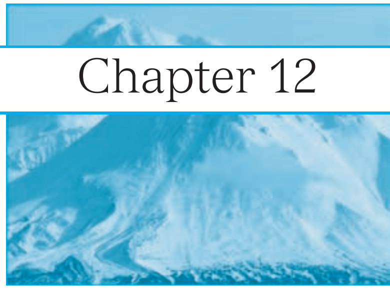


Chapter 12

Bessel Functions

12.1 Bessel Functions of the First Kind, $J_\nu(x)$

Bessel functions appear in a wide variety of physical problems. When one analyzes the sound vibrations of a drum, the partial differential wave equation (PDE) is solved in cylindrical coordinates. By separating the radial and angular variables, $R(r)e^{in\varphi}$, one is led to the Bessel ordinary differential equation (ODE) for $R(r)$ involving the integer n as a parameter (see Example 12.1.4). The Wentzel-Kramers-Brioullin (WKB) approximation in quantum mechanics involves Bessel functions. A spherically symmetric square well potential in quantum mechanics is solved by spherical Bessel functions. Also, the extraction of phase shifts from atomic and nuclear scattering data requires spherical Bessel functions. In Sections 8.5 and 8.6 series solutions to Bessel's equation were developed. In Section 8.9 we have seen that the Laplace equation in cylindrical coordinates also leads to a form of Bessel's equation. Bessel functions also appear in integral form—integral representations. This may result from integral transforms (Chapter 15).

Bessel functions and closely related functions form a rich area of mathematical analysis with many representations, many interesting and useful properties, and many interrelations. Some of the major interrelations are developed in Section 12.1 and in succeeding sections. Note that Bessel functions are not restricted to Chapter 12. The asymptotic forms are developed in Section 7.3 as well as in Section 12.3, and the series solutions are discussed in Sections 8.5 and 8.6.

Biographical Data

Bessel, Friedrich Wilhelm. Bessel, a German astronomer, was born in Minden, Prussia, in 1784 and died in Königsberg, Prussia (now Russia) in 1846. At the age of 20, he recalculated the orbit of Halley's comet, impressing the well-known astronomer Olbers sufficiently to support him in 1806 for a post at an observatory. There he developed the functions named after him in refinements of astronomical calculations. The first parallax measurement of a star, 61 Cygni about 6 light-years away from Earth, due to him in 1838, proved definitively that Earth was moving in accord with Copernican theory. His calculations of irregularities in the orbit of Uranus paved the way for the later discovery of Neptune by Leverrier and J. C. Adams, a triumph of Newton's theory of gravity.

Generating Function for Integral Order

Although Bessel functions $J_\nu(x)$ are of interest primarily as solutions of Bessel's differential equation, Eq. (8.62),

$$x^2\frac{d^2J_\nu}{dx^2}+x\frac{dJ_\nu}{dx}+(x^2-\nu^2)J_\nu=0,$$

it is instructive and convenient to develop them from a generating function, just as for Legendre polynomials in Chapter 11.[1] This approach has the advantages of finding recurrence relations, special values, and normalization integrals and focusing on the functions themselves rather than on the differential equation they satisfy. Since there is no physical application that provides the generating function in closed form, such as the electrostatic potential for Legendre polynomials in Chapter 11, we have to find it from a suitable differential equation.

We therefore start by deriving from Bessel's series [Eq. (8.70)] for integer index $\nu = n$,

$$J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(s+n)!}\left(\frac{x}{2}\right)^{2s+n},$$

that converges absolutely for all x, the recursion relation

$$\frac{d}{dx}(x^{-n}J_n(x))=-x^{-n}J_{n+1}(x). \tag{12.1}$$

This can also be written as

$$\frac{n}{x}J_n(x)-J_{n+1}(x)=J_n'(x). \tag{12.2}$$

To show this, we replace the summation index $s \to s-1$ in the Bessel function series [Eq. (8.70)] for $J_{n+1}(x)$,

$$J_{n+1}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(s+n+1)!}\left(\frac{x}{2}\right)^{2s+n+1}, \tag{12.3}$$

[1] Generating functions were also used in Chapter 5. In Section 5.6, the generating function $(1+x)^n$ defines the binomial coefficients; $x/(e^x-1)$ generates the Bernoulli numbers in the same sense.

in order to change the denominator $(s+n+1)!$ to $(s+n)!$. Thus, we obtain the series

$$J_{n+1}(x) = -\frac{1}{x}\sum_{s=0}^{\infty}\frac{(-1)^s 2s}{s!(s+n)!}\left(\frac{x}{2}\right)^{n+2s}, \tag{12.4}$$

which is almost the series for $J_n(x)$, except for the factor s. If we divide by x^n and differentiate, this factor s is produced so that we get from Eq. (12.4)

$$x^{-n}J_{n+1}(x) = -\frac{d}{dx}\sum_{n}\frac{(-1)^s}{s!(s+n)!}\left(\frac{x}{2}\right)^{2s}2^{-n} = -\frac{d}{dx}[x^{-n}J_n(x)], \tag{12.5}$$

that is, Eq. (12.1).

A similar argument for J_{n-1}, with summation index s replaced first by $s-n$ and then by $s \to s+1$, yields

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x), \tag{12.6}$$

which can be written as

$$J_{n-1}(x) - \frac{n}{x}J_n(x) = J_n'(x). \tag{12.7}$$

Eliminating J_n' from Eqs. (12.2) and (12.7), we obtain the recurrence

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x), \tag{12.8}$$

which we substitute into the **generating series**

$$g(x, t) = \sum_{n=-\infty}^{\infty} J_n(x)t^n. \tag{12.9}$$

This gives the ODE in t (with x a parameter)

$$\sum_{n=-\infty}^{\infty} t^n\left(J_{n-1}(x) + J_{n+1}(x)\right) = \left(t + \frac{1}{t}\right)g(x, t) = \frac{2t}{x}\frac{\partial g}{\partial t}. \tag{12.10}$$

Writing it as

$$\frac{1}{g}\frac{\partial g}{\partial t} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right), \tag{12.11}$$

and integrating we get

$$\ln g = \frac{x}{2}\left(t - \frac{1}{t}\right) + \ln c,$$

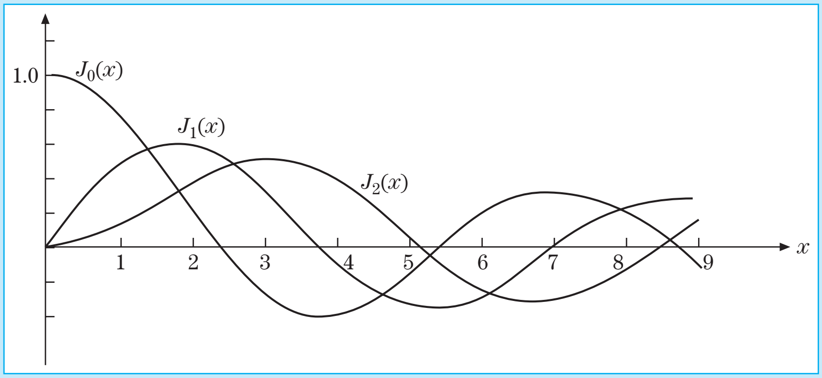
which, when exponentiated, leads to

$$g(x, t) = e^{(x/2)(t-1/t)}c(x), \tag{12.12}$$

where c is the integration constant that may depend on the parameter x. Now taking $x = 0$ and using $J_n(0) = \delta_{n0}$ (from Example 12.1.1) in Eq. (12.9) gives $g(0, t) = 1$ and $c(0) = 1$. To determine $c(x)$ for **all** x, we expand the

Figure 12.1

Bessel Functions, $J_0(x)$, $J_1(x)$, and $J_2(x)$



exponential in Eq. (12.12). The $1/t$ term leads to a Laurent series (see Section 6.5). Incidentally, we understand why the summation in Eq. (12.9) has to run over negative integers as well. So we have a product of Maclaurin series in $xt/2$ and $-x/2t$,

$$e^{xt/2} \cdot e^{-x/2t} = \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}. \tag{12.13}$$

For a given s we get $t^n (n \geq 0)$ from $r = n + s$

$$\left(\frac{x}{2}\right)^{n+s} \frac{t^{n+s}}{(n+s)!} (-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}. \tag{12.14}$$

The coefficient of t^n is then[2]

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s} = \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!} + \cdots \tag{12.15}$$

so that $c(x) \equiv 1$ for all x in Eq. (12.12) by comparing the coefficient of t^0, $J_0(x)$, with Eq. (12.3) for $n = -1$. Thus, the generating function is

$$g(x, t) = e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n. \tag{12.16}$$

This series form, Eq. (12.15), exhibits the behavior of the Bessel function $J_n(x)$ for all x, converging everywhere, and permits numerical evaluation of $J_n(x)$. The results for J_0, J_1, and J_2 are shown in Fig. 12.1. From Section 5.3, the error in using only a finite number of terms in numerical evaluation is less than the first term omitted. For instance, if we want $J_n(x)$ to $\pm 1\%$ accuracy, the first term alone of Eq. (12.15) will suffice, provided the ratio of the second term to the first is less than 1% (in magnitude) or $x < 0.2(n+1)^{1/2}$. The Bessel

[2]From the steps leading to this series and from its absolute convergence properties it should be clear that this series converges absolutely, with x replaced by z and with z any point in the finite complex plane.

functions oscillate but are **not periodic**; however, in the limit as $x \to \infty$ the zeros become equidistant (Section 12.3). The amplitude of $J_n(x)$ is not constant but decreases asymptotically as $x^{-1/2}$. [See Eq. (12.106) for this envelope.]

Equation (12.15) actually holds for $n < 0$, also giving

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s-n)!}\left(\frac{x}{2}\right)^{2s-n}, \tag{12.17}$$

which amounts to replacing n by $-n$ in Eq. (12.15). Since n is an integer (here), $(s-n)! \to \infty$ for $s = 0, \ldots, (n-1)$. Hence, the series may be considered to start with $s = n$. Replacing s by $s + n$, we obtain

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{s!(s+n)!}\left(\frac{x}{2}\right)^{n+2s},$$

showing that $J_n(x)$ and $J_{-n}(x)$ are not independent but are related by

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{(integral } n\text{)}. \tag{12.18}$$

These series expressions [Eqs. (12.15) and (12.17)] may be used with n replaced by ν to **define** $J_\nu(x)$ and $J_{-\nu}(x)$ for nonintegral ν (compare Exercise 12.1.11).

EXAMPLE 12.1.1

Special Values Setting $x = 0$ in Eq. (12.12), using the series [Eq. (12.9)] yields

$$1 = \sum_{n=-\infty}^{\infty} J_n(0)t^n,$$

from which we infer (uniqueness of Laurent expansion)

$$J_0(0) = 1, \qquad J_n(0) = 0 = J_{-n}(0), \quad n \ge 1.$$

From $t = 1$ we find the identity

$$1 = \sum_{n=-\infty}^{\infty} J_n(x) = J_0(x) + 2\sum_{n=1}^{\infty} J_n(x)$$

using the symmetry relation [Eq. (12.18)].

Finally, the identity $g(-x, t) = g(x, -t)$ implies

$$\sum_{n=-\infty}^{\infty} J_n(-x)t^n = \sum_{n=-\infty}^{\infty} J_n(x)(-t)^n,$$

and again the parity relations $J_n(-x) = (-1)^n J_n(x)$. These results can also be extracted from the identity $g(-x, 1/t) = g(x, t)$. ■

Applications of Recurrence Relations

We have already derived the basic recurrence relations Eqs. (12.1), (12.2), (12.6), and (12.7) that led us to the generating function. Many more can be derived as follows.

EXAMPLE 12.1.2

Addition Theorem The linearity of the generating function in the exponent x suggests the identity

$$g(u+v,t)=e^{(u+v)/2(t-1/t)}=e^{(u/2)(t-1/t)}e^{(v/2)(t-1/t)}=g(u,t)g(v,t),$$

which implies the Bessel expansions

$$\sum_{n=-\infty}^{\infty}J_n(u+v)t^n=\sum_{l=-\infty}^{\infty}J_l(u)t^l\cdot\sum_{k=-\infty}^{\infty}J_k(v)t^k=\sum_{k,l=-\infty}^{\infty}J_l(u)J_k(v)t^{l+k}$$
$$=\sum_{m=-\infty}^{\infty}t^m\sum_{l=-\infty}^{\infty}J_l(u)J_{m-l}(v)$$

denoting $m=k+l$. Comparing coefficients yields the **addition theorem**

$$J_m(u+v)=\sum_{l=-\infty}^{\infty}J_l(u)J_{m-l}(v). \tag{12.19}$$

■

Differentiating Eq. (12.16) partially with respect to x, we have

$$\frac{\partial}{\partial x}g(x,t)=\frac{1}{2}\left(t-\frac{1}{t}\right)e^{(x/2)(t-1/t)}=\sum_{n=-\infty}^{\infty}J_n'(x)t^n. \tag{12.20}$$

Again, substituting in Eq. (12.16) and equating the coefficients of like powers of t, we obtain

$$J_{n-1}(x)-J_{n+1}(x)=2J_n'(x), \tag{12.21}$$

which can also be obtained by adding Eqs. (12.2) and (12.7). As a special case of this recurrence relation,

$$J_0'(x)=-J_1(x). \tag{12.22}$$

Bessel's Differential Equation

Suppose we consider a set of functions $Z_\nu(x)$ that satisfies the basic recurrence relations [Eqs. (12.8) and (12.21)], but with ν not necessarily an integer and Z_ν not necessarily given by the series [Eq. (12.15)]. Equation (12.7) may be rewritten ($n\to\nu$) as

$$xZ_\nu'(x)=xZ_{\nu-1}(x)-\nu Z_\nu(x). \tag{12.23}$$

On differentiating with respect to x, we have

$$xZ_\nu''(x)+(\nu+1)Z_\nu'-xZ_{\nu-1}'-Z_{\nu-1}=0. \tag{12.24}$$

Multiplying by x and then subtracting Eq. (12.23) multiplied by ν gives us

$$x^2Z_\nu''+xZ_\nu'-\nu^2Z_\nu+(\nu-1)xZ_{\nu-1}-x^2Z_{\nu-1}'=0. \tag{12.25}$$

Now we rewrite Eq. (12.2) and replace n by $\nu - 1$:

$$xZ'_{\nu-1} = (\nu - 1)Z_{\nu-1} - xZ_\nu. \tag{12.26}$$

Using Eq. (12.26) to eliminate $Z_{\nu-1}$ and $Z'_{\nu-1}$ from Eq. (12.25), we finally get

$$x^2 Z''_\nu + xZ'_\nu + (x^2 - \nu^2)Z_\nu = 0. \tag{12.27}$$

This is **Bessel's ODE.** Hence, any functions, $Z_\nu(x)$, that satisfy the recurrence relations [Eqs. (12.2) and (12.7), (12.8) and (12.21), or (12.1) and (12.6)] satisfy Bessel's equation; that is, the unknown Z_ν are Bessel functions. In particular, we have shown that the functions $J_n(x)$, defined by our generating function, satisfy Bessel's ODE. Under the parity transformation, $x \to -x$, Bessel's ODE stays invariant, thereby relating $Z_\nu(-x)$ to $Z_\nu(x)$, up to a phase factor. If the argument is $k\rho$ rather than x, which is the case in many physics problems, then Eq. (12.27) becomes

$$\rho^2 \frac{d^2}{d\rho^2} Z_\nu(k\rho) + \rho \frac{d}{d\rho} Z_\nu(k\rho) + (k^2\rho^2 - \nu^2)Z_\nu(k\rho) = 0. \tag{12.28}$$

Integral Representations

A particularly useful and powerful way of treating Bessel functions employs integral representations. If we return to the generating function [Eq. (12.16)] and substitute $t = e^{i\theta}$, we get

$$\begin{aligned} e^{ix\sin\theta} &= J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots] \\ &\quad + 2i[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots], \end{aligned} \tag{12.29}$$

in which we have used the relations

$$\begin{aligned} J_1(x)e^{i\theta} + J_{-1}(x)e^{-i\theta} &= J_1(x)(e^{i\theta} - e^{-i\theta}) \\ &= 2iJ_1(x)\sin\theta, \\ J_2(x)e^{2i\theta} + J_{-2}(x)e^{-2i\theta} &= 2J_2(x)\cos 2\theta, \end{aligned} \tag{12.30}$$

and so on. In summation notation, equating real and imaginary parts of Eq. (12.29), we have

$$\begin{aligned} \cos(x\sin\theta) &= J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta), \\ \sin(x\sin\theta) &= 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin[(2n-1)\theta]. \end{aligned} \tag{12.31}$$

It might be noted that angle θ (in radians) has no dimensions, just as x. Likewise, $\sin\theta$ has no dimensions and the function $\cos(x\sin\theta)$ is perfectly proper from a dimensional standpoint.

If n and m are **positive** integers (zero is excluded),[3] we recall the orthogonality properties of cosine and sine:[4]

$$\int_0^{\pi} \cos n\theta \cos m\theta \, d\theta = \frac{\pi}{2}\delta_{nm}, \tag{12.32}$$

$$\int_0^{\pi} \sin n\theta \sin m\theta \, d\theta = \frac{\pi}{2}\delta_{nm}. \tag{12.33}$$

Multiplying Eq. (12.31) by $\cos n\theta$ and $\sin n\theta$, respectively, and integrating we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta \, d\theta = \begin{cases} J_n(x) & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{12.34}$$

$$\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta \, d\theta = \begin{cases} 0, & n \text{ even}, \\ J_n(x), & n \text{ odd}, \end{cases} \tag{12.35}$$

upon employing the orthogonality relations Eqs. (12.32) and (12.33). If Eqs. (12.34) and (12.35) are added together, we obtain

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^{\pi} [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta] d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta) d\theta, \qquad n = 0, 1, 2, 3, \ldots. \end{aligned} \tag{12.36}$$

As a special case,

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta) d\theta. \tag{12.37}$$

Noting that $\cos(x\sin\theta)$ repeats itself in all four quadrants ($\theta_1 = \theta, \theta_2 = \pi - \theta, \theta_3 = \pi + \theta, \theta_4 = -\theta$), $\cos(x\sin\theta_2) = \cos(x\sin\theta)$, etc., we may write Eq. (12.37) as

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta) d\theta. \tag{12.38}$$

On the other hand, $\sin(x\sin\theta)$ reverses its sign in the third and fourth quadrants so that

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(x\sin\theta) d\theta = 0. \tag{12.39}$$

Adding Eq. (12.38) and i times Eq. (12.39), we obtain the complex exponential representation

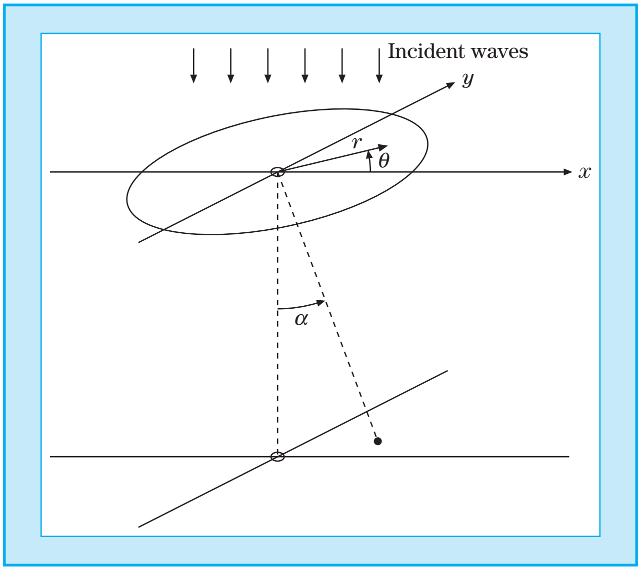
$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta} d\theta. \tag{12.40}$$

[3] Equations (12.32) and (12.33) hold for either m or $n = 0$. If both m and $n = 0$, the constant in Eq. (12.32) becomes π; the constant in Eq. (12.33) becomes 0.

[4] They are eigenfunctions of a self-adjoint equation (oscillator ODE of classical mechanics) and satisfy appropriate boundary conditions (compare Section 9.2).

Figure 12.2

Fraunhofer Diffraction-Circular Aperture



This integral representation [Eq. (12.40)] may be obtained more directly by employing contour integration.[5] Many other integral representations exist.

EXAMPLE 12.1.3

Fraunhofer Diffraction, Circular Aperture In the theory of diffraction through a circular aperture we encounter the integral

$$\Phi \sim \int_0^a \int_0^{2\pi} e^{ibr\cos\theta} d\theta r\, dr \tag{12.41}$$

for Φ, the amplitude of the diffracted wave. Here, the parameter b is defined as

$$b = \frac{2\pi}{\lambda} \sin\alpha, \tag{12.42}$$

where λ is the wavelength of the incident wave, α is the angle defined by a point on a screen below the circular aperture relative to the normal through the center point,[6] and θ is an azimuth angle in the plane of the circular aperture of radius a. The other symbols are defined in Fig. 12.2. From Eq. (12.40) we get

$$\Phi \sim 2\pi \int_0^a J_0(br) r\, dr. \tag{12.43}$$

[5]For $n = 0$ a simple integration over θ from 0 to 2π will convert Eq. (12.29) into Eq. (12.40).
[6]The exponent $ibr\cos\theta$ gives the phase of the wave on the distant screen at angle α relative to the phase of the wave incident on the aperture at the point (r, θ). The imaginary exponential form of this integrand means that the integral is technically a Fourier transform (Chapter 15). In general, the Fraunhofer diffraction pattern is given by the Fourier transform of the aperture.

Table 12.1

Zeros of the Bessel Functions and Their First Derivatives

Number of Zeros	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178
	$J_0'(x)^a$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$		
1	3.8317	1.8412	3.0542	4.2012		
2	7.0156	5.3314	6.7061	8.0152		
3	10.1735	8.5363	9.9695	11.3459		

[a] $J_0'(x) = -J_1(x)$.

Equation (12.6) enables us to integrate Eq. (12.43) immediately to obtain

$$\Phi \sim \frac{2\pi ab}{b^2} J_1(ab) \sim \frac{\lambda a}{\sin\alpha} J_1\left(\frac{2\pi a}{\lambda}\sin\alpha\right). \tag{12.44}$$

The intensity of the light in the diffraction pattern is proportional to Φ^2 and

$$\Phi^2 \sim \left\{\frac{J_1[(2\pi a/\lambda)\sin\alpha]}{\sin\alpha}\right\}^2. \tag{12.45}$$

From Table 12.1, which lists some zeros of the Bessel functions and their first derivatives,[7] Eq. (12.45) will have its smallest zero at

$$\frac{2\pi a}{\lambda}\sin\alpha = 3.8317\ldots \tag{12.46}$$

or

$$\sin\alpha = \frac{3.8317\lambda}{2\pi a}. \tag{12.47}$$

For green light $\lambda = 5.5 \times 10^{-7}$ m. Hence, if $a = 0.5$ cm,

$$\alpha \approx \sin\alpha = 6.7 \times 10^{-5}(\text{radian}) \approx 14 \text{ sec of arc}, \tag{12.48}$$

which shows that the bending or spreading of the light ray is extremely small, because most of the intensity of light is in the principal maximum. If this analysis had been known in the 17th century, the arguments against the wave theory of light would have collapsed. In the mid-20th century this same diffraction pattern appears in the scattering of nuclear particles by atomic nuclei—a striking demonstration of the wave properties of the nuclear particles. ■

[7] Additional roots of the Bessel functions and their first derivatives may be found in C. L. Beattie, Table of first 700 zeros of Bessel functions. *Bell Syst. Tech. J.* **37**, 689 (1958), and *Bell Monogr.* **3055**.

Orthogonality

If Bessel's equation [Eq. (12.28)] is divided by ρ, we see that it becomes self-adjoint, and therefore by the Sturm–Liouville theory of Section 9.2 the solutions are expected to be orthogonal—if we can arrange to have appropriate boundary conditions satisfied. To take care of the boundary conditions, for a finite interval $[0, a]$, we introduce parameters a and $\alpha_{\nu m}$ into the argument of J_ν to get $J_\nu(\alpha_{\nu m}\rho/a)$. Here, a is the upper limit of the cylindrical radial coordinate ρ. From Eq. (12.28),

$$\rho\frac{d^2}{d\rho^2}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)+\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)+\left(\frac{\alpha_{\nu m}^2\rho}{a^2}-\frac{\nu^2}{\rho}\right)J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)=0. \quad (12.49)$$

Changing the parameter $\alpha_{\nu m}$ to $\alpha_{\nu n}$, we find that $J_\nu(\alpha_{\nu n}\rho/a)$ satisfies

$$\rho\frac{d^2}{d\rho^2}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)+\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)+\left(\frac{\alpha_{\nu n}^2\rho}{a^2}-\frac{\nu^2}{\rho}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)=0. \quad (12.50)$$

Proceeding as in Section 9.2, we multiply Eq. (12.49) by $J_\nu(\alpha_{\nu n}\rho/a)$ and Eq. (12.50) by $J_\nu(\alpha_{\nu m}\rho/a)$ and subtract, obtaining

$$\begin{aligned}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]-J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\right]\\ =\frac{\alpha_{\nu n}^2-\alpha_{\nu m}^2}{a^2}\rho J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right).\end{aligned} \quad (12.51)$$

Integrating from $\rho = 0$ to $\rho = a$, we obtain

$$\begin{aligned}&\int_0^a J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]d\rho\\ &-\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\right]d\rho\\ &=\frac{\alpha_{\nu n}^2-\alpha_{\nu m}^2}{a^2}\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\rho\, d\rho.\end{aligned} \quad (12.52)$$

Upon integrating by parts, the left-hand side of Eq. (12.52) becomes

$$\rho J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\Bigg|_0^a-\rho J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\Bigg|_0^a. \quad (12.53)$$

For $\nu \geq 0$ the factor ρ guarantees a zero at the lower limit, $\rho = 0$. Actually, the lower limit on the index ν may be extended down to $\nu > -1$.[8] At $\rho = a$, each expression vanishes if we choose the parameters $\alpha_{\nu n}$ and $\alpha_{\nu m}$ to be zeros or

[8]The case $\nu = -1$ reverts to $\nu = +1$, Eq. (12.18).

roots of J_ν; that is, $J_\nu(\alpha_{\nu m}) = 0$. The subscripts now become meaningful: $\alpha_{\nu m}$ is the mth zero of J_ν.

With this choice of parameters, the left-hand side vanishes (the Sturm–Liouville boundary conditions are satisfied) and for $m \neq n$

$$\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\rho\, d\rho = 0. \tag{12.54}$$

This gives us orthogonality over the interval $[0, a]$.

Normalization

The normalization integral may be developed by returning to Eq. (12.53), setting $\alpha_{\nu n} = \alpha_{\nu m} + \varepsilon$, and taking the limit $\varepsilon \to 0$. With the aid of the recurrence relation, Eq. (12.2), the result may be written as

$$\int_0^a \left[J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]^2 \rho\, d\rho = \frac{a^2}{2}[J_{\nu+1}(\alpha_{\nu m})]^2. \tag{12.55}$$

Bessel Series

If we assume that the set of Bessel functions $J_\nu(\alpha_{\nu m}\rho/a)$($\nu$ fixed, $m = 1, 2, 3, \ldots$) is complete, then any well-behaved, but otherwise arbitrary, function $f(\rho)$ may be expanded in a Bessel series (Bessel–Fourier or Fourier–Bessel)

$$f(\rho) = \sum_{m=1}^{\infty} c_{\nu m} J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right), \qquad 0 \le \rho \le a, \quad \nu > -1. \tag{12.56}$$

The coefficients $c_{\nu m}$ are determined by using Eq. (12.55),

$$c_{\nu m} = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu m})]^2}\int_0^a f(\rho) J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\rho\, d\rho. \tag{12.57}$$

An application of the use of Bessel functions and their roots is provided by drumhead vibrations in Example 12.1.4 and the electromagnetic resonant cavity, Example 12.1.5, and the exercises of Section 12.1.

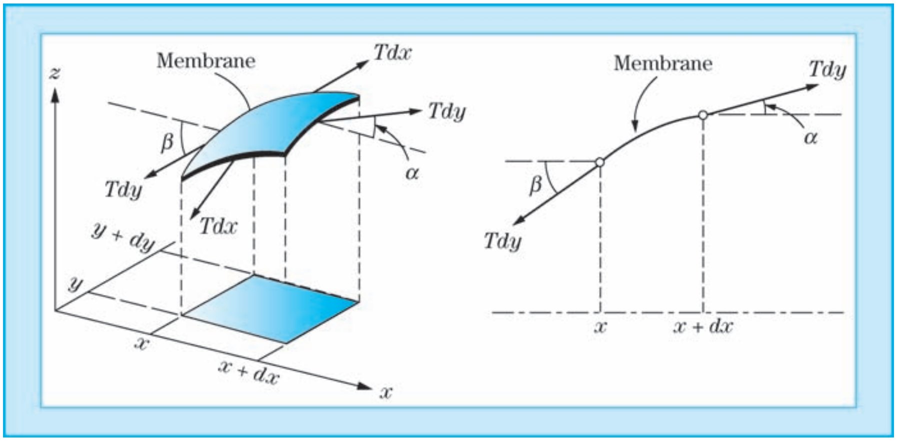
EXAMPLE 12.1.4

Vibrations of a Plane Circular Membrane Vibrating membranes are of great practical importance because they occur not only in drums but also in telephones, microphones, pumps, and other devices. We will show that their vibrations are governed by the two-dimensional wave equation and then solve this PDE in terms of Bessel functions by separating variables.

We assume that the membrane is made of elastic material of constant mass per unit area, ρ, without resistance to (slight) bending. It is stretched along all of its boundary in the xy-plane generating **constant tension** T per unit length in all relevant directions, which does not change while the membrane vibrates. The deflected membrane surface is denoted by $z = z(x, y, t)$, and $|z|$, $|\partial z/\partial x|$, $|\partial z/\partial y|$ are small compared to a, the **radius of the drumhead** for all times t. For small transverse (to the membrane surface) vibrations of a **thin elastic membrane** these assumptions are valid and lead to an accurate description of a drumhead.

Figure 12.3

Rectangular Patch of a Vibrating Membrane



To derive the PDE we analyze a small rectangular patch of lengths dx, dy and area $dx\,dy$. The forces on the sides of the patch are $T\,dx$ and $T\,dy$ acting tangentially while the membrane is deflected slightly from its horizontal equilibrium position (Fig. 12.3). The angles α, β at opposite ends of the deflected membrane patch are small so that $\cos\alpha \sim \cos\beta \sim 1$, upon keeping only terms linear in α. Hence, the horizontal force components, $T\cos\alpha$, $T\cos\beta \sim T$ are practically equal so that **horizontal movements are negligible**.

The z components of the forces are $T\,dy\,\sin\alpha$ and $-T\,dy\,\sin\beta$ at opposite sides in the y-direction, up (positive) at $x + dx$ and down (negative) at x. Their sum is

$$T\,dy(\sin\alpha - \sin\beta) \sim T\,dy(\tan\alpha - \tan\beta)$$
$$= T\,dy\left[\frac{\partial z(x + dx, y, t)}{\partial x} - \frac{\partial z(x, y, t)}{\partial x}\right]$$

because $\tan\alpha$, $\tan\beta$ are the slopes of the deflected membrane in the x-direction at $x + dx$ and x, respectively. Similarly, the sum of the forces at opposite sides in the other direction is

$$T\,dx\left[\frac{\partial z(x, y + dy, t)}{\partial y} - \frac{\partial z(x, y, t)}{\partial y}\right].$$

According to Newton's force law the sum of these forces is equal to the mass of the undeflected membrane area, $\rho\,dx\,dy$, times the acceleration; that is,

$$\rho\,dx\,dy\frac{\partial^2 z}{\partial t^2} = T\,dx\left[\frac{\partial z(x, y + dy, t)}{\partial y} - \frac{\partial z(x, y, t)}{\partial y}\right]$$
$$+ T\,dy\left[\frac{\partial z(x + dx, y, t)}{\partial x} - \frac{\partial z(x, y, t)}{\partial x}\right].$$

Dividing by the patch area $dx\,dy$ we obtain the second-order PDE

$$\frac{\rho}{T}\frac{\partial^2 z}{\partial t^2} = \frac{\frac{\partial z(x,y+dy,t)}{\partial y} - \frac{\partial z(x,y,t)}{\partial y}}{dy} + \frac{\frac{\partial z(x+dx,y,t)}{\partial x} - \frac{\partial z(x,y,t)}{\partial x}}{dx},$$

or, with the constant $c^2 = T/\rho$ and in the limit $dx, dy \to 0$

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}, \tag{12.58}$$

which is called the **two-dimensional wave equation**.

Because there are no mixed derivatives, such as $\frac{\partial^2 z}{\partial t \partial x}$, we can separate the time dependence in a **product form of the solution** $z = v(t)w(x, y)$. Substituting this z and its derivatives into our wave equation and dividing by $v(t)w(x, y)$ yields

$$\frac{1}{c^2 v(t)}\frac{\partial^2 v}{\partial t^2} = \frac{1}{w(x, y)}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right).$$

Here, the left-hand side depends on the variable t only, whereas the right-hand side contains the spatial variables only. This implies that both sides must be equal to a constant, $-k^2$, leading to the harmonic oscillator ODE in t and the two-dimensional Helmholtz equation in x and y:

$$\frac{d^2 v}{dt^2} + k^2 c^2 v(t) = 0, \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + k^2 w(x, y) = 0. \tag{12.59}$$

Further steps will depend on our boundary conditions and their symmetry. We have chosen the negative sign for the **separation constant** $-k^2$ because this sign will correspond to oscillatory solutions,

$$v(t) = A\cos(kct) + B\sin(kct),$$

in time rather than exponentially damped ones we would get for a positive separation constant. Note that, in our derivation of the wave equation, we have tacitly assumed **no damping** of the membrane (which could lead to a more complicated PDE). Besides the dynamics, this choice is also dictated by the **boundary conditions** that we have in mind and will discuss in more detail next.

The circular shape of the membrane suggests using cylindrical coordinates so that $z = z(t, \rho, \varphi)$. Invariance under rotations about the z-axis suggests no dependence of the deflected membrane on the azimuthal angle φ. Therefore, $z = z(t, \rho)$, provided the **initial** deflection (dent) and its velocity at time $t = 0$ (needed because our PDE is second order in time)

$$z(0, \rho) = f(\rho), \qquad \frac{\partial z}{\partial t}(0, \rho) = g(\rho)$$

also have no angular dependence. Moreover, the **membrane is fixed at the circular boundary** $\rho = a$ **at all times**, $z(t, a) \equiv 0$.

In cylindrical coordinates, using Eq. (2.21) for the two-dimensional Laplacian, the wave equation becomes

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 z}{\partial \varphi^2}, \tag{12.60}$$

where we delete the angular dependence, reducing the wave equation to

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial z}{\partial \rho}.$$

Now we separate variables again seeking a solution of the product form $z = v(t)w(\rho)$. Substituting this z and its derivatives into our wave equation [Eq. (12.60)] and dividing by $v(t)w(\rho)$ yields the same harmonic oscillator ODE for $v(t)$ and

$$\frac{d^2 w}{d\rho^2} + \frac{1}{\rho}\frac{dw}{d\rho} + k^2 w(\rho) = 0$$

instead of Eq. (12.59). Dimensional arguments suggest rescaling $\rho \to r = k\rho$ and dividing by k^2, yielding

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + w(r) = 0,$$

which is Bessel's ODE for $\nu = 0$.

We adopt the solution $J_0(r)$ because it is finite everywhere, whereas the second independent solution is singular at the origin, which is ruled out by our initial and boundary conditions.

The boundary condition $w(a) = 0$ requires $J_0(ka) = 0$ so that

$$k = k_n = \gamma_n/a, \quad \text{with} \quad J_0(\gamma_n) = 0, \; n = 1, 2, \ldots.$$

The zeros $\gamma_1 = 2.4048, \ldots$ are listed in Table 12.1. The general solution

$$z(t,\rho) = \sum_{n=1}^{\infty}[A_n \cos(\gamma_n ct/a) + B_n \sin(\gamma_n ct/a)]J_0(\gamma_n \rho/a) \tag{12.61}$$

follows from the superposition principle. The initial conditions at $t = 0$ require expanding

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_0(\gamma_n\rho/a), \quad g(\rho) = \sum_{n=1}^{\infty}\frac{\gamma_n c}{a}B_n J_0(\gamma_n\rho/a), \tag{12.62}$$

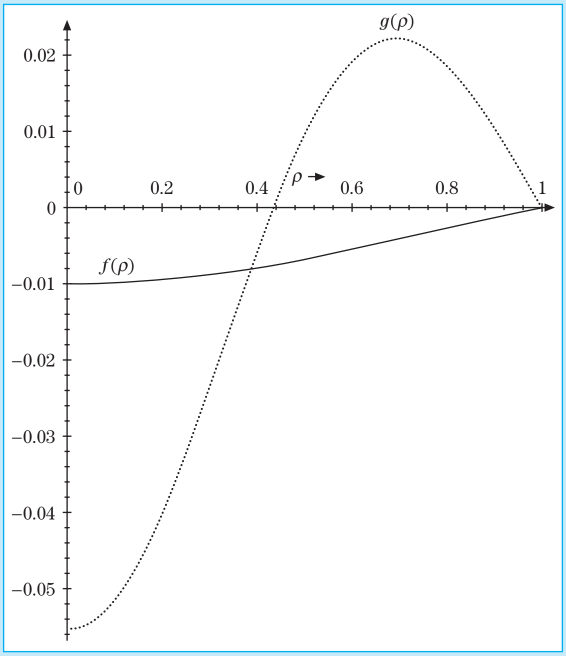
where the coefficients A_n, B_n may be obtained by projection from these Bessel series expansions of the given functions $f(\rho)$, $g(\rho)$ using orthogonality properties of the Bessel functions [see Eq. (12.57)] in the general case

$$A_n = \frac{2}{a^2[J_1(\gamma_n)]^2}\int_0^a f(\rho)J_0(\gamma_n\rho/a)\rho\, d\rho. \tag{12.63}$$

A similar relation holds for the B_n involving $g(\rho)$.

Figure 12.4

Initial Central Bang on Drumhead



To illustrate a simpler case, let us assume initial conditions

$$f(\rho) = -0.01aJ_0(\gamma_1\rho/a), \quad g(\rho) = -0.1\frac{\gamma_2 c}{a}J_0(\gamma_2\rho/a),$$

corresponding to an initial central bang on the drumhead (Fig. 12.4) at $t = 0$. Then our complete solution (with $c^2 = T/\rho$)

$$z(t,\rho) = -0.01aJ_0\left(2.4048\frac{\rho}{a}\right)\cos\frac{2.4048ct}{a} - 0.1J_0\left(5.5201\frac{\rho}{a}\right)\sin\frac{5.5201ct}{a}$$

contains only the $n = 1$ and $n = 2$ terms as dictated by the initial conditions (Fig. 12.5). ■

EXAMPLE 12.1.5

Cylindrical Resonant Cavity The propagation of electromagnetic waves in hollow metallic cylinders is important in many practical devices. If the cylinder has end surfaces, it is called a cavity. Resonant cavities play a crucial role in many particle accelerators.

We take the z-axis along the center of the cavity with end surfaces at $z = 0$ and $z = l$ and use cylindrical coordinates suggested by the geometry. Its walls are perfect conductors so that the tangential electric field vanishes on them (as in Fig. 12.6):

$$E_z = 0 \quad \text{for} \quad \rho = a, \quad E_\rho = 0 = E_\varphi \quad \text{for} \quad z = 0, l. \tag{12.64}$$

Figure 12.5

Drumhead Solution, $a = 1, c = 0.1$

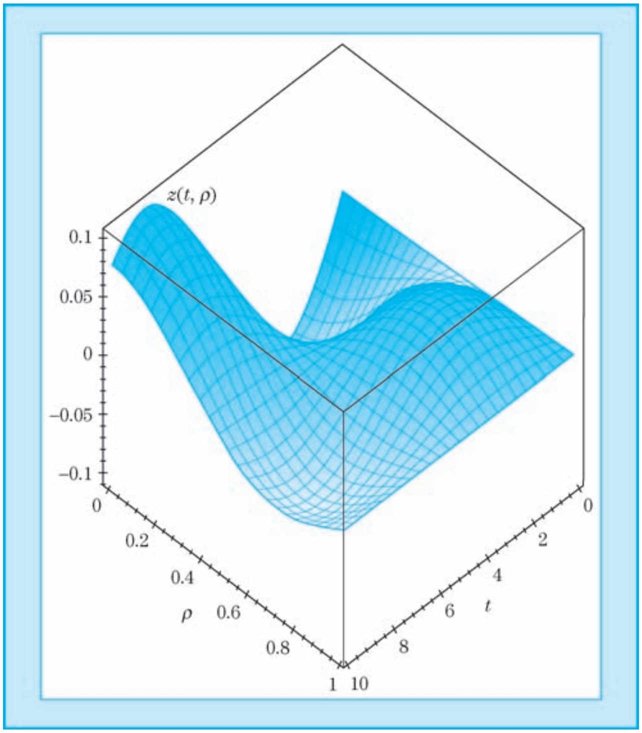
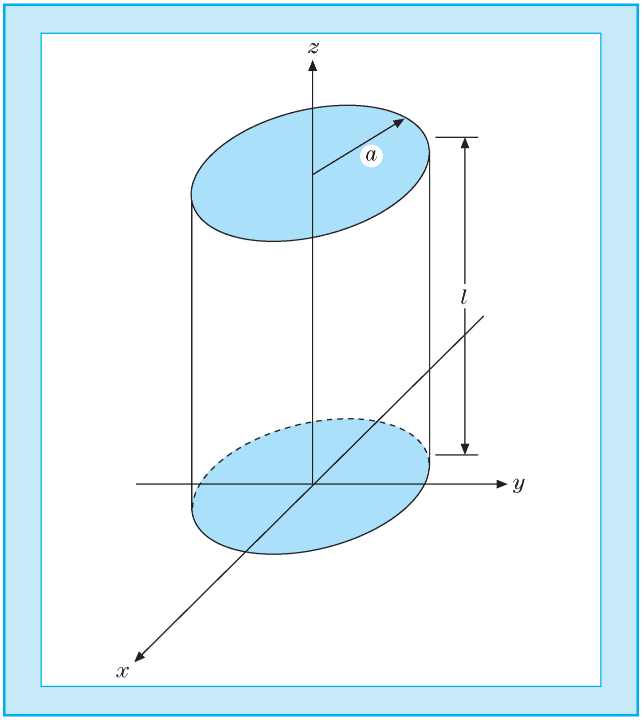


Figure 12.6

Cylindrical Resonant Cavity



Inside the cavity we have a vacuum so that $\varepsilon_0\mu_0 = 1/c^2$. In the interior of a resonant cavity electromagnetic waves oscillate with harmonic time dependence $e^{-i\omega t}$, which follows from separating the time from the spatial variables in Maxwell's equations (Section 1.8) so that

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = k_0^2 \mathbf{E}, \quad k_0^2 = \omega^2/c^2.$$

With $\nabla \cdot \mathbf{E} = 0$ (vacuum, no charges) and Eq. (1.96), we obtain for the space part of the electric field

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0,$$

which is called the vector Helmholtz PDE. The z component (E_z, space part only) satisfies the scalar Helmholtz equation [three-dimensional generalization of Eq. (12.59) in Example 12.1.4]

$$\nabla^2 E_z + k_0^2 E_z = 0. \tag{12.65}$$

The transverse electric field components $\mathbf{E}_\perp = (E_\rho, E_\varphi)$ obey the same PDE but different boundary conditions given previously.

As in Example 12.1.4, we can separate the z variable from ρ and φ because there are no mixed derivatives $\frac{\partial^2 E_z}{\partial z \partial \rho}$, etc. The product solution $E_z = v(\rho, \varphi)w(z)$ is substituted into Eq. (12.65) using Eq. (2.21) for ∇^2 in cylindrical coordinates; then we divide by vw, yielding

$$-\frac{1}{w(z)}\frac{d^2 w}{dz^2} = \frac{1}{v(\rho,\varphi)}\left(\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \varphi^2} + k_0^2 v\right) = k^2,$$

where k^2 is the separation constant because the left- and right-hand sides depend on different variables. For $w(z)$ we find the harmonic oscillator ODE with standing wave solution

$$w(z) = A \sin kz + B \cos kz,$$

with A, B constants. For $v(\rho, \varphi)$ we obtain

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \varphi^2} + \gamma^2 v = 0, \quad \gamma^2 = k_0^2 - k^2.$$

In this PDE we can separate the ρ and φ variables because there is no mixed term $\frac{\partial^2 v}{\partial \rho \partial \varphi}$. The product form $v = u(\rho)\Phi(\varphi)$ yields

$$\frac{\rho^2}{u(\rho)}\left(\frac{d^2 u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} + \gamma^2 u\right) = -\frac{1}{\Phi(\varphi)}\frac{d^2\Phi}{d\varphi^2} = m^2,$$

where the **separation constant** m^2 **must be an integer** because the angular solution, $\Phi = e^{im\varphi}$ of the ODE

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0,$$

must be periodic in the azimuthal angle.

This leaves us with the radial ODE

$$\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} + \left(\gamma^2 - \frac{m^2}{\rho^2}\right)^u = 0.$$

Dimensional arguments suggest rescaling $\rho \to r = \gamma\rho$ and dividing by γ^2, which yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(1 - \frac{m^2}{r^2}\right)^u = 0.$$

This is Bessel's ODE for $\nu = m$.

We use the regular solution $J_m(\gamma\rho)$ because the (irregular) second independent solution is singular at the origin, which is unacceptable. The complete solution is

$$E_z = J_m(\gamma\rho)e^{im\varphi}(A\sin kz + B\cos kz), \tag{12.66}$$

where the constant γ is determined from the **boundary condition** $E_z = 0$ on the cavity surface $\rho = a$ (i.e., that γa be a root of the Bessel function J_m) (Table 12.1). This gives a discrete set of values $\gamma = \gamma_{mn}$, where n designates the nth root of J_m (Table 12.1).

For the transverse magnetic (TM) mode of oscillation with $H_z = 0$, Maxwell's equations imply (see "Resonant Cavities" in J. D. Jackson's *Electrodynamics*)

$$\mathbf{E}_\perp \sim \boldsymbol{\nabla}_\perp \frac{\partial E_z}{\partial z}, \quad \boldsymbol{\nabla}_\perp = \left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\varphi}\right).$$

The form of this result suggests $E_z \sim \cos kz$, that is, setting $A = 0$, so that $\mathbf{E}_\perp \sim \sin kz = 0$ at $z = 0, l$ can be satisfied by

$$k = \frac{p\pi}{l}, \quad p = 0, 1, 2, \ldots. \tag{12.67}$$

Thus, the **tangential** electric fields E_ρ and E_φ vanish at $z = 0$ and l. In other words, $A = 0$ corresponds to $dE_z/dz = 0$ at $z = 0$ and $z = a$ for the TM mode. Altogether then, we have

$$\gamma^2 = \frac{\omega^2}{c^2} - k^2 = \frac{\omega^2}{c^2} - \frac{p^2\pi^2}{l^2}, \tag{12.68}$$

with

$$\gamma = \gamma_{mn} = \frac{\alpha_{mn}}{a}, \tag{12.69}$$

where α_{mn} is the nth zero of J_m. The general solution

$$E_z = \sum_{m,n,p} J_m(\gamma_{mn}\rho)e^{\pm im\varphi} B_{mnp} \cos\frac{p\pi z}{l}, \tag{12.70}$$

with constants B_{mnp}, now follows from the superposition principle.

The consequence of the two boundary conditions and the separation constant m^2 is that the angular frequency of our oscillation depends on three discrete parameters:

$$\omega_{mnp} = c\sqrt{\frac{\alpha_{mn}^2}{a^2} + \frac{p^2\pi^2}{l^2}}, \qquad \begin{cases} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \\ p = 0, 1, 2, \ldots \end{cases} \tag{12.71}$$

These are the allowed resonant frequencies for the TM mode. Feynman *et al.*[9] develops Bessel functions from cavity resonators. ■

Bessel Functions of Nonintegral Order

The generating function approach is very convenient for deriving two recurrence relations, Bessel's differential equation, integral representations, addition theorems (Example 12.1.2), and upper and lower bounds (Exercise 12.1.1). However, the generating function defines only Bessel functions of integral order J_0, J_1, J_2, and so on. This is a limitation of the generating function approach that can be avoided by using a contour integral (Section 12.3) instead. However, the Bessel function of the first kind, $J_\nu(x)$, may easily be defined for nonintegral ν by using the series [Eq. (12.15)] as a new definition.

We have verified the recurrence relations by substituting in the series form of $J_\nu(x)$. From these relations Bessel's equation follows. In fact, if ν is not an integer, there is actually an important simplification. It is found that J_ν and $J_{-\nu}$ are independent because no relation of the form of Eq. (12.18) exists. On the other hand, for $\nu = n$, an integer, we need another solution. The development of this second solution and an investigation of its properties are the subject of Section 12.2.

EXERCISES

12.1.1 From the product of the generating functions $g(x,t)\cdot g(x,-t)$ show that

$$1 = [J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots$$

and therefore that $|J_0(x)| \le 1$ and $|J_n(x)| \le 1/\sqrt{2}$, $n = 1, 2, 3, \ldots$.
Hint. Use uniqueness of power series (Section 5.7).

[9] Feynman, R. P., Leighton, R. B., and Sands, M. (1964). *The Feynman Lectures on Physics*, Vol. 2, Chap. 23. Addison-Wesley, Reading, MA.

12.1.2 Derive the Jacobi–Anger expansion

$$e^{iz\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(z) e^{im\theta}.$$

This is an expansion of a plane wave in a series of cylindrical waves.

12.1.3 Show that

(a) $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$,

(b) $\sin x = 2\sum_{n=1}^{\infty}(-1)^{n+1} J_{2n+1}(x)$.

12.1.4 Prove that

$$\frac{\sin x}{x} = \int_0^{\pi/2} J_0(x\cos\theta)\cos\theta\, d\theta, \qquad \frac{1-\cos x}{x} = \int_0^{\pi/2} J_1(x\cos\theta)\, d\theta.$$

Hint. The definite integral

$$\int_0^{\pi/2} \cos^{2s+1}\theta\, d\theta = \frac{2\cdot 4\cdot 6\cdots(2s)}{1\cdot 3\cdot 5\cdots(2s+1)}$$

may be useful.

12.1.5 Show that

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\, dt.$$

This integral is a Fourier cosine transform (compare Section 15.4). The corresponding Fourier sine transform,

$$J_0(x) = \frac{2}{\pi}\int_1^{\infty} \frac{\sin xt}{\sqrt{t^2-1}}\, dt,$$

is established using a Hankel function integral representation.

12.1.6 Derive

$$J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x).$$

Hint. Rewrite the recursion so that it serves to step from n to $n+1$ in a proof by mathematical induction.

12.1.7 Show that between any two consecutive zeros of $J_n(x)$ there is one and only one zero of $J_{n+1}(x)$.
Hint. Equations (12.1) and (12.6) may be useful.

12.1.8 An analysis of antenna radiation patterns for a system with a circular aperture involves the equation

$$g(u) = \int_0^1 f(r) J_0(ur)\, r\, dr.$$

If $f(r) = 1 - r^2$, show that

$$g(u) = \frac{2}{u^2} J_2(u).$$

12.1.9 The differential cross section in a nuclear scattering experiment is given by $d\sigma/d\Omega = |f(\theta)|^2$. An approximate treatment leads to

$$f(\theta) = \frac{-ik}{2\pi}\int_0^{2\pi}\int_0^R \exp[ik\rho\sin\theta\sin\varphi]\rho\, d\rho\, d\varphi,$$

where θ is an angle through which the scattered particle is scattered, and R is the nuclear radius. Show that

$$\frac{d\sigma}{d\Omega} = (\pi R^2)\frac{1}{\pi}\left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2.$$

12.1.10 A set of functions $C_n(x)$ satisfies the recurrence relations

$$C_{n-1}(x) - C_{n+1}(x) = \frac{2n}{x}C_n(x),$$

$$C_{n-1}(x) + C_{n+1}(x) = 2C_n'(x).$$

(a) What linear second-order ODE does the $C_n(x)$ satisfy?
(b) By a change of variable, transform your ODE into Bessel's equation. This suggests that $C_n(x)$ may be expressed in terms of Bessel functions of transformed argument.

12.1.11 (a) From

$$J_\nu(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\nu \int t^{-\nu-1}e^{t-x^2/4t}dt$$

with a suitably defined contour, derive the recurrence relation

$$J_\nu'(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x).$$

(b) From

$$J_\nu(x) = \frac{1}{2\pi i}\int t^{-\nu-1}e^{(x/2)(t-1/t)}\, dt$$

with the same contour, derive the recurrence relation

$$J_\nu'(x) = \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)].$$

12.1.12 Show that the recurrence relation

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

follows directly from differentiation of

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta.$$

12.1.13 Evaluate

$$\int_0^\infty e^{-ax}J_0(bx)\, dx, \qquad a, b > 0.$$

Actually the results hold for $a \geq 0$, $-\infty < b < \infty$. This is a Laplace transform of J_0.

Hint. Either an integral representation of J_0 or a series expansion or a Laplace transformation of Bessel's ODE will be helpful.

12.1.14 Using trigonometric forms [Eq. (12.29)], verify that

$$J_0(br) = \frac{1}{2\pi}\int_0^{2\pi} e^{ibr\sin\theta}\,d\theta.$$

12.1.15 The fraction of light incident on a circular aperture (normal incidence) that is transmitted is given by

$$T = \int_0^{2ka} J_2(x)\left(\frac{2}{x} - \frac{1}{2ka}\right)dx,$$

where a is the radius of the aperture, and k is the wave number, $2\pi/\lambda$. Show that

(a) $T = 1 - \dfrac{1}{ka}\displaystyle\sum_{n=0}^{\infty} J_{2n+1}(2ka)$, (b) $T = 1 - \dfrac{1}{2ka}\displaystyle\int_0^{2ka} J_0(x)\,dx$.

12.1.16 Show that, defining

$$I_{m,n}(a) \equiv \int_0^a x^m J_n(x)\,dx, \qquad m \geq n \geq 0,$$

(a) $I_{3,0}(x) = \displaystyle\int_0^x t^3 J_0(t)\,dt = x^3 J_1(x) - 2x^2 J_2(x)$;

(b) is integrable in terms of Bessel functions and powers of x [such as $a^p J_q(a)$] for $m + n$ odd;

(c) may be reduced to integrated terms plus $\int_0^a J_0(x)\,dx$ for $m + n$ even.

12.1.17 Solve the ODE $x^2 y''(x) + a x y'(x) + (1 + b^2 x^2) y(x) = 0$, where $a,\ b$ are real parameters, using the substitution $y(x) = x^{-n} v(x)$. Adjust n and a so that the ODE for v becomes Bessel's ODE for J_0. Find the general solution $y(x)$ for this value of a.

12.2 Neumann Functions, Bessel Functions of the Second Kind

From the theory of ODEs it is known that Bessel's second-order ODE has two independent solutions. Indeed, for nonintegral order ν we have already found two solutions and labeled them $J_\nu(x)$ and $J_{-\nu}(x)$ using the infinite series [Eq. (12.15)]. The trouble is that when ν is integral, Eq. (12.18) holds and we have but one independent solution. A second solution may be developed by the methods of Section 8.6. This yields a perfectly good second solution of Bessel's equation but is not the standard form.

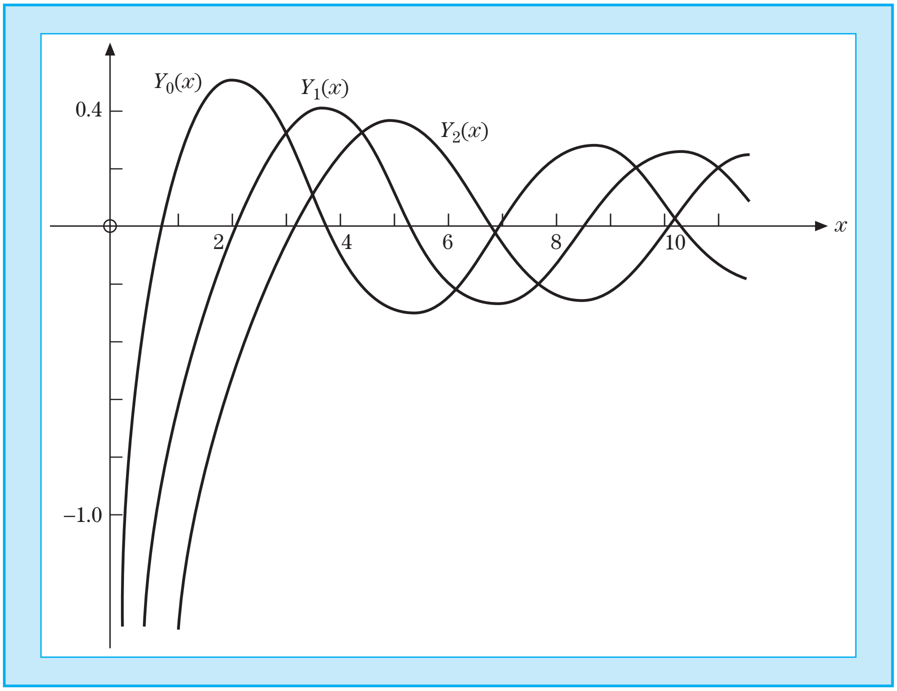
Definition and Series Form

As an alternate approach, we take the particular linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}. \tag{12.72}$$

Figure 12.7

Neumann Functions $Y_0(x)$, $Y_1(x)$, and $Y_2(x)$



This is the Neumann function (Fig. 12.7).[10] For nonintegral ν, $Y_\nu(x)$ clearly satisfies Bessel's equation because it is a linear combination of known solutions, $J_\nu(x)$ and $J_{-\nu}(x)$. Substituting the power series [Eq. (12.15)] for $n \to \nu$ yields

$$Y_\nu(x) = -\frac{(\nu-1)!}{\pi}\left(\frac{2}{x}\right)^\nu + \cdots \tag{12.73}$$

for $\nu > 0$. However, for integral ν, Eq. (12.18) applies and Eq. (12.58)[11] becomes indeterminate. The definition of $Y_\nu(x)$ was chosen deliberately for this indeterminate property. Again substituting the power series and evaluating $Y_\nu(x)$ for $\nu \to 0$ by l'Hôpital's rule for indeterminate forms, we obtain the limiting value

$$Y_0(x) = \frac{2}{\pi}(\ln x + \gamma - \ln 2) + \mathcal{O}(x^2) \tag{12.74}$$

for $n = 0$ and $x \to 0$, using

$$\nu!(-\nu)! = \frac{\pi\nu}{\sin\pi\nu} \tag{12.75}$$

from Eq. (10.32). The first and third terms in Eq. (12.74) come from using $(d/d\nu)(x/2)^\nu = (x/2)^\nu \ln(x/2)$, whereas γ comes from $(d/d\nu)\nu!$ for $\nu \to 0$

[10] We use the notation in AMS-55 and in most mathematics tables.

[11] Note that this limiting form applies to both integral and nonintegral values of the index ν.

using Eqs. (10.38) and (10.40). For $n > 0$ we obtain similarly

$$Y_n(x) = -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n + \cdots + \frac{2}{\pi}\left(\frac{x}{2}\right)^n \frac{1}{n!} \ln\left(\frac{x}{2}\right) + \cdots. \tag{12.76}$$

Equations (12.74) and (12.76) exhibit the logarithmic dependence that was to be expected. This, of course, verifies the independence of J_n and Y_n.

Other Forms

As with all the other Bessel functions, $Y_\nu(x)$ has integral representations. For $Y_0(x)$ we have

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\, dt = -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{(t^2-1)^{1/2}}\, dt, \qquad x > 0.$$

These forms can be derived as the imaginary part of the Hankel representations of Section 12.3. The latter form is a Fourier cosine transform.

The most general solution of Bessel's ODE for any ν can be written as

$$y(x) = AJ_\nu(x) + BY_\nu(x). \tag{12.77}$$

It is seen from Eqs. (12.74) and (12.76) that Y_n diverges at least logarithmically. Some boundary condition that requires the solution of a problem with Bessel function solutions to be finite at the origin automatically excludes $Y_n(x)$. Conversely, in the absence of such a requirement $Y_n(x)$ must be considered.

Biographical Data

Neumann, Karl. Neumann, a German mathematician and physicist, was born in 1832 and died in 1925. He was appointed a professor of mathematics at the University of Leipzig in 1868. His main contributions were to potential theory and partial differential and integral equations.

Recurrence Relations

Substituting Eq. (12.72) for $Y_\nu(x)$ (nonintegral ν) or Eq. (12.76) (integral ν) into the recurrence relations [Eqs. (12.8) and (12.21)] for $J_n(x)$, we see immediately that $Y_\nu(x)$ satisfies these same recurrence relations. This actually constitutes another proof that Y_ν is a solution. Note that the converse is not necessarily true. All solutions need not satisfy the same recurrence relations because Y_ν for nonintegral ν also involves $J_{-\nu} \neq J_\nu$ obeying recursions with $\nu \to -\nu$.

Wronskian Formulas

From Section 8.6 and Exercise 9.1.3 we have the Wronskian formula[12] for solutions of the Bessel equation

$$u_\nu(x)v_\nu'(x) - u_\nu'(x)v_\nu(x) = \frac{A_\nu}{x}, \tag{12.78}$$

[12]This result depends on $P(x)$ of Section 8.5 being equal to $p'(x)/p(x)$, the corresponding coefficient of the self-adjoint form of Section 9.1.

in which A_ν is a parameter that depends on the particular Bessel functions $u_\nu(x)$ and $v_\nu(x)$ being considered. It is a constant in the sense that it is independent of x. Consider the special case

$$u_\nu(x) = J_\nu(x), \quad v_\nu(x) = J_{-\nu}(x), \tag{12.79}$$

$$J_\nu J'_{-\nu} - J'_\nu J_{-\nu} = \frac{A_\nu}{x}. \tag{12.80}$$

Since A_ν is a constant, it may be identified using the leading terms in the power series expansions [Eqs. (12.15) and (12.17)]. All other powers of x cancel. We obtain

$$\begin{aligned} J_\nu &\to x^\nu/(2^\nu \nu!), & J_{-\nu} &\to 2^\nu x^{-\nu}/(-\nu)!, \\ J'_\nu &\to \nu x^{\nu-1}/(2^\nu \nu!), & J'_{-\nu} &\to -\nu 2^\nu x^{-\nu-1}/(-\nu)!. \end{aligned} \tag{12.81}$$

Substitution into Eq. (12.80) yields

$$J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x) = \frac{-2\nu}{x\nu!(-\nu)!} = -\frac{2\sin\nu\pi}{\pi x}. \tag{12.82}$$

Note that A_ν vanishes for integral ν, as it must since the nonvanishing of the Wronskian is a test of the independence of the two solutions. By Eq. (12.18), J_n and J_{-n} are clearly linearly dependent.

Using our recurrence relations, we may readily develop a large number of alternate forms, among which are

$$J_\nu J_{-\nu+1} + J_{-\nu}J_{\nu-1} = \frac{2\sin\nu\pi}{\pi x}, \tag{12.83}$$

$$J_\nu J_{-\nu-1} + J_{-\nu}J_{\nu+1} = -\frac{2\sin\nu\pi}{\pi x}, \tag{12.84}$$

$$J_\nu Y'_\nu - J'_\nu Y_\nu = \frac{2}{\pi x}, \tag{12.85}$$

$$J_\nu Y_{\nu+1} - J_{\nu+1}Y_\nu = -\frac{2}{\pi x}. \tag{12.86}$$

Many more can be found in Additional Reading.

The reader will recall that in Chapter 8 Wronskians were of great value in two respects: (i) in establishing the linear independence or linear dependence of solutions of differential equations and (ii) in developing an integral form of a second solution. Here, the specific forms of the Wronskians and Wronskian-derived combinations of Bessel functions are useful primarily to illustrate the general behavior of the various Bessel functions. Wronskians are of great use in checking tables of Bessel functions numerically.

EXAMPLE 12.2.1

Coaxial Waveguides We are interested in an electromagnetic wave confined between the concentric, conducting cylindrical surfaces $\rho = a$ and $\rho = b$. Most of the mathematics is worked out in Example 12.1.5. That is, we work in cylindrical coordinates and separate the time dependence as before, which is that of a traveling wave $e^{i(kz-\omega t)}$ now instead of standing waves in Example 12.1.5.

To implement this, we let $A = iB$ in the solution $w(z) = A\sin kz + B\cos kz$ and obtain

$$E_z = \sum_{m,n} b_{mn} J_m(\gamma\rho) e^{\pm im\varphi} e^{i(kz-\omega t)}. \tag{12.87}$$

For the coaxial waveguide both the Bessel and Neumann functions contribute because the origin $\rho = 0$ can no longer be used to exclude the Neumann functions because it is not part of the physical region ($0 < a \le \rho \le b$). It is consistent that there are now two boundary conditions, at $\rho = a$ and $\rho = b$. With the Neumann function $Y_m(\gamma\rho)$, $E_z(\rho, \varphi, z, t)$ becomes

$$E_z = \sum_{m,n} [b_{mn} J_m(\gamma_{mn}\rho) + c_{mn} Y_m(\gamma_{mn}\rho)] e^{\pm im\varphi} e^{i(kz-\omega t)}, \tag{12.88}$$

where γ_{mn} will be determined from boundary conditions. With the transverse magnetic field condition

$$H_z = 0 \tag{12.89}$$

everywhere, we have the basic equations for a TM wave.

The (tangential) electric field must vanish at the conducting surfaces (Dirichlet boundary condition), or

$$b_{mn} J_m(\gamma_{mn} a) + c_{mn} Y_m(\gamma_{mn} a) = 0, \tag{12.90}$$

$$b_{mn} J_m(\gamma_{mn} b) + c_{mn} Y_m(\gamma_{mn} b) = 0. \tag{12.91}$$

For a nontrivial solution b_{mn}, c_{mn} of these homogeneous linear equations to exist, their determinant must be zero. The resulting transcendental equation, $J_m(\gamma_{mn} a) Y_m(\gamma_{mn} b) = J_m(\gamma_{mn} b) Y_m(\gamma_{mn} a)$, may be solved for γ_{mn}, and then the ratio c_{mn}/b_{mn} can be determined. From Example 12.1.5,

$$k^2 = \omega^2 \mu_0 \varepsilon_0 - \gamma_{mn}^2 = \frac{\omega^2}{c^2} - \gamma_{mn}^2, \tag{12.92}$$

where c is the velocity of light. Since k^2 must be positive for an oscillatory solution, the minimum frequency that will be propagated (in this TM mode) is

$$\omega = \gamma_{mn} c, \tag{12.93}$$

with γ_{mn} fixed by the boundary conditions, Eqs. (12.90) and (12.91). This is the cutoff frequency of the waveguide. In general, at any given frequency only a finite number of modes can propagate. The dimensions ($a < b$) of the cylindrical guide are often chosen so that, at given frequency, only the lowest mode k can propagate.

There is also a transverse electric mode with $E_z = 0$ and H_z given by Eq. (12.88). ■

SUMMARY

To conclude this discussion of Neumann functions, we introduce the Neumann function, $Y_\nu(x)$, for the following reasons:

1. It is a second, independent solution of Bessel's equation, which completes the general solution.
2. It is required for specific physical problems, such as electromagnetic waves in coaxial cables and quantum mechanical scattering theory.
3. It leads directly to the two Hankel functions (Section 12.3).

EXERCISES

12.2.1 Prove that the Neumann functions Y_n (with n an integer) satisfy the recurrence relations

$$Y_{n-1}(x) + Y_{n+1}(x) = \frac{2n}{x} Y_n(x)$$

$$Y_{n-1}(x) - Y_{n+1}(x) = 2Y_n'(x).$$

Hint. These relations may be proved by differentiating the recurrence relations for J_ν or by using the limit form of Y_ν but **not** dividing everything by zero.

12.2.2 Show that

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

12.2.3 Show that

$$Y_0'(x) = -Y_1(x).$$

12.2.4 If Y and Z are any two solutions of Bessel's equation, show that

$$Y_\nu(x)Z_\nu'(x) - Y_\nu'(x)Z_\nu(x) = \frac{A_\nu}{x},$$

in which A_ν may depend on ν but is independent of x. This is a special case of Exercise 9.1.3.

12.2.5 Verify the Wronskian formulas

$$J_\nu(x)J_{-\nu+1}(x) + J_{-\nu}(x)J_{\nu-1}(x) = \frac{2\sin\nu\pi}{\pi x},$$

$$J_\nu(x)Y_\nu'(x) - J_\nu'(x)Y_\nu(x) = \frac{2}{\pi x}.$$

12.2.6 As an alternative to letting x approach zero in the evaluation of the Wronskian constant, we may invoke uniqueness of power series (Section 5.7). The coefficient of x^{-1} in the series expansion of $u_\nu(x)v_\nu'(x) - u_\nu'(x)v_\nu(x)$ is then A_ν. Show by series expansion that the coefficients of x^0 and x^1 of $J_\nu(x)J_{-\nu}'(x) - J_\nu'(x)J_{-\nu}(x)$ are each zero.

12.2.7 (a) By differentiating and substituting into Bessel's ODE, show that

$$\int_0^\infty \cos(x\cosh t)\,dt$$

is a solution.
Hint. You can rearrange the final integral as

$$\int_0^\infty \frac{d}{dt}\{x\sin(x\cosh t)\sinh t\}\,dt.$$

(b) Show that

$$Y_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt$$

is linearly independent of $J_0(x)$.

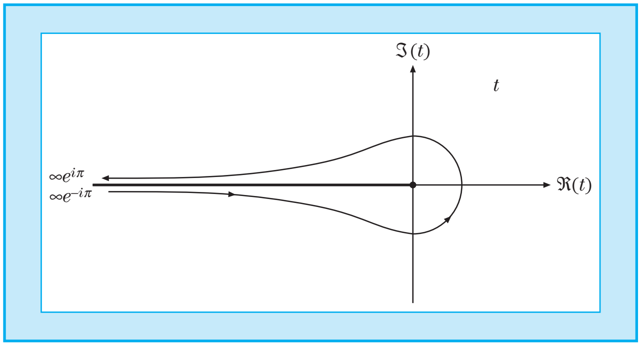
12.3 Asymptotic Expansions

Frequently, in physical problems there is a need to know how a given Bessel function behaves for large values of the argument, that is, the asymptotic behavior. This is one occasion when computers are not very helpful, except in matching numerical solutions to known asymptotic forms or checking an asymptotic guess numerically. One possible approach is to develop a power series solution of the differential equation, as in Section 8.5, but now using negative powers. This is Stokes's method. The limitation is that starting from some positive value of the argument (for convergence of the series), we do not know what mixture of solutions or multiple of a given solution we have. The problem is to relate the asymptotic series (useful for large values of the variable) to the power series or related definition (useful for small values of the variable). This relationship can be established by introducing a suitable **integral representation** and then using either the method of steepest descent (Section 7.3) or the direct expansion as developed in this section.

Integral representations have appeared before: Eq. (10.35) for $\Gamma(z)$ and various representations of $J_\nu(z)$ in Section 12.1. With these integral representations of the Bessel (and Hankel) functions, it is perhaps appropriate to ask why we are interested in integral representations. There are at least four reasons. The first is simply aesthetic appeal. Second, the integral representations help to distinguish between two linearly independent solutions (Section 7.3). Third, the integral representations facilitate manipulations, analysis, and the development of relations among the various special functions. Fourth, and probably most important, the integral representations are extremely useful in developing asymptotic expansions. One approach, the method of steepest descents, appears in Section 7.3 and is used here.

Figure 12.8

Bessel Function Contour



Hankel functions are introduced here for the following reasons:

- As Bessel function analogs of $e^{\pm ix}$ they are useful for describing traveling waves.
- They offer an alternate (contour integral) and an elegant definition of Bessel functions.

Expansion of an Integral Representation

As a direct approach, consider the integral representation (Schlaefli integral)

$$J_\nu(z) = \frac{1}{2\pi i}\int_C e^{(z/2)(t-1/t)} t^{-\nu-1} dt, \tag{12.94}$$

with the contour C around the origin in the positive mathematical sense displayed in Fig. 12.8. This formula follows from Cauchy's theorem, applied to the defining Eq. (12.9) of the generating function given by Eq. (12.16) as the exponential in the integral. This proves Eq. (12.94) for $-\pi < \arg z < 2\pi$, but only for $\nu =$ integer. If ν is not an integer, the integrand is not single-valued and a cut line is needed in our complex t plane. Choosing the negative real axis as the cut line and using the contour shown in Fig. 12.8, we can extend Eq. (12.94) to nonintegral ν. For this case, we still need to verify Bessel's ODE by substituting the integral representation [Eq. (12.94)],

$$\begin{aligned} &z^2 J''(z)_\nu + zJ'_\nu(z) + (z^2-\nu^2)J_\nu(z) \\ &= \frac{1}{2\pi i}\int_C e^{(z/2)(t-1/t)} t^{-\nu-1}\left[\frac{z^2}{4}\left(t+\frac{1}{t}\right)^2 + \frac{z}{2}\left(t-\frac{1}{t}\right) - \nu^2\right] dt, \end{aligned} \tag{12.95}$$

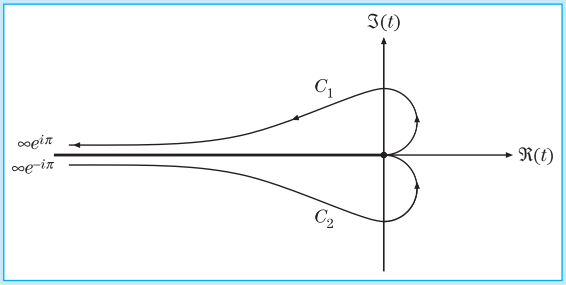
where the integrand can be verified to be the following exact derivative that vanishes as $t \to \infty e^{\pm i\pi}$:

$$\frac{d}{dt}\left\{\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] t^{-\nu}\left[\nu + \frac{z}{2}\left(t+\frac{1}{t}\right)\right]\right\}. \tag{12.96}$$

Hence, the integral in Eq. (12.95) vanishes and Bessel's ODE is satisfied.

Figure 12.9

Hankel Function Contours



We now deform the contour so that it approaches the origin along the positive real axis, as shown in Fig. 12.9. This particular approach guarantees that the exact derivative in Eq. (12.96) will vanish as $t \to 0$ because of the $e^{-z/2t}$ factor. Hence, each of the separate portions corresponding to $\infty e^{-i\pi}$ to 0 and 0 to $\infty e^{i\pi}$ is a solution of Bessel's ODE. We define

$$H_\nu^{(1)}(z) = \frac{1}{\pi i} \int_0^{\infty e^{i\pi}} e^{(z/2)(t-1/t)} \frac{dt}{t^{\nu+1}}, \tag{12.97}$$

$$H_\nu^{(2)}(z) = \frac{1}{\pi i} \int_{\infty e^{-i\pi}}^{0} e^{(z/2)(t-1/t)} \frac{dt}{t^{\nu+1}} \tag{12.98}$$

so that

$$J_\nu(z) = \frac{1}{2}\left[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right]. \tag{12.99}$$

These expressions are particularly convenient because they may be handled by the method of steepest descents (Section 7.3). $H_\nu^{(1)}(z)$ has a saddle point at $t = +i$, whereas $H_\nu^{(2)}(z)$ has a saddle point at $t = -i$. To leading order Eq. (7.84) yields

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left(i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right) \tag{12.100}$$

for large $|z|$ in the region $-\pi < \arg z < 2\pi$. The second Hankel function is just the complex conjugate of the first (for real argument z) so that

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left(-i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right) \tag{12.101}$$

for large $|z|$ with $-2\pi < \arg z < \pi$.

In addition to Eq. (12.99) we can also show that

$$Y_\nu(z) = \frac{1}{2i}\left[H_\nu^{(1)}(z) - H_\nu^{(2)}(z)\right]. \tag{12.102}$$

This may be accomplished by the following steps:

1. With the substitutions $t = e^{i\pi}/s$ for $H_\nu^{(1)}$ in Eq. (12.97) and $t = e^{-i\pi}/s$ for $H_\nu^{(2)}$ in Eq. (12.98), we obtain

$$H_\nu^{(1)}(z) = e^{-i\nu\pi} H_{-\nu}^{(1)}(z), \tag{12.103}$$

$$H_\nu^{(2)}(z) = e^{i\nu\pi} H_{-\nu}^{(2)}(z). \tag{12.104}$$

2. From Eqs. (12.99) ($\nu \to -\nu$), (12.103), and (12.104), we get

$$J_{-\nu}(z) = \frac{1}{2}\left[e^{i\nu\pi} H_\nu^{(1)}(z) + e^{-i\nu\pi} H_\nu^{(2)}(z)\right]. \tag{12.105}$$

3. Finally, substitute J_ν [Eq. (12.99)] and $J_{-\nu}$ [Eq. (12.105)] into the defining equation for Y_ν, Eq. (12.72). This leads to Eq. (12.102) and establishes the contour integrals [Eqs. (12.97) and (12.98)] as the standard Hankel functions.

Since $J_\nu(z)$ is the real part of $H_\nu^{(1)}(z)$ for real z,

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] \tag{12.106}$$

for large $|z|$ with $-\pi < \arg z < \pi$. The Neumann function is the imaginary part of $H_\nu^{(1)}(z)$ or

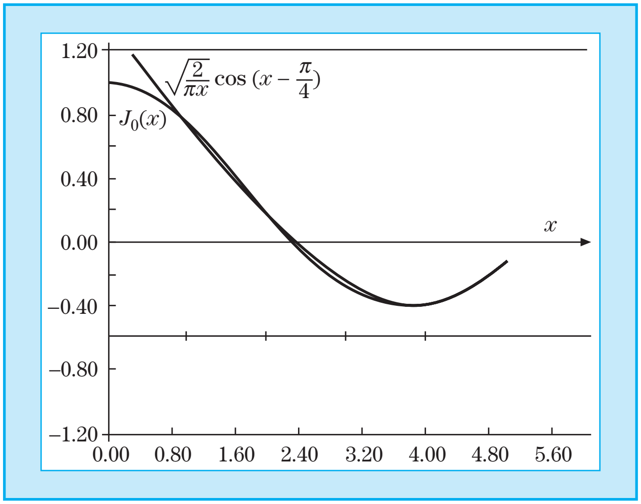
$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] \tag{12.107}$$

for large $|z|$ with $-\pi < \arg z < \pi$.

It is of interest to consider the accuracy of the asymptotic forms, taking only the first term [Eq. (12.106)], for example (Fig. 12.10). Clearly, the condition for the validity of Eq. (12.106) is that the next (a nonleading sine) term in Eq. (12.106) be negligible; estimating this leads to $8x \gg 4n^2 - 1$.

Figure 12.10

Asymptotic Approximation of $J_0(x)$



To a certain extent the definition of the Neumann function $Y_n(x)$ is arbitrary. Equations (12.72) and (12.76) contain terms of the form $a_n J_n(x)$. Clearly, any finite value of the constant a_n would still give us a second solution of Bessel's equation. Why should a_n have the particular value implicit in Eqs. (12.72) and (12.76)? The answer is given by the asymptotic dependence developed here. If J_n corresponds to a cosine wave asymptotically [Eq. (12.106)], then Y_n corresponds to a sine wave [Eq. (12.107)]. This simple and convenient asymptotic phase relationship is a consequence of the particular admixture of J_n in Y_n.

This completes our determination of the asymptotic expansions. However, it is worth noting the primary characteristics. Apart from the ubiquitous $z^{-1/2}$, $J_\nu(z)$, and $Y_\nu(z)$ behave as cosine and sine, respectively. The zeros are **almost** evenly spaced at intervals of π; the spacing becomes exactly π in the limit as $z \to \infty$. The Hankel functions have been defined to behave like the imaginary exponentials. This asymptotic behavior may be sufficient to eliminate immediately one of these functions as a solution for a physical problem. This is illustrated in the next example.

EXAMPLE 12.3.1

Cylindrical Traveling Waves As an illustration of the use of Hankel functions, consider a two-dimensional wave problem similar to the vibrating circular membrane of Example 12.1.4. Now imagine that the waves are generated at $\rho = 0$ and move outward to infinity. We replace our standing waves by traveling ones. The differential equation remains the same, but the boundary conditions change. We now demand that for large ρ the solution behaves like

$$U \to e^{i(k\rho - \omega t)} \tag{12.108}$$

to describe an outgoing wave. As before, k is the wave number. This assumes, for simplicity, that there is no azimuthal dependence, that is, no angular momentum, or $m = 0$. In Sections 7.4 and 12.3, $H_0^{(1)}(k\rho)$ is shown to have the asymptotic behavior (for $\rho \to \infty$)

$$H_0^{(1)}(k\rho) \to e^{ik\rho}. \tag{12.109}$$

This boundary condition at infinity then determines our wave solution as

$$U(\rho, t) = H_0^{(1)}(k\rho) e^{-i\omega t}. \tag{12.110}$$

This solution diverges as $\rho \to 0$, which is just the behavior to be expected with a source at the origin representing a singularity reflected by singular behavior of the solution. ■

The choice of a two-dimensional wave problem to illustrate the Hankel function $H_0^{(1)}(z)$ is not accidental. Bessel functions may appear in a variety of ways, such as in the separation in conical coordinates. However, they enter most commonly from the radial equations from the separation of variables in the Helmholtz equation in cylindrical and in spherical polar coordinates. We have used a degenerate form of cylindrical coordinates for this illustration.

Had we used spherical polar coordinates (spherical waves), we should have encountered index $\nu = n + \frac{1}{2}$, n an integer. These special values yield the spherical Bessel functions discussed in Section 12.4.

Finally, as pointed out in Section 12.2, the asymptotic forms may be used to evaluate the various Wronskian formulas (compare Exercise 12.3.2).

Numerical Evaluation

When a computer program calls for one of the Bessel or modified Bessel functions, the programmer has two alternatives: to store all the Bessel functions and tell the computer how to locate the required value or to instruct the computer to simply calculate the needed value. The first alternative would be fairly slow and would place unreasonable demands on storage capacity. Thus, our programmer adopts the "compute it yourself" alternative.

Let us discuss the computation of $J_n(x)$ using the recurrence relation [Eq. (12.8)]. Given J_0 and J_1, for example, J_2 (and any other integral order J_n) may be computed from Eq. (12.8). With the opportunities offered by computers, Eq. (12.8) has acquired an interesting new application. In computing a numerical value of $J_N(x_0)$ for a given x_0, one could use the series form of Eq. (12.15) for small x or the asymptotic form [Eq. (12.106)] for large x. A better way, in terms of accuracy and machine utilization, is to use the recurrence relation [Eq. (12.8)] and work **down**.[13] With $n \gg N$ and $n \gg x_0$, assume

$$J_{n+1}(x_0) = 0 \quad \text{and} \quad J_n(x_0) = \alpha,$$

where α is some small number. Then Eq. (12.8) leads to $J_{n-1}(x_0)$, $J_{n-2}(x_0)$, and so on, and finally to $J_0(x_0)$. Since α is arbitrary, the J_n are all off by a common factor. This factor is determined by the condition

$$J_0(x_0) + 2\sum_{m=1}^{\infty} J_{2m}(x_0) = 1.$$

(See Example 12.1.1.) The accuracy of this calculation is checked by trying again at $n' = n + 3$. This technique yields the desired $J_N(x_0)$ and all the lower integral index J_n down to J_0, and it avoids the fatal **accumulation of rounding errors in a recursion relation that works up**. High-precision numerical computation is more or less an art. Modifications and refinements of this and other numerical techniques are proposed every year. For information on the current "state of the art," the student will have to consult the literature, such as *Numerical Recipes* in Additional Reading of Chapter 8, *Atlas for Computing Mathematical Functions* in Additional Reading of Chapter 13, or the journal *Mathematics of Computation*.

[13]Stegun, I.A., and Abramowitz, M. (1957). Generation of Bessel functions on computers. *Math. Tables Aids Comput.* **11**, 255–257.

Table 12.2

Equations for the Computation of Neumann Functions

Note: In practice, it is convenient to limit the series (power or asymptotic) computation of $Y_n(x)$ to $n = 0, 1$. Then $Y_n(x)$, $n \geq 2$ is computed using the recurrence relation, Eq. (12.8).

	Power Series	Asymptotic Series
$Y_n(x)$	Eq. (12.76), $x \leq 4$	Eq. (12.107), $x > 4$

For Y_n, the preferred methods are the series if x is small and the asymptotic forms (with many terms in the series of negative powers) if x is large. The criteria of large and small may vary as shown in Table 12.2.

EXERCISES

12.3.1 In checking the normalization of the integral representation of $J_\nu(z)$ [Eq. (12.94)], we assumed that $Y_\nu(z)$ was not present. How do we know that the integral representation [Eq. (12.94)] does not yield $J_\nu(z) + \varepsilon Y_\nu(z)$ with $\varepsilon \neq 0$ albeit small?

12.3.2 Use the asymptotic expansions to verify the following Wronskian formulas:

(a) $J_\nu(x)J_{-\nu-1}(x) + J_{-\nu}(x)J_{\nu+1}(x) = -2\sin \nu\pi/\pi x$,
(b) $J_\nu(x)Y_{\nu+1}(x) - J_{\nu+1}(x)Y_\nu(x) = -2/\pi x$,
(c) $J_\nu(x)H_{\nu}^{(2)}(x) - J_{\nu-1}(x)H_\nu^{(2)}(x) = 2/i\pi x$.

12.3.3 Stokes's method.

(a) Replace the Bessel function in Bessel's equation by $x^{-1/2}y(x)$ and show that $y(x)$ satisfies

$$y''(x) + \left(1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) y(x) = 0.$$

(b) Develop a power series solution with negative powers of x starting with the assumed form

$$y(x) = e^{ix}\sum_{n=0}^{\infty} a_n x^{-n}.$$

Determine the recurrence relation giving a_{n+1} in terms of a_n. Check your result against the asymptotic formula, Eq. (12.106).

(c) From the results of Section 7.3, determine the initial coefficient, a_0.

12.3.4 (a) Write a subroutine that will generate Bessel functions $J_n(x)$, that is, will generate the numerical value of $J_n(x)$ given x and n.

(b) Check your subroutine by using symbolic software, such as Maple and Mathematica. If possible, compare the machine time needed for this check for several n and x with the time required for your subroutine.

12.4 Spherical Bessel Functions

When the Helmholtz equation is separated in spherical coordinates the radial equation has the form

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}+[k^2r^2-n(n+1)]R=0. \tag{12.111}$$

This is Eq. (8.62) of Section 8.5, as we now show. The parameter k enters from the original Helmholtz equation, whereas $n(n+1)$ is a separation constant. From the behavior of the polar angle function (Legendre's equation; Sections 4.3, 8.9, and 11.2), the separation constant must have this form, with n a nonnegative integer. Equation (12.111) has the virtue of being self-adjoint, but clearly it is not Bessel's ODE. However, if we substitute

$$R(kr)=\frac{Z(kr)}{(kr)^{1/2}},$$

Equation (12.111) becomes

$$r^2\frac{d^2Z}{dr^2}+r\frac{dZ}{dr}+\left[k^2r^2-\left(n+\frac{1}{2}\right)^2\right]Z=0, \tag{12.112}$$

which **is Bessel's equation**. Z is a Bessel function of order $n+\frac{1}{2}$ (n an integer). Because of the importance of spherical coordinates, this combination,

$$\frac{Z_{n+1/2}(kr)}{(kr)^{1/2}},$$

occurs often in physics problems.

Definitions

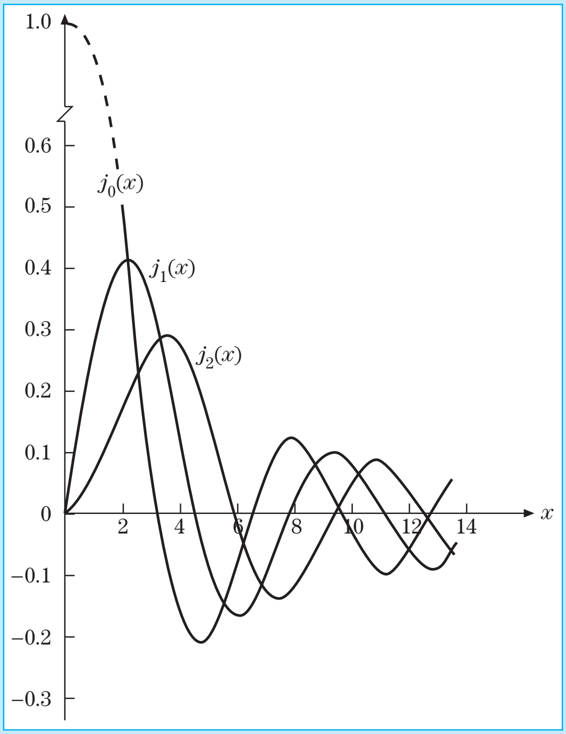
It is convenient to label these functions spherical Bessel functions, with the following defining equations:

$$\begin{aligned}
j_n(x)&=\sqrt{\frac{\pi}{2x}}J_{n+1/2}(x),\\
y_n(x)&=\sqrt{\frac{\pi}{2x}}Y_{n+1/2}(x)=(-1)^{n+1}\sqrt{\frac{\pi}{2x}}J_{-n-1/2}(x),^{14}\\
h_n^{(1)}(x)&=\sqrt{\frac{\pi}{2x}}H^{(1)}_{n+1/2}(x)=j_n(x)+iy_n(x),\\
h_n^{(2)}(x)&=\sqrt{\frac{\pi}{2x}}H^{(2)}_{n+1/2}(x)=j_n(x)-iy_n(x).
\end{aligned} \tag{12.113}$$

[14]This is possible because $\cos(n+\frac{1}{2})\pi=0$ for n an integer.

Figure 12.11

Spherical Bessel Functions



These spherical Bessel functions (Figs. 12.11 and 12.12) can be expressed in series form by using the series [Eq. (12.15)] for J_n, replacing n with $n+\frac{1}{2}$:

$$J_{n+1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\left(s+n+\frac{1}{2}\right)!} \left(\frac{x}{2}\right)^{2s+n+1/2}. \tag{12.114}$$

Using the Legendre duplication formula (Section 10.1),

$$z!\left(z+\frac{1}{2}\right)! = 2^{-2z-1}\pi^{1/2}(2z+1)!, \tag{12.115}$$

we have

$$\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} \frac{(-1)^s 2^{2s+2n+1}(s+n)!}{\pi^{1/2}(2s+2n+1)!s!}\left(\frac{x}{2}\right)^{2s+n+1/2} \\ &= 2^n x^n \sum_{s=0}^{\infty} \frac{(-1)^s (s+n)!}{s!(2s+2n+1)!} x^{2s}. \end{aligned} \tag{12.116}$$

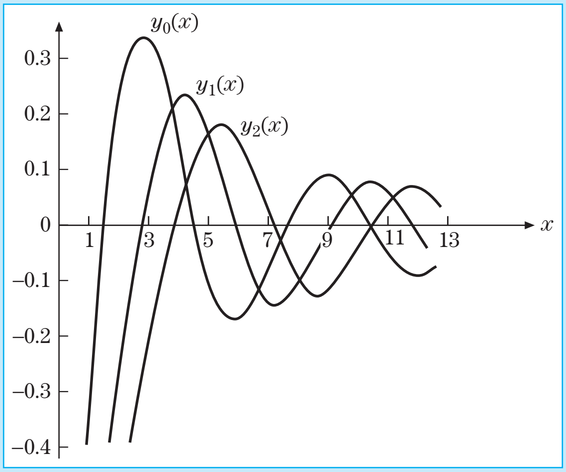


Figure 12.12

Spherical Neumann Functions

Now

$$Y_{n+1/2}(x) = (-1)^{n+1} J_{-n-1/2}(x) = (-1)^{n+1} j_{-n-1}(x) \sqrt{\frac{2x}{\pi}} \tag{12.117}$$

from Eq. (12.72), and from Eq. (12.15) we find that

$$J_{-n-1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\left(s-n-\frac{1}{2}\right)!} \left(\frac{x}{2}\right)^{2s-n-1/2}. \tag{12.118}$$

This yields

$$y_n(x) = (-1)^{n+1} \frac{2^n \pi^{1/2}}{x^{n+1}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\left(s-n-\frac{1}{2}\right)!} \left(\frac{x}{2}\right)^{2s}. \tag{12.119}$$

We can also use Eq. (12.117), replacing $n \to -n-1$ in Eq. (12.116) to give

$$y_n(x) = \frac{(-1)^{n+1}}{2^n x^{n+1}} \sum_{s=0}^{\infty} \frac{(-1)^s (s-n)!}{s!(2s-2n)!} x^{2s}. \tag{12.120}$$

These series forms, Eqs. (12.116) and (12.120), are useful in two ways:

- limiting values as $x \to 0$,
- closed form representations for $n = 0$.

For the special case $n = 0$ we find from Eq. (12.116)

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}, \tag{12.121}$$

whereas for y_0, Eq. (12.120) yields

$$y_0(x) = -\frac{\cos x}{x}. \tag{12.122}$$

From the definition of the spherical Hankel functions [Eq. (12.113)],

$$h_0^{(1)}(x) = \frac{1}{x}(\sin x - i\cos x) = -\frac{i}{x}e^{ix}$$

$$h_0^{(2)}(x) = \frac{1}{x}(\sin x + i\cos x) = \frac{i}{x}e^{-ix}. \tag{12.123}$$

Equations (12.121) and (12.122) suggest expressing all spherical Bessel functions as combinations of sine, cosine, and inverse powers of x. The appropriate combinations can be developed from the power series solutions, Eqs. (12.116) and (12.120), but this approach is awkward. We will use recursion relations instead.

Limiting Values

For $x \ll 1$,[15] Eqs. (12.116) and (12.120) yield

$$j_n(x) \approx \frac{2^n n!}{(2n+1)!}x^n = \frac{x^n}{(2n+1)!!} \tag{12.124}$$

$$y_n(x) \approx \frac{(-1)^{n+1}}{2^n} \cdot \frac{(-n)!}{(-2n)!}x^{-n-1}$$

$$= -\frac{(2n)!}{2^n n!}x^{-n-1} = -(2n-1)!!x^{-n-1}. \tag{12.125}$$

The transformation of factorials in the expressions for $y_n(x)$ employs Exercise 10.1.3. The limiting values of the spherical Hankel functions go as $\pm i y_n(x)$.

The asymptotic values of j_n, y_n, $h_n^{(2)}$, and $h_n^{(1)}$ may be obtained from the Bessel asymptotic forms (Section 12.3). We find

$$j_n(x) \sim \frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right), \tag{12.126}$$

$$y_n(x) \sim -\frac{1}{x}\cos\left(x - \frac{n\pi}{2}\right), \tag{12.127}$$

$$h_n^{(1)}(x) \sim (-i)^{n+1}\frac{e^{ix}}{x} = -i\frac{e^{i(x-n\pi/2)}}{x}, \tag{12.128a}$$

$$h_n^{(2)}(x) \sim i^{n+1}\frac{e^{-ix}}{x} = i\frac{e^{-i(x-n\pi/2)}}{x}. \tag{12.128b}$$

The condition for these spherical Bessel forms is that $x \gg n(n+1)/2$. From these asymptotic values we see that $j_n(x)$ and $y_n(x)$ are appropriate for a description of **standing spherical waves**; $h_n^{(1)}(x)$ and $h_n^{(2)}(x)$ correspond to **traveling spherical waves**. If the time dependence for the traveling waves is taken to be $e^{-i\omega t}$, then $h_n^{(1)}(x)$ yields an outgoing traveling spherical wave and

[15] The condition that the second term in the series be negligible compared to the first is actually $x \ll 2[(2n+2)(2n+3)/(n+1)]^{1/2}$ for $j_n(x)$.

$h_n^{(2)}(x)$ an incoming wave. Radiation theory in electromagnetism and scattering theory in quantum mechanics provide many applications.

EXAMPLE 12.4.1

Particle in a Sphere An illustration of the use of the spherical Bessel functions is provided by the problem of a quantum mechanical particle in a sphere of radius a. Quantum theory requires that the wave function ψ, describing our particle, satisfy

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi, \tag{12.129}$$

and the boundary conditions (i) $\psi(r \le a)$ remains finite, (ii) $\psi(a) = 0$. This corresponds to a square well potential $V = 0, r \le a$, and $V = \infty, r > a$. Here, $\hbar$ is Planck's constant divided by 2π, m is the mass of our particle, and E is its energy. Let us determine the **minimum** value of the energy for which our wave equation has an acceptable solution. Equation (12.129) is Helmholtz's equation with a radial part (compare Example 12.1.4):

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{n(n+1)}{r^2}\right]R = 0, \tag{12.130}$$

with $k^2 = 2mE/\hbar^2$. Hence, by Eq. (12.111), with $n = 0$,

$$R = Aj_0(kr) + By_0(kr).$$

We choose the orbital angular momentum index $n = 0$. Any angular dependence would raise the energy because of the repulsive angular momentum barrier [involving $n(n+1) > 0$]. The spherical Neumann function is rejected because of its divergent behavior at the origin. To satisfy the second boundary condition (for all angles), we require

$$ka = \frac{\sqrt{2mE}}{\hbar}a = \alpha, \tag{12.131}$$

where α is a root of j_0; that is, $j_0(\alpha) = 0$. This has the effect of limiting the allowable energies to a certain discrete set; in other words, application of boundary condition (ii) quantizes the energy E. The smallest α is the first zero of j_0,

$$\alpha = \pi$$

and

$$E_{\min} = \frac{\pi^2\hbar^2}{2ma^2} = \frac{h^2}{8ma^2}, \tag{12.132}$$

which means that for any finite sphere the particle energy will have a positive minimum or zero point energy. Compare this energy with $E = \frac{h^2}{8m}(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2})$ of an infinite rectangular square well of lengths $a,\ b,\ c$. This example is an illustration of the Heisenberg uncertainty principle for $\Delta p \sim \hbar\pi/a$ from de Broglie's relation and $\Delta r \sim a$ so that $\Delta p \Delta r \sim h/2$. ■

Recurrence Relations

The recurrence relations to which we now turn provide a convenient way of developing the higher order spherical Bessel functions. These recurrence relations may be derived from the series, but as with the modified Bessel functions, it is easier to substitute into the known recurrence relations [Eqs. (12.8) and (12.21)]. This gives

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x), \tag{12.133}$$

$$n f_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) f_n'(x). \tag{12.134}$$

Rearranging these relations [or substituting into Eqs. (12.1) and (12.6)], we obtain

$$\frac{d}{dx}[x^{n+1} f_n(x)] = x^{n+1} f_{n-1}(x) \tag{12.135}$$

$$\frac{d}{dx}[x^{-n} f_n(x)] = -x^{-n} f_{n+1}(x), \tag{12.136}$$

where f_n may represent j_n, y_n, $h_n^{(1)}$, or $h_n^{(2)}$.

Specific forms may also be readily obtained from Eqs. (12.133) and (12.134):

$$h_1^{(1)}(x) = e^{ix}\left(-\frac{1}{x} - \frac{i}{x^2}\right), \tag{12.137a}$$

$$h_2^{(1)}(x) = e^{ix}\left(\frac{i}{x} - \frac{3}{x^2} - \frac{3i}{x^3}\right), \tag{12.137b}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x},$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x, \tag{12.138}$$

$$y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x},$$

$$y_2(x) = -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x, \tag{12.139}$$

and so on.

By mathematical induction one may establish the Rayleigh formulas

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right), \tag{12.140}$$

$$y_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right), \tag{12.141}$$

$$h_n^{(1)}(x) = -i(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right),$$

$$h_n^{(2)}(x) = i(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{-ix}}{x}\right). \tag{12.142}$$

Numerical Computation

The spherical Bessel functions are computed using the same techniques described in Sections 12.1 and 12.3 for evaluating the Bessel functions. For $j_n(x)$ it is convenient to use Eq. (12.133) and work **downward**, as is done for $J_n(x)$. Normalization is accomplished by comparing with the known forms of $j_0(x)$, Eqs. (12.121) and Exercise 12.4.11. For $y_n(x)$, Eq. (12.120) is used again, but this time working upward, starting with the known forms of $y_0(x)$, $y_1(x)$ [Eqs. (12.122) and (12.139)].

EXAMPLE 12.4.2

Phase Shifts Here we show that spherical Bessel functions are needed to define phase shifts for scattering from a spherically symmetric potential $V(r)$. The radial wave function $R_l(r) = u(r)/r$ of the Schrödinger equation satisfies the ODE

$$\left[-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}V(r) - k^2\right]R_l(r) = 0, \qquad (12.143)$$

where m is the reduced mass, $E = \hbar^2k^2/2m$ the energy eigenvalue, and the scattering potential V goes to zero (exponentially) for $r \to \infty$. At large values of r, where V is negligible, this ODE is identical to Eqs. (12.111), with the general solution a linear combination of the regular and irregular solutions; that is,

$$R_l(r) = A_l j_l(kr) + B_l y_l(kr), \quad r \to \infty. \qquad (12.144)$$

Using the asymptotic expansions Eqs. (12.126) and (12.127) we have

$$R_l(r) \sim A_l\frac{\sin(kr - l\pi/2)}{kr} - B_l\frac{\cos(kr - l\pi/2)}{kr}, \quad r \to \infty. \qquad (12.145)$$

If there is no scattering, that is, $V(r) = 0$, then the incident plane wave is our solution and $B_l = 0$ because it has no y_l contribution, being finite everywhere. Therefore, $B_l/A_l \equiv -\tan\delta_l(k)$ is a measure for the amount of scattering at momentum $\hbar k$.

Because we can rewrite Eq. (12.145) as

$$R_l(r) \sim C_l\frac{\sin(kr - l\pi/2 + \delta_l)}{kr}, \quad C_l = \frac{A_l}{\cos\delta_l}, \qquad (12.146)$$

δ_l is called the phase shift; it depends on the incident energy.

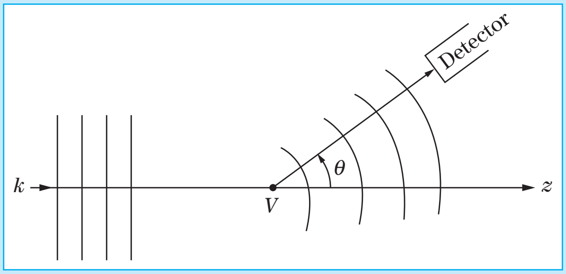
We expand the scattering wave function ψ in Legendre polynomials $P_l(\cos\theta)$ with the scattering angle θ defined by the position $\mathbf{r}$ of the detector (Fig. 12.13). Then we compare with the asymptotic expression of the wave function

$$\psi(\mathbf{r}) \sim e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

by substituting the Rayleigh expansion (Exercise 12.4.21) for the incident plane wave and replacing the spherical Bessel functions in this expansion by their asymptotic form. For further details, see Griffiths, *Introduction to Quantum Mechanics*, Section 11.2. Prentice-Hall, New York (1994). As a result, one finds

Figure 12.13

Incident Plane Wave Is Scattered by a Potential V into an Outgoing Radial Wave



the partial wave expansion of the scattering amplitude as the coefficient of the outgoing radial wave,

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\sin\delta_l P_l(\cos\theta). \tag{12.147}$$

Upon integrating $|f(\theta)|^2$ over the scattering angle using the orthogonality of Legendre polynomials we obtain the total scattering cross section

$$\sigma = \int |f(\theta|^2\, d\Omega = \frac{4\pi}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l. \tag{12.148}$$

SUMMARY

Bessel functions of integer order are defined by a power series expansion of their generating function

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

Bessel and Neumann functions are the regular and irregular solutions of Bessel's ODE, which arises in the separation of variables in prominent PDEs, such as Laplace's equation with spherical or cylindrical symmetry and the heat and Helmholtz equations. Many of the properties of Bessel functions are consequences of the Sturm–Liouville theory of their differential equation.

EXERCISES

12.4.1 Show that if

$$y_n(x) = \sqrt{\frac{\pi}{2x}}Y_{n+1/2}(x),$$

it automatically equals

$$(-1)^{n+1}\sqrt{\frac{\pi}{2x}}J_{-n-1/2}(x).$$

12.4.2 Derive the trigonometric–polynomial forms of $j_n(z)$ and $y_n(z)$:[16]

$$j_n(z) = \frac{1}{z}\sin\left(z - \frac{n\pi}{2}\right)\sum_{s=0}^{[n/2]} \frac{(-1)^s (n+2s)!}{(2s)!(2z)^{2s}(n-2s)!}$$
$$+ \frac{1}{z}\cos\left(z - \frac{n\pi}{2}\right)\sum_{s=0}^{[(n-1)/2]} \frac{(-1)^s (n+2s+1)!}{(2s+1)!(2z)^{2s}(n-2s-1)!}$$
$$y_n(z) = \frac{(-1)^{n+1}}{z}\cos\left(z + \frac{n\pi}{2}\right)\sum_{s=0}^{[n/2]} \frac{(-1)^s (n+2s)!}{(2s)!(2z)^{2s}(n-2s)!}$$
$$+ \frac{(-1)^{n+1}}{z}\sin\left(z + \frac{n\pi}{2}\right)\sum_{s=0}^{[(n-1)/2]} \frac{(-1)^s (n+2s+1)!}{(2s+1)!(2z)^{2s+1}(n-2s-1)!}.$$

12.4.3 Use the integral representation of $J_\nu(x)$,

$$J_\nu(x) = \frac{1}{\pi^{1/2}\left(\nu - \frac{1}{2}\right)!}\left(\frac{x}{2}\right)^\nu \int_{-1}^{1} e^{\pm ixp}(1-p^2)^{\nu-1/2}\,dp,$$

to show that the spherical Bessel functions $j_n(x)$ are expressible in terms of trigonometric functions; that is, for example,

$$j_0(x) = \frac{\sin x}{x}, \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}.$$

12.4.4 (a) Derive the recurrence relations

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x),$$
$$n f_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) f_n'(x),$$

satisfied by the spherical Bessel functions $j_n(x)$, $y_n(x)$, $h_n^{(1)}(x)$, and $h_n^{(2)}(x)$.

(b) Show from these two recurrence relations that the spherical Bessel function $f_n(x)$ satisfies the differential equation

$$x^2 f_n''(x) + 2x f_n'(x) + [x^2 - n(n+1)] f_n(x) = 0.$$

12.4.5 Prove by mathematical induction that

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

for n an arbitrary nonnegative integer.

12.4.6 From the discussion of orthogonality of the spherical Bessel functions, show that a Wronskian relation for $j_n(x)$ and $y_n(x)$ is

$$j_n(x) y_n'(x) - j_n'(x) y_n(x) = \frac{1}{x^2}.$$

[16]The upper limit on the summation $[n/2]$ means the largest **integer** that does not exceed $n/2$.

12.4.7 Verify

$$h_n^{(1)}(x)h_n^{(2)\prime}(x) - h_n^{(1)\prime}(x)h_n^{(2)}(x) = -\frac{2i}{x^2}.$$

12.4.8 Verify Poisson's integral representation of the spherical Bessel function,

$$j_n(z) = \frac{z^n}{2^{n+1}n!}\int_0^\pi \cos(z\cos\theta)\sin^{2n+1}\theta\, d\theta.$$

12.4.9 Show that

$$\int_0^\infty J_\mu(x)J_\nu(x)\frac{dx}{x} = \frac{2}{\pi}\frac{\sin[(\mu-\nu)\pi/2]}{\mu^2-\nu^2}, \qquad \mu+\nu>0.$$

12.4.10 Derive

$$\int_{-\infty}^\infty j_m(x)j_n(x)\,dx = 0, \qquad \begin{array}{l} m \neq n \\ m, n \geq 0. \end{array}$$

12.4.11 Derive

$$\int_{-\infty}^\infty [j_n(x)]^2\,dx = \frac{\pi}{2n+1}.$$

12.4.12 Set up the orthogonality integral for $j_l(kr)$ in a sphere of radius R with the boundary condition

$$j_l(kR) = 0.$$

The result is used in classifying electromagnetic radiation according to its angular momentum l.

12.4.13 The Fresnel integrals (Fig. 12.14) occurring in diffraction theory are given by

$$x(t) = \int_0^t \cos(v^2)\,dv, \qquad y(t) = \int_0^t \sin(v^2)\,dv.$$

Show that these integrals may be expanded in series of spherical Bessel functions

$$x(s) = \frac{1}{2}\int_0^s j_{-1}(u)u^{1/2}\,du = s^{1/2}\sum_{n=0}^\infty j_{2n}(s),$$

$$y(s) = \frac{1}{2}\int_0^s j_0(u)u^{1/2}\,du = s^{1/2}\sum_{n=0}^\infty j_{2n+1}(s).$$

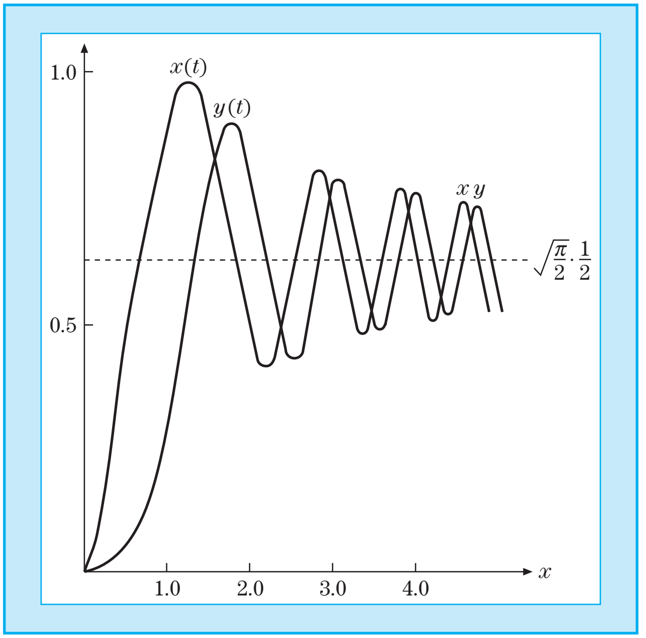
Hint. To establish the equality of the integral and the sum, you may wish to work with their derivatives. The spherical Bessel analogs of Eqs. (12.7) and (12.21) are helpful.

12.4.14 A hollow sphere of radius a (Helmholtz resonator) contains standing sound waves. Find the minimum frequency of oscillation in terms of the radius a and the velocity of sound v. The sound waves satisfy the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

Figure 12.14

Fresnel Integrals



and the boundary condition

$$\frac{\partial \psi}{\partial r} = 0, \qquad r = a.$$

This is a Neumann boundary condition. Example 12.4.1 has the same PDE but with a Dirichlet boundary condition.

ANS. $\nu_{\text{min}} = 0.3313v/a, \quad \lambda_{\text{max}} = 3.018a.$

12.4.15 A quantum particle is trapped in a spherically symmetric well of radius a. The Schrödinger equation potential is

$$V(r) = \begin{cases} -V_0, & 0 \le r < a \\ 0, & r > a. \end{cases}$$

The particle's energy E is negative (an eigenvalue).

(a) Show that the radial part of the wave function is given by $j_l(k_1 r)$ for $0 \le r < a$ and $j_l^{\text{out}}(k_2 r)$ for $r > a$. [We require that $\psi(0)$ and $\psi(\infty)$ be finite.] Here, $k_1^2 = 2M(E + V_0)/\hbar^2$, $k_2^2 = -2ME/\hbar^2$, and l is the angular momentum [n in Eq. (12.111)].

(b) The boundary condition at $r = a$ is that the wave function $\psi(r)$ and its first derivative be continuous. Show that this means

$$\left.\frac{(d/dr)j_l(k_1 r)}{j_l(k_1 r)}\right|_{r=a} = \left.\frac{(d/dr)j_l^{\text{out}}(k_2 r)}{j_l^{\text{out}}(k_2 r)}\right|_{r=a}.$$

This equation determines the energy eigenvalues.

Note. This is a generalization of the deuteron Example 9.1.3.

12.4.16 The quantum mechanical radial wave function for a scattered wave is given by

$$\psi_k = \frac{\sin(kr + \delta_0)}{kr},$$

where k is the wave number, $k = \sqrt{2mE}/\hbar$, and δ_0 is the scattering phase shift. Show that the normalization integral is

$$\int_0^\infty \psi_k(r)\psi_{k'}(r) r^2\, dr = \frac{\pi}{2k}\delta(k - k').$$

Hint. You can use a sine representation of the Dirac delta function.

12.4.17 Derive the spherical Bessel function closure relation

$$\frac{2a^2}{\pi}\int_0^\infty j_n(ar) j_n(br) r^2\, dr = \delta(a - b).$$

Note. An interesting derivation involving Fourier transforms, the Rayleigh plane wave expansion, and spherical harmonics has been given by P. Ugincius, *Am. J. Phys.* **40**, 1690 (1972).

12.4.18 The wave function of a particle in a sphere (Example 12.4.2) with angular momentum l is $\psi(r, \theta, \varphi) = Aj_l((\sqrt{2ME})r/\hbar)Y_l^m(\theta, \varphi)$. The $Y_l^m(\theta, \varphi)$ is a spherical harmonic, described in Section 11.5. From the boundary condition $\psi(a, \theta, \varphi) = 0$ or $j_l((\sqrt{2ME})a/\hbar) = 0$, calculate the 10 lowest energy states. Disregard the m degeneracy ($2l + 1$ values of m for each choice of l). Check your results against Maple, Mathematica, etc.

Check values. $j_l(\alpha_{ls}) = 0$

$$\begin{aligned}
\alpha_{01} &= 3.1416 \\
\alpha_{11} &= 4.4934 \\
\alpha_{21} &= 5.7635 \\
\alpha_{02} &= 6.2832.
\end{aligned}$$

12.4.19 Let Example 12.4.1 be modified so that the potential is a finite V_0 outside ($r > a$). Use symbolic software (Mathematica, Maple, etc.).

(a) For $E < V_0$ show that

$$\psi_{\text{out}}(r, \theta, \varphi) \sim j_l^{\text{out}}\left(\frac{r}{\hbar}\sqrt{2M(V_0 - E)}\right).$$

(b) The new boundary conditions to be satisfied at $r = a$ are

$$\begin{aligned}
\psi_{\text{in}}(a, \theta, \varphi) &= \psi_{\text{out}}(a, \theta, \varphi) \\
\frac{\partial}{\partial r}\psi_{\text{in}}(a, \theta, \varphi) &= \frac{\partial}{\partial r}\psi_{\text{out}}(a, \theta, \varphi)
\end{aligned}$$

or

$$\left.\frac{1}{\psi_{\text{in}}}\frac{\partial \psi_{\text{in}}}{\partial r}\right|_{r=a} = \left.\frac{1}{\psi_{\text{out}}}\frac{\partial \psi_{\text{out}}}{\partial r}\right|_{r=a}.$$

For $l = 0$ show that the boundary condition at $r = a$ leads to

$$f(E) = k\left\{\cot ka - \frac{1}{ka}\right\} + k'\left\{1 + \frac{1}{k'a}\right\} = 0,$$

where $k = \sqrt{2ME}/\hbar$ and $k' = \sqrt{2M(V_0 - E)}/\hbar$.

(c) With $a = \hbar^2/Me^2$ (Bohr radius) and $V_0 = 4Me^4/2\hbar^2$, compute the possible bound states $(0 < E < V_0)$.
Hint. Call a root-finding subroutine after you know the approximate location of the roots of

$$f(E) = 0, \quad (0 \le E \le V_0).$$

(d) Show that when $a = \hbar^2/Me^2$ the minimum value of V_0 for which a bound state exists is $V_0 = 2.4674Me^4/2\hbar^2$.

12.4.20 In some nuclear stripping reactions the differential cross section is proportional to $j_l(x)^2$, where l is the angular momentum. The location of the maximum on the curve of experimental data permits a determination of l if the location of the (first) maximum of $j_l(x)$ is known. Compute the location of the first maximum of $j_1(x)$, $j_2(x)$, and $j_3(x)$.
Note. For better accuracy, look for the first zero of $j_l'(x)$. Why is this more accurate than direct location of the maximum?

12.4.21 A plane wave may be expanded in a series of spherical waves by the Rayleigh equation

$$e^{ikr\cos\gamma} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\gamma).$$

Show that $a_n = i^n(2n + 1)$.
Hint.

- Use the orthogonality of the P_n to solve for $a_n j_n(kr)$.
- Differentiate n times with respect to kr and set $r = 0$ to eliminate the r dependence.
- Evaluate the remaining integral by Exercise 11.4.4.

12.4.22 Verify the Rayleigh expansion of Exercise 12.4.21 by starting with the following steps:

- Differentiate with respect to kr to establish

$$\sum_n j_n'(kr) P_n(\cos\gamma) = i \sum_n a_n j_n(kr) \cos\gamma\, P_n(\cos\gamma).$$

- Use a recurrence relation to replace $\cos\gamma\, P_n(\cos\gamma)$ by a linear combination of P_{n-1} and P_{n+1}.
- Use a recurrence relation to replace j_n' by a linear combination of j_{n-1} and j_{n+1}.

12.4.23 The Legendre polynomials and the spherical Bessel functions are related by

$$j_n(z) = \frac{1}{2}(-1)^n \int_0^{\pi} e^{iz\cos\theta} P_n(\cos\theta) \sin\theta\, d\theta, \quad n = 0, 1, 2 \ldots.$$

Verify this relation by transforming the right-hand side into

$$\frac{z^n}{2^{n+1}n!}\int_0^\pi \cos(z\cos\theta)\sin^{2n+1}\theta\,d\theta$$

using Exercise 12.4.21.

Additional Reading

McBride, E. B. (1971). *Obtaining Generating Functions*. Springer-Verlag, New York. An introduction to methods of obtaining generating functions.

Watson, G. N. (1922). *A Treatise on the Theory of Bessel Functions*. Cambridge Univ. Press, Cambridge, UK.

Watson, G. N. (1952). *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge Univ. Press, Cambridge, UK. This is the definitive text on Bessel functions and their properties. Although difficult reading, it is invaluable as the ultimate reference.