and the corresponding values of $y(x)$ as $y(x_n) = y_n$. If x_0 and y_0 are given, the problem is to find y_1, then to find y_2, and so on.

Taylor Series Solution

Consider the ordinary (possibly nonlinear) first-order differential equation

$$\frac{d}{dx} y(x) = f(x, y), \tag{8.99}$$

with the initial condition $y(x_0) = y_0$. In principle, a step-by-step solution of the first-order equation [Eq. (8.99)] may be developed to any degree of accuracy by a Taylor expansion

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \cdots + \frac{h^n}{n!}y^{(n)}(x_0) + \cdots, \quad (8.100)$$

assuming the derivatives exist and the series is convergent. The initial value $y(x_0)$ is known and $y'(x_0)$ is given as $f(x_0, y_0)$. In principle, the higher derivatives may be obtained by differentiating $y'(x) = f(x, y)$. In practice, this differentiation may be tedious. Now, however, this differentiation can be done by computer using symbolic software, such as Mathematica, Maple, or Reduce, or numerical packages. For equations of the form encountered in this chapter, a computer has no trouble generating and evaluating 10 or more derivatives.

The Taylor series solution is a form of analytic continuation (Section 6.5). If the right-hand side of Eq. (8.100) is truncated after two terms, we have

$$y_1 = y_0 + hy_0' = y_0 + hf(x_0, y_0), \ldots, y_{n+1} = y_n + hf(x_n, y_n), \quad (8.101)$$

neglecting the terms of order h^2. Equation (8.101) is often called the **Euler solution**. Clearly, it is subject to serious error with the neglect of terms of order h^2. Let us discuss a specific case.

EXAMPLE 8.7.1

Taylor Series Approximation for First-Order ODE Because there is no general method for solving first-order ODEs, we often resort to numerical approximations. From $y' = f(x, y)$, we obtain by differentiation of the ODE

$$y'' = \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y)y', \quad y''' = \frac{\partial^2 f}{\partial x^2}(x, y) + \cdots + \frac{\partial f}{\partial y}(x, y)y'',$$

etc. Starting from the point (x_0, y_0), we determine $y(x_0), y'(x_0), y''(x_0), \ldots$ from these derivatives of the ODE and plug them into the Taylor expansion in order to get to a neighboring point, from which we continue the process.

To be specific, consider the ODE $y' + y^2 = 0$, whose analytic solution through the point $(x = 1, y = 1)$ is the hyperbola $y(x) = 1/x$. Following the approximation method we just outlined, we find

$$y'' = -2yy', \quad y''' = -2(yy'' + y'^2), \quad y^{(IV)} = -2(yy''' + 3y'y''), \ldots.$$

The resulting Taylor series

$$\begin{aligned} y(x) &= y(1) + (x-1)y'(1) + \frac{(x-1)^2}{2}y''(1) + \frac{(x-1)^3}{3!}y'''(1) + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \cdots = \frac{1}{1+(x-1)} = \frac{1}{x}, \end{aligned}$$

for $0 < x < 2$ indeed confirms the exact solution and extends its validity beyond the interval of convergence. ■

Runge–Kutta Method

The Runge–Kutta method is a refinement of Euler's approximation [Eq. (8.101)]. The fourth-order Runge–Kutta approximation has an error of order h^5. The relevant formulas are

$$y_{n+1} = y_n + \frac{1}{6}[k_0 + 2k_1 + 2k_2 + k_3], \tag{8.102}$$

where

$$\begin{aligned} k_0 &= hf(x_n, y_n), \\ k_1 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0\right), \\ k_2 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right), \\ k_3 &= hf(x_n + h, y_n + k_2). \end{aligned} \tag{8.103}$$

The basic idea of the Runge–Kutta method is to eliminate the error terms order by order. A derivation of these equations appears in Ralston and Wilf[16] (see Chapter 9 by M. J. Romanelli) and in Press *et al.*[17]

Equations (8.102) and (8.103) define what might be called the classic fourth-order Runge–Kutta method (accurate through terms of order h^4). This is the form followed in Sections 15.1 and 15.2 of Press *et al.* Many other Runge–Kutta methods exist. Lapidus and Seinfeld (see Additional Reading) analyze and compare other possibilities and recommend a fifth-order form due to Butcher as slightly superior to the classic method. However, for applications not demanding high precision and for not so smooth ODEs the fourth-order Runge–Kutta method **with adaptive step size control** (see Press *et al.*, Chapter 15) is the method of choice for numerical solutions of ODEs. In general, but not always, fourth-order Runge–Kutta is superior to second-order **and higher order** Runge–Kutta schemes. From this Taylor expansion viewpoint the Runge–Kutta method is also an example of analytic continuation.

For the special case in which dy/dx is a function of x alone [$f(x, y)$ in Eq. (8.99) $\to f(x)$], the last term in Eq. (8.102) reduces to a Simpson rule numerical integration from x_n to x_{n+1}.

The Runge–Kutta method is stable, meaning that small errors do not get amplified. It is self-starting, meaning that we just take the x_0 and y_0 and away we go. However, it has disadvantages. Four separate calculations of $f(x, y)$ are required at each step. The errors, although of order h^5 per step, are not known. One checks the numerical solution by cutting h in half and repeating the calculation. If the second result agrees with the first, then h was small enough. Finally, the Runge–Kutta method can be extended to a set of coupled

[16]Ralston, A., and Wilf, H. S. (Eds.) (1960). *Mathematical Methods for Digital Computers*. Wiley, New York.
[17]Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T. (1992). *Numerical Recipes*, 2nd ed. Cambridge Univ. Press, Cambridge, UK.

first-order equations:

$$\frac{du}{dx} = f_1(x, u, v), \qquad \frac{dv}{dx} = f_2(x, u, v), \quad \text{and so on,} \tag{8.104}$$

with as many **dependent** variables as desired. Again, Eq. (8.104) may be nonlinear, an advantage of the numerical solution.

For high-precision applications one can also use either Richardson's extrapolation in conjunction with the Burlish–Stoer method[18] or the predictor–corrector method described later. Richardson's extrapolation is based on approximating the numerical solution by a rational function that can then be evaluated in the limit of step size $h \to 0$. This often allows for a large actual step size in applications.

Predictor–Corrector Methods

As an alternate attack on Eq. (8.99), we might estimate or **predict** a tentative value of y_{n+1} by

$$\bar{y}_{n+1} = y_{n-1} + 2hy'_n = y_{n-1} + 2hf(x_n, y_n). \tag{8.105}$$

This is not quite the same as Eq. (8.101). Rather, it may be interpreted as

$$y'_n \approx \frac{\Delta y}{\Delta x} = \frac{y_{n+1} - y_{n-1}}{2h}, \tag{8.106}$$

the derivative as a tangent being replaced by a chord. Next, we **calculate**

$$y'_{n+1} = f(x_{n+1}, \bar{y}_{n+1}). \tag{8.107}$$

Then to correct for the crudeness of Eq. (8.105), we take

$$y_{n+1} = y_n + \frac{h}{2}(\bar{y}_{n+1} + y'_n). \tag{8.108}$$

Here, the finite difference ratio $\Delta y/h$ is approximated by the average of the two derivatives. This technique—a prediction followed by a correction (and iteration until agreement is reached)—is the heart of the predictor–corrector method. It should be emphasized that the preceding set of equations is intended only to illustrate the predictor–corrector method. The accuracy of this set (to order h^3) is usually inadequate.

The iteration [substituting y_{n+1} from Eq. (8.108) back into Eq. (8.107) and recycling until y_{n+1} settles down to some limit] is time-consuming in a computer run. Consequently, the iteration is usually replaced by an intermediate step (the **modifier**) between Eqs. (8.105) and (8.107). This modified predictor–corrector method has the major advantage over the Runge–Kutta method of requiring only two computations of $f(x, y)$ per step instead of four. Unfortunately, the method as originally developed was unstable—small errors (round-off and truncation) tended to propagate and become amplified.

[18] See Section 15.4 of Press *et al.* and also Stoer, J., and Burlirsch, R. (1980). *Introduction to Numerical Analysis* (Chap. 7). Springer-Verlag, New York.

This very serious problem of instability has been overcome in a version of the predictor–corrector method devised by Hamming. The formulas (which are moderately involved), a partial derivation, and detailed instructions for starting the solution are all given by Ralston (Chapter 8 of Ralston and Wilf). Hamming's method is accurate to order h^4. It is stable for all reasonable values of h and provides an estimate of the error. Unlike the Runge–Kutta method, it is **not** self-starting. For example, Eq. (8.105) requires both y_{n-1} and y_n. Starting values (y_0, y_1, y_2, y_3) for the Hamming predictor–corrector method may be computed by series solution (power series for small x and asymptotic series for large x) or by the Runge–Kutta method. The Hamming predictor–corrector method may be extended to cover a set of coupled first-order ODEs—that is, Eq. (8.104).

Second-Order ODEs

Any second-order differential equation,

$$y''(x) + P(x)y'(x) + Q(x)y(x) = F(x), \tag{8.109}$$

may be split into two first-order ODEs by writing

$$y'(x) = z(x) \tag{8.110}$$

and then

$$z'(x) + P(x)z(x) + Q(x)y(x) = F(x). \tag{8.111}$$

These coupled first-order ODEs may be solved by either the Runge–Kutta or Hamming predictor–corrector techniques previously described. The Runge–Kutta–Nyström method for second-order ODEs is a more accurate version that proceeds via an intermediate auxiliary y'_{n+1}. The form of Eqs. (8.102) and (8.103) is assumed and the parameters are adjusted to fit a Taylor expansion through h^4.

As a final note, a thoughtless "turning the crank" application of these powerful numerical techniques is an invitation to disaster. The solution of a new and different differential equation will usually involve a combination of analysis and numerical calculation. There is little point in trying to force a Runge–Kutta solution through a singular point (see Section 8.4) where the solution (or y' or y'') may blow up. For a more extensive treatment of computational methods we refer the reader to Garcia (see Additional Reading).

EXERCISES

8.7.1 The Runge–Kutta method, Eq. (8.102), is applied to a first-order ODE $dy/dx = f(x)$. Note that this function $f(x)$ is independent of y. Show that in this special case the Runge–Kutta method reduces to Simpson's rule for numerical quadrature.

8.7.2 (a) A body falling through a resisting medium is described by

$$\frac{dv}{dt} = g - av$$

(for a retarding force proportional to the velocity). Take the constants to be $g = 9.80$ (m/sec^2) and $a = 0.2$ (sec^{-1}). The initial conditions are $t = 0$, $v = 0$. Integrate this equation out to $t = 20.0$ in steps of 0.1 sec. Tabulate the value of the velocity for each whole second, $v(1.0)$, $v(2.0)$, and so on. If a plotting routine is available, plot $v(t)$ versus t.

(b) Calculate the ratio of $v(20.0)$ to the terminal velocity $v(\infty)$.
Check value. $v(10) = 42.369$ m/sec.

ANS. (b) 0.9817.

8.7.3 Integrate Legendre's differential equation (Exercise 8.5.5) from $x = 0$ to $x = 1$ with the initial conditions $y(0) = 1$, $y'(0) = 0$ (even solution). Tabulate $y(x)$ and dy/dx at intervals of 0.05. Take $n = 2$.

8.7.4 The Lane–Emden equation of astrophysics is

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y^s = 0.$$

Take $y(0) = 1$, $y'(0) = 0$, and investigate the behavior of $y(x)$ for $s = 0, 1, 2, 3, 4, 5$, and 6. In particular, locate the first zero of $y(x)$.
Hint. From a power series solution $y''(0) = -\frac{1}{3}$.
Note. For $s = 0$, $y(x)$ is a parabola; for $s = 1$, a spherical Bessel function, $j_0(x)$. As $s \to 5$, the first zero moves out to ∞, and for $s > 5$, $y(x)$ never crosses the positive x-axis.

ANS. For $y(x_s) = 0$, $x_0 = 2.45(\approx\sqrt{6})$,
$x_1 = 3.14(\approx\pi)$, $x_2 = 4.35$,
$x_3 = 6.90$.

8.7.5 As a check on Exercise 8.6.10(a), integrate Hermite's equation ($x = 0$)

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 0$$

from $x = 0$ out to $x = 3$. The initial conditions are $y(0) = 0$, $y'(0) = 1$. Tabulate $y(1)$, $y(2)$, and $y(3)$.

ANS. $y(1) = 1.463$, $y(2) = 16.45$, $y(3) = 1445$.

8.7.6 Solve numerically ODE 3 of Exercise 8.3.15 using the Euler method, then compare with the Runge–Kutta method. For several fixed x, plot on a log–log scale $|y(x) - y_{\text{num}}(x)|$ versus step size, where $y(x)$ is your analytic and y_{num} your numerical solution. Find the best step size.

8.7.7 Solve numerically Bessel's ODE in Eq. (8.62) for $n = 0, 1, 2$ and calculate the location of the first two roots $J_n(\alpha_{ns}) = 0$.
Check value. $\alpha_{12} = 7.01559$.

8.7.8 Solve numerically the pendulum ODE

$$l\frac{d^2\theta}{dt^2} = -(g + a\sin\omega t)\sin\theta$$

with a harmonically driven pivot. Choose your step size according to the driving frequency ω and pick suitable parameters l, a, ω. Include and discuss the case $g \ll a$.

8.7.9 Solve numerically the ODE of Exercise 8.2.20. Compare with a Runge–Kutta result.

8.7.10 Solve numerically the ODE of Exercise 8.2.21.

8.7.11 Solve numerically the ODE of Exercise 8.2.22.

8.8 Introduction to Partial Differential Equations

The dynamics of many physical systems involve second-order derivatives, such as the acceleration in classical mechanics and the kinetic energy, $\sim\nabla^2$, in quantum mechanics, and lead to partial differential equations (PDEs) in time and one or more spatial variables.

Partial derivatives are linear operators:

$$\frac{\partial(a\varphi(x, y) + b\psi(x, y))}{\partial x} = a\frac{\partial\varphi(x, y)}{\partial x} + b\frac{\partial\psi(x, y)}{\partial x},$$

where a, b are constants. Similarly, if $\mathcal{L}$ is an operator consisting of (partial) derivatives, and the operator $\mathcal{L}$ is linear,

$$\mathcal{L}(a\psi_1 + b\psi_2) = a\mathcal{L}\psi_1 + b\mathcal{L}\psi_2,$$

and the PDE may be cast in the form

$$\mathcal{L}\psi = F,$$

where F is the external source, independent of $\psi(t, \mathbf{r})$, the unknown function. If $F \equiv 0$ the PDE is called **homogeneous**; if $F \neq 0$ the PDE is **inhomogeneous**. For homogeneous PDEs the **superposition principle** is valid: If ψ_1, ψ_2 are solutions of the PDE, so is any linear combination $a\psi_1 + b\psi_2$. If $\mathcal{L}$ contains first-order partial derivatives at most, the PDE is called **first order**; if $\mathcal{L}$ contains second-order derivatives, such as ∇^2, but no higher derivatives, we have a second-order PDE, etc. Second-order PDEs with constant coefficients occur often in physics. They are classified further into **elliptic** PDEs if they involve ∇^2 or $\nabla^2 + c^{-2}\partial^2/\partial t^2$, **parabolic** with $a\partial/\partial t + \nabla^2$, and **hyperbolic** with operators such as $c^{-2}\partial^2/\partial t^2 - \nabla^2$. Hyperbolic (and some parabolic) PDEs have waves as solutions.

8.9 Separation of Variables

Our first technique for solving PDEs splits the PDE of n variables into n ordinary differential equations. Each separation introduces an arbitrary constant of separation. If we have n variables, we have to introduce $n - 1$ constants, determined by the conditions imposed in the problem being solved. Let us start with the heat flow equation for a rectangular metal slab. The geometry suggests using Cartesian coordinates.[19]

Cartesian Coordinates

In a homogeneous medium at temperature $\psi(\mathbf{r})$ that varies with location, heat flows from sites at high temperature to lower temperature in the direction of negative temperature gradient. We assume that appropriate heat sources are present on the boundaries to produce the boundary conditions. The heat flow must be of the form $\mathbf{j} = -\kappa \boldsymbol{\nabla}\psi$, where the proportionality constant κ measures the heat conductivity of the medium, a rectangular metal slab in our case. The current density is proportional to the velocity of the heat flow. If the temperature increases somewhere, this is due to more heat flowing into that particular volume element d^3r than leaves it. From Section 1.6, we know that the difference is given by the negative divergence of the heat flow; that is, $-\boldsymbol{\nabla}\cdot\mathbf{j}d^3r = \kappa\boldsymbol{\nabla}^2\psi d^3r$. On the other hand, the increase of energy with time is proportional to the change of temperature with time, the specific heat σ, and the mass ρd^3r, where ρ is the density, taken to be constant in space and time. In the absence of sources, we obtain the **heat flow equation**

$$\frac{\partial\psi}{\partial t} = \frac{\kappa}{\sigma\rho}\boldsymbol{\nabla}^2\psi, \tag{8.112}$$

a parabolic PDE. For the simplest, **time-independent steady-state** case we have $\partial\psi/\partial t = 0$, and the Laplace equation results. In Cartesian coordinates the Laplace equation becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0, \tag{8.113}$$

using Eq. (2.5) for the Laplacian. Perhaps the simplest way of treating a PDE such as Eq. (8.113) is to split it into a set of ODEs. This may be done with a product ansatz, or trial form, for

$$\psi(x, y, z) = X(x)Y(y)Z(z), \tag{8.114}$$

and then substitute it back into Eq. (8.113). How do we know Eq. (8.114) is valid? We are proceeding with trial and error. If our attempt succeeds, then Eq. (8.114) will be justified. If it does not succeed, we shall find out soon enough and then we try another attack. With ψ assumed given by Eq. (8.114), Eq. (8.113) becomes

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0. \tag{8.115}$$

Dividing by $\psi = XYZ$ and rearranging terms, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2}. \tag{8.116}$$

[19]Boundary conditions are part of the geometry, assumed Euclidean as well, and will be discussed later and in Chapter 16 in more detail.

Equation (8.116) exhibits one separation of variables. The left-hand side is a function of the variable x alone, whereas the right-hand side depends only on y and z. However, x, y, and z are all independent coordinates. This independence means that the recipe of Eq. (8.114) worked; the left-hand side of Eq. (8.116) depends on x only, etc. That is, the behavior of x as an independent variable is not determined by y and z. Therefore, each side must be equal to a constant—a **constant of separation**.

The choice of sign, completely arbitrary here, will be fixed in specific problems by the need to satisfy **specific boundary conditions**, which we need to discuss now. Let us put one corner of the slab in the coordinate origin with its sides along the coordinate axes and its bottom sitting at $z = 0$ with a given temperature distribution $\psi(x, y, z = 0) = \psi_0(x, y)$. To simplify the problem further, we assume that the slab is finite in the x- and y-directions but infinitely long in the z-direction with zero temperature as $z \to +\infty$. This is a reasonable assumption, as long as we are not interested in the temperature near the end of the slab ($z \to \infty$). We take the lengths of the slab in the transverse x, y-directions to be the same, 2π. Now we choose

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2 \tag{8.117}$$

because the $\sin lx$, $\cos lx$ solutions with integer l implied by the boundary condition allow for a Fourier expansion to fit the x-dependence of the temperature distribution ψ_0 at $z = 0$ and fixed y. [Note that if a is the length of the slab in the x-direction, we write the separation constant as $-(2\pi l/a)^2$ with integer l, and this boundary condition gives the solutions $a_l \sin(2\pi lx/a) + b_l \cos(2\pi lx/a)$. If the temperature at $x = 0$, $x = a$ and $z = 0$ is zero, then all $b_l = 0$ and the cosine solutions are ruled out.]

Returning to the other half of Eq. (8.116), it becomes

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} = -l^2. \tag{8.118}$$

Rewriting it so as to separate the y- and z-dependent parts, we obtain

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = l^2 - \frac{1}{Z}\frac{d^2Z}{dz^2}, \tag{8.119}$$

and a second separation has been achieved. Here we have a function of y equated to a function of z as before. We resolve the situation as before by equating each side to another (negative) constant of separation, $-m^2$,

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2, \tag{8.120}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = l^2 + m^2 = n^2, \tag{8.121}$$

introducing a positive constant n^2 by $n^2 = l^2 + m^2$ to produce a symmetric set of equations in x, y. As a consequence, the separation constant in the z-direction is positive, which implies that its solution is exponential, $e^{\pm nz}$.

We discard solutions containing e^{+nz} because they violate the boundary condition for large values of z, where the temperature goes to zero. Now we have three ODEs [Eqs. (8.117), (8.120), and (8.121)] to replace the PDE [Eq. (8.113)]. Our assumption [Eq. (8.114)] has succeeded and is thereby justified.

Our solution should be labeled according to the choice of our constants l, m, and n; that is,

$$\psi_{lm}(x, y, z) = X_l(x)Y_m(y)Z_n(z), \tag{8.122}$$

subject to the conditions of the problem being solved and to the condition $n^2 = l^2 + m^2$. We may choose l and m as we like and Eq. (8.122) will still be a solution of Eq. (8.113), provided $X_l(x)$ is a solution of Eq. (8.117), and so on. We may develop **the most general solution** of Eq. (8.113) by taking a **linear combination of solutions** ψ_{lm}, **by the superposition principle**

$$\Psi(x, y, z) = \sum_{l,m} a_{lm}\psi_{lm}, \tag{8.123}$$

because the Laplace equation is homogeneous and linear. The constant coefficients a_{lm} are chosen to permit Ψ to satisfy the **boundary condition** of the problem at $z = 0$, where all Z_n are normalized to $Z_n(0) = 1$, so that

$$\psi_0(x, y) = \sum_{l,m} a_{lm}X_l(x)Y_m(y). \tag{8.124}$$

In other words, the expansion coefficients a_{lm} of our solution Ψ in Eq. (8.123) are uniquely determined as Fourier coefficients of the given temperature distribution at the boundary $z = 0$ of the metal slab. A specific case is treated in the next example.

EXAMPLE 8.9.1

Cartesian Boundary Conditions Let us consider a case in which, at $z = 0$, $\psi(x, y, 0) = \psi_0(x, y) = 100°\text{C}$ (the boiling point of water) in the area $-1 < x < 1, -1 < y < 1$, an input temperature distribution on the $z = 0$ plane. Moreover, the temperature ψ is held at zero, $\psi = 0$ (the freezing point of water) at the end points $x = \pm 1$ for all y and z, and at $y = \pm 1$ for all x and z, a boundary condition that restricts the temperature spread to the finite area of the slab in the x, y-directions. Also, $\psi(x, y, z) \to 0$ for $z \to \infty$ for all x, y. The entire slab, except the $z = 0$ plane, is in contact with a constant-temperature heat bath, whose temperature can be taken as the zero of ψ; Eq. (8.113) is invariant with respect to the addition of a constant to ψ.

Because of the adopted boundary conditions, we choose the solutions $\cos\frac{\pi l x}{2}$ with integer l that vanish at the end points $x = \pm 1$ and the corresponding ones in the y-direction, excluding $l = 0$ and $X, Y = \text{const}$. Inside the interval at $z = 0$, therefore, we have the (Fourier) expansion

$$X(x) = \sum_{l=1}^{\infty} a_l \cos\left(\frac{\pi l x}{2}\right) = 1, \quad -1 < x < 1,$$

with coefficients (see Section 14.1)

$$\begin{aligned} a_l &= \int_{-1}^{1} 100 \cdot \cos\frac{\pi l x}{2} dx = \frac{200}{l\pi} \sin\frac{\pi l x}{2}\bigg|_{x=-1}^{1} \\ &= \frac{400}{\pi l} \sin\frac{l\pi}{2} = \frac{400(-1)^\mu}{(2\mu+1)\pi}, \quad l = 2\mu + 1; \\ a_l &= 0, \quad l = 2\mu \end{aligned}$$

for integer μ. The same Fourier expansion (now without the factor 100) applies to the y-direction involving the integer summation index ν in $Y(y)$, whereas the z-dependence is given by Eq. (8.121), so that the complete solution becomes

$$\begin{aligned} \psi(x, y, z) = XYZ &= 100 \sum_{\mu=0}^{\infty} \frac{4(-1)^\mu}{(2\mu+1)\pi} \cos\left((2\mu+1)\frac{\pi x}{2}\right) \\ &\times \sum_{\nu=0}^{\infty} \frac{4(-1)^\nu}{(2\nu+1)\pi} \cos\left((2\nu+1)\frac{\pi y}{2}\right) e^{-\pi n z/2}, \\ n^2 &= (2\mu+1)^2 + (2\nu+1)^2. \end{aligned}$$

For $z > 0$ this solution converges absolutely, but at $z = 0$ there is only conditional convergence for each sum that is caused by the discontinuity at $z = 0$, $x = \pm 1$, $y = \pm 1$. ■

Circular Cylindrical Coordinates

Let us now consider a cylindrical, infinitely long metal rod with a heat source at $z = 0$ that generates a given steady-state temperature distribution $\psi_0(\rho, \varphi) = \psi(\rho, \varphi, 0)$ at $z = 0$ in a circular area about the origin and zero temperature at large values of z for all ρ and φ. For the method of separation of variables to apply, the radial boundary condition needs to be independent of z and φ. Choices that lead to reasonable physical situations are (i) zero temperature at $\rho = a$ (radius of the rod, corresponding to immersion of the rod in a constant-temperature bath) and (ii) zero gradient of temperature at $\rho = a$ (corresponding to no lateral flow out of the rod). This choice will lead to a situation in which the lateral temperature distribution for large z will approach a uniform value equal to the average temperature of the area at $z = 0$. We want to find the temperature distribution for $z > 0$ for all ρ and φ under such conditions.

With our unknown function ψ dependent on ρ, φ, and z, the Laplace equation becomes (see Section 2.2 for ∇^2)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} = 0. \tag{8.125}$$

As before, we assume a factored form for ψ,

$$\psi(\rho, \varphi, z) = P(\rho)\Phi(\varphi)Z(z). \tag{8.126}$$

Substituting into Eq. (8.125), we have

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{PZ}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + P\Phi\frac{d^2 Z}{dz^2} = 0. \tag{8.127}$$

All the partial derivatives have become ordinary derivatives because each function depends on a single variable. Dividing by the product $P\Phi Z$ and moving the z derivative to the right-hand side yields

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2}=-\frac{1}{Z}\frac{d^2Z}{dz^2}. \tag{8.128}$$

Again, a function of z on the right depends on a function of ρ and φ on the left. We resolve this paradox by setting each side of Eq. (8.128) equal to the same constant. Let us choose $-n^2$. Then

$$\frac{d^2Z}{dz^2}=n^2Z, \tag{8.129}$$

and

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2}=-n^2. \tag{8.130}$$

From Eq. (8.130) we find the already familiar exponential solutions $Z \sim e^{\pm nz}$, from which we discard e^{+nz} again because the temperature goes to zero at large values of $z \geq 0$.

Returning to Eq. (8.131), multiplying by ρ^2 and rearranging terms, we obtain[20]

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+n^2\rho^2=-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}. \tag{8.131}$$

We then set the right-hand side to the positive constant m^2 to obtain

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}=-m^2\Phi(\varphi). \tag{8.132}$$

Finally, as an illustration of how the constant m in Eq. (8.133) is restricted, we note that φ in cylindrical and spherical polar coordinates is an azimuth angle. If this is a classical problem, we shall certainly require that the azimuthal solution $\Phi(\varphi)$ be single-valued; that is,

$$\Phi(\varphi+2\pi)=\Phi(\varphi), \tag{8.133}$$

which yields the periodic solutions $\Phi(\varphi) \sim e^{\pm im\varphi}$ for integer m. This is equivalent to requiring the azimuthal solution to have a period of 2π or some integral multiple of it.[21] Therefore, m must be an integer. Which integer it is depends on the details of the boundary conditions of the problem. This is discussed later and in Chapter 9. Whenever a coordinate corresponds to an azimuth angle the separated equation always has the form of Eq. (8.132).

Finally, for the ρ dependence we have

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+(n^2\rho^2-m^2)P=0. \tag{8.134}$$

[20]The choice of sign of the separation constant is arbitrary. However, a minus sign is chosen for the axial coordinate z in expectation of a possible exponential dependence on $z \geq 0$. A positive sign is chosen for the azimuthal coordinate φ in expectation of a periodic dependence on φ.

[21]This also applies in most quantum mechanical problems but the argument is much more involved. If m is not an integer, or half an integer for spin $\frac{1}{2}$ particles, rotation group relations and ladder operator relations (Section 4.3) are disrupted. Compare Merzbacher, E. (1962). Single valuedness of wave functions. *Am. J. Phys.* **30**, 237.

This is Bessel's ODE. The solutions and their properties are presented in Chapter 12. We emphasize here that we can rescale the variable ρ by a constant in Eq. (8.134) so that P must be a function of $n\rho$ and also depend on the parameter m; hence the notation $P_m(n\rho)$. Because the temperature is finite at the center of the rod, $\rho = 0$, P_m must be the regular solution J_m of Bessel's ODE rather than the irregular solution, the Neumann function Y_m. In case of boundary condition (i), $J_m(na) = 0$ will require na to be a zero of the Bessel function, thereby restricting n to a discrete set of values. The alternative (ii) requires $dJ_m/d\rho|_{n\rho=a} = 0$ instead. To fit the solution to a distribution ψ_0 at $z = 0$ one needs an expansion in Bessel functions and to use the associated orthogonality relations.

The separation of variables of Laplace's equation in parabolic coordinates also gives rise to Bessel's equation. The Bessel equation is notorious for the variety of disguises it may assume. For an extensive tabulation of possible forms the reader is referred to *Tables of Functions* by Jahnke and Emde.[22]

The original Laplace equation, a three-dimensional PDE, has been replaced by three ODEs [Eqs. (8.129), (8.132), and (8.134)]. A solution of the Laplace equation is

$$\psi_{mn}(\rho, \varphi, z) = P_m(n\rho)\Phi_m(\varphi)Z_n(z). \tag{8.135}$$

Identifying the specific P, Φ, Z solutions by subscripts, we see that the **most general solution of the Laplace equation** is a linear combination of the product solutions:

$$\Psi(\rho, \varphi, z) = \sum_{m,n} a_{mn}P_m(n\rho)\Phi_m(\varphi)Z_n(z). \tag{8.136}$$

Here, the coefficients a_{mn} are determined by the Bessel–Fourier expansion of the boundary condition at $z = 0$, where the given temperature distribution ψ_0 has to obey

$$\psi_0(\rho, \varphi) = \sum_{m,n} a_{mn}P_m(n\rho)\Phi_m(\varphi) \tag{8.137}$$

because all $Z_n(z = 0) = 1$. Recall that na is restricted to a discrete set of values by the radial boundary condition, and m are integers.

Spherical Polar Coordinates

For a large metal sphere and spherical boundary conditions, with a temperature distribution on the surface $\psi(r = a, \theta, \varphi) = \psi_0(\theta, \varphi)$ generated by a heat source at the surface $r = a$, let us separate the Laplace equation in spherical

[22] Jahnke, E., and Emde, F. (1945). *Tables of Functions*, 4th rev. ed. (p. 146). Dover, New York; also, Jahnke, E., Emde, F., and Lösch, F. (1960). *Tables of Higher Functions*, 6th ed. McGraw-Hill, New York.

polar coordinates. Using Eq. (2.77), we obtain

$$\frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \varphi^2}\right] = 0. \tag{8.138}$$

Now, in analogy with Eq. (8.114) we try a product solution

$$\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi). \tag{8.139}$$

By substituting back into Eq. (8.138) and dividing by $R\Theta\Phi$, we have

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0. \tag{8.140}$$

Note that all derivatives are now ordinary derivatives rather than partials. By multiplying by $r^2 \sin^2\theta$, we can isolate $(1/\Phi)(d^2\Phi/d\varphi^2)$ to obtain[23]

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = r^2 \sin^2\theta \left[-\frac{1}{r^2 R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{1}{r^2 \sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)\right]. \tag{8.141}$$

Equation (8.141) relates a function of φ alone to a function of r and θ alone. Since r, θ, and φ are independent variables, we equate each side of Eq. (8.141) to a constant. In almost all physical problems φ will appear as an azimuth angle. This suggests a periodic solution rather than an exponential so that Φ is single-valued. With this in mind, let us use $-m^2$ as the separation constant. Then

$$\frac{1}{\Phi}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2 \tag{8.142}$$

and

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{r^2 \sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{r^2 \sin^2\theta} = 0. \tag{8.143}$$

Multiplying Eq. (8.143) by r^2 and rearranging terms, we obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = -\frac{1}{\sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta}. \tag{8.144}$$

Again, the variables are separated. We equate each side to a constant Q and finally obtain

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + Q\Theta = 0, \tag{8.145}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{QR}{r^2} = 0. \tag{8.146}$$

Once more we have replaced a PDE of three variables by three ODEs. The solutions of these ODEs are discussed in Chapters 11 and 12. In Chapter 11, for example, Eq. (8.145) is identified as the **associated Legendre equation** in which the constant Q becomes $l(l+1)$; l is a positive integer. The radial

[23]The order in which the variables are separated here is not unique. Many quantum mechanics texts show the r dependence split off first.

Eq. (8.146) has powers $R \sim r^l, r^{-l-1}$ for solutions so that $Q = l(l+1)$ is maintained. These power solutions occur in the multipole expansions of electrostatic and gravitational potentials, the most important physical applications. The corresponding positive power solutions are called harmonic polynomials, but the negative powers are required for a complete solution. The boundary conditions usually determine whether or not the negative powers are retained as (irregular) solutions.

Again, our most general solution may be written as

$$\psi(r, \theta, \varphi) = \sum_{l,m} a_{lm} R_l(r)\Theta_{lm}(\theta)\Phi_m(\varphi). \tag{8.147}$$

The great importance of this separation of variables in spherical polar coordinates stems from the fact that the method covers a tremendous amount of physics—many of the theories of gravitation, electrostatics, atomic, nuclear, and particle physics, where the **angular dependence** is isolated in the **same** Eqs. (8.142) and (8.145), **which can be solved exactly**. In the hydrogen atom problem, one of the most important examples of the Schrödinger wave equation with a closed form solution, the analog of Eq. (8.146) for the hydrogen atom becomes the associated Laguerre equation.

Whenever a coordinate z corresponds to an axis of translation the separated equation always has the form

$$\frac{d^2 Z(z)}{dz^2} = \pm a^2 Z(z)$$

in one of the cylindrical coordinate systems. The solutions, of course, are $\sin az$ and $\cos az$ for $-a^2$ and the corresponding hyperbolic function (or exponentials) $\sinh az$ and $\cosh az$ for $+a^2$.

Other occasionally encountered ODEs include the Laguerre and associated Laguerre equations from the supremely important hydrogen atom problem in quantum mechanics:

$$x\frac{d^2 y}{dx^2} + (1-x)\frac{dy}{dx} + \alpha y = 0, \tag{8.148}$$

$$x\frac{d^2 y}{dx^2} + (1+k-x)\frac{dy}{dx} + \alpha y = 0. \tag{8.149}$$

From the quantum mechanical theory of the linear oscillator we have Hermite's equation,

$$\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2\alpha y = 0. \tag{8.150}$$

Finally, occasionally we find the Chebyshev differential equation

$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0. \tag{8.151}$$

More ODEs and two generalizations of them will be examined and systematized in Chapter 16. General properties following from the form of the differential equations are discussed in Chapter 9. The individual solutions are developed and applied in Chapters 11–13.

The practicing physicist probably will encounter other second-order ODEs, some of which may possibly be transformed into the examples studied here. Some of these ODEs may be solved by the techniques of Sections 8.5 and 8.6. Others may require a computer for a numerical solution.

- To put the separation method of solving PDEs in perspective, let us view it as a consequence of a symmetry of the PDE. Take the stationary Schrödinger equation $H\psi = E\psi$ as an example with a potential $V(r)$ depending only on the radial distance r. Then this PDE is invariant under rotations that comprise the group SO(3). Its diagonal generator is the orbital angular momentum operator $L_z = -i\frac{\partial}{\partial\varphi}$, and its quadratic (Casimir) invariant is $\mathbf{L}^2$. Since both commute with H (see Section 4.3), we end up with three separate eigenvalue equations:

$$H\psi = E\psi, \qquad \mathbf{L}^2\psi = l(l+1)\psi, \qquad L_z\psi = m\psi.$$

  Upon replacing L_z^2 in $\mathbf{L}^2$ by its eigenvalue m^2, the $\mathbf{L}^2$ PDE becomes Legendre's ODE (see Exercise 2.5.12), and similarly $H\psi = E\psi$ becomes the radial ODE of the separation method in spherical polar coordinates upon substituting the eigenvalue $l(l+1)$ for $\mathbf{L}^2$.
- For cylindrical coordinates the PDE is invariant under rotations about the z-axis only, which form a subgroup of SO(3). This invariance yields the generator $L_z = -i\partial/\partial\varphi$ and separate azimuthal ODE $L_z\psi = m\psi$ as before. Invariance under translations along the z-axis with the generator $-i\partial/\partial z$ gives the separate ODE in the z-variable provided the boundary conditions obey the same symmetry. The potential $V = V(\rho)$ or $V = V(z)$ depends on one variable, as a rule.
- In general, there are n mutually commuting generators H_i with eigenvalues m_i of the (classical) Lie group G of rank n and the corresponding Casimir invariants C_i with eigenvalues c_i, which yield the separate ODEs

$$H_i\psi = m_i\psi, \qquad C_i\psi = c_i\psi$$

  in addition to the radial ODE $H\psi = E\psi$.

EXERCISES

8.9.1 An atomic (quantum mechanical) particle is confined inside a rectangular box of sides a, b, and c. The particle is described by a wave function ψ that satisfies the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi.$$

The wave function is required to vanish at each surface of the box (but not to be identically zero). This condition imposes constraints on the separation constants and therefore on the energy E. What is the smallest value of E for which such a solution can be obtained?

$$ANS. \quad E = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

8.9.2 The quantum mechanical angular momentum operator is given by $L = -i(\mathbf{r} \times \boldsymbol{\nabla})$. Show that

$$\mathbf{L} \cdot \mathbf{L}\psi = l(l+1)\psi$$

leads to the associated Legendre equation.
Hint. Exercises 1.8.6 and 2.5.13 may be helpful.

8.9.3 The one-dimensional Schrödinger wave equation for a particle in a potential field $V = \frac{1}{2}kx^2$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi(x).$$

(a) Using $\xi = ax$ and a constant λ, we have

$$a = \left(\frac{mk}{\hbar^2}\right)^{1/4}, \qquad \lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{1/2}.$$

Show that

$$\frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\psi(\xi) = 0.$$

(b) Substituting

$$\psi(\xi) = y(\xi)e^{-\xi^2/2},$$

show that $y(\xi)$ satisfies the Hermite differential equation.

8.9.4 Verify that the following are solutions of Laplace's equation:

$$\text{(a) } \psi_1 = 1/r, \qquad \text{(b) } \psi_2 = \frac{1}{2r}\ln\frac{r+z}{r-z}.$$

Note. The z derivatives of $1/r$ generate the Legendre polynomials, $P_n(\cos\theta)$ (Exercise 11.1.7). The z derivatives of $(1/2r)\ln[(r+z)/(r-z)]$ generate the Legendre functions, $Q_n(\cos\theta)$.

8.9.5 If Ψ is a solution of Laplace's equation, $\nabla^2\Psi = 0$, show that $\partial\Psi/\partial z$ is also a solution.

Additional Reading

Cohen, H. (1992). *Mathematics for Scientists and Engineers*. Prentice-Hall, Englewood Cliffs, NJ.

Davis, P. J., and Rabinowitz, P. (1967). *Numerical Integration*. Blaisdell, Waltham, MA. This book covers a great deal of material in a relatively easy-to-read form. Appendix 1 (*On the Practical Evaluation of Integrals* by M. Abramowitz) is excellent as an overall view.

Garcia, A. L. (1994). *Numerical Methods for Physics*. Prentice-Hall, Englewood Cliffs, NJ.

Hamming, R. W. (1973). *Numerical Methods for Scientists and Engineers*, 2nd ed. McGraw-Hill, New York. Reprinted, Dover (1987). This well-written text discusses a wide variety of numerical methods from zeros of functions to the fast Fourier transform. All topics are selected and developed with a modern high-speed computer in mind.

Hubbard, J., and West, B. H. (1995). *Differential Equations*. Springer, Berlin.

Ince, E. L. (1956). *Ordinary Differential Equations*. Dover, New York. The classic work on the theory of ordinary differential equations.

Lapidus, L., and Seinfeld, J. H. (1971). *Numerical Solutions of Ordinary Differential Equations*. Academic Press, New York. A detailed and comprehensive discussion of numerical techniques with emphasis on the Runge–Kutta and predictor–corrector methods. Work on the improvement of characteristics such as stability is clearly presented.

Margenau, H., and Murphy, G. M. (1956). *The Mathematics of Physics and Chemistry*, 2nd ed. Van Nostrand, Princeton, NJ. Chapter 5 covers curvilinear coordinates and 13 specific coordinate systems.

Miller, R. K., and Michel, A. N. (1982). *Ordinary Differential Equations*. Academic Press, New York.

Morse, P. M., and Feshbach, H. (1953). *Methods of Theoretical Physics*. McGraw-Hill, New York. Chapter 5 includes a description of several different coordinate systems. Note that Morse and Feshbach are not above using left-handed coordinate systems even for Cartesian coordinates. Elsewhere in this excellent (and difficult) book there are many examples of the use of the various coordinate systems in solving physical problems. Eleven additional fascinating but seldom encountered orthogonal coordinate systems are discussed in the second edition of *Mathematical Methods for Physicists* (1970).

Murphy, G. M. (1960). *Ordinary Differential Equations and Their Solutions*. Van Nostrand, Princeton, NJ. A thorough, relatively readable treatment of ordinary differential equations, both linear and nonlinear.

Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T. (1992). *Numerical Recipes*, 2nd ed. Cambridge Univ. Press, Cambridge, UK.

Ralston, A., and Wilf, H. (Eds.) (1960). *Mathematical Methods for Digital Computers*. Wiley, New York.

Ritger, P. D., and Rose, N. J. (1968). *Differential Equations with Applications*. McGraw-Hill, New York.

Stoer, J., and Burlirsch, R. (1992). *Introduction to Numerical Analysis*. Springer-Verlag, New York.

Stroud, A. H. (1974). *Numerical Quadrature and Solution of Ordinary Differential Equations*, Applied Mathematics Series Vol. 10. Springer-Verlag, New York. A balanced, readable, and very helpful discussion of various methods of integrating differential equations. Stroud provides numerous references.